A UNIFIED FORMALISM FOR VARIOUS COUPLING ANALYSES OF COMPOSITE LAMINATES AND ITS APPLICATIONS TO CRACK PROBLEMS

Chyanbin Hwu

ABSTRACT

In this paper, a unified formalism is proposed to deal with several different coupling analyses of composites. This formalism is established from the Stroh formalism for two-dimensional problems and the Stroh-like formalism for coupled stretching-bending problems and their extensions to the magneto-electro-elastic coupling analysis. Since different coupling conditions can be represented by one unified formalism, if for particular problems the boundary conditions of different coupling problems can be written in one unified form, the final solution should also have the same form for different types of problems. To show the usefulness and powerfulness of this unified formalism, a typical example of crack problems is presented in this paper.

Key Words: Stroh formalism, composite laminates, cracks, complex variable formulation

I. INTRODUCTION

The mechanical properties of materials are described by constitutive laws. There are a wide variety of materials existing in the world. We are not surprised that there are a great many constitutive laws describing an almost infinite variety of materials. What is surprising is that a simple idealized stress-strain relationship gives a good description of the mechanical properties of many elastic materials around us. Additionally, some materials such as piezoelectric materials have the ability of converting energy from one form (mechanical to electric) to another. To consider this kind of material, we need to expand the pure stress-strain laws to include the electric field. For other coupling conditions such as the magnetic 2 field and hygrothermal environment, etc., further expansion of the constitutive laws is necessary. It seems that the more coupling of the constitutive laws, the more complex the mathematical modeling. To provide a simple way to solve the various coupling problems, in this paper we try to provide a unified formalism for various coupling analyses of composite laminates.

Due to the nature of anisotropy, composite materials are usually modeled as anisotropic elastic solids. For two-dimensional linear anisotropic elasticity, there are two major complex variable formalisms in the literature. One is Lekhnitskii formalism (Lekhnitskii, 1963; 1968), the other is Stroh formalism (Stroh, 1958). The analysis of piezoelectric materials in two-dimensional elasticity was extended from Stroh's six-dimensional framework to an eight-dimensional formalism (Liang and Hwu, 1996). By extending Stroh formalism for two-dimensional linear anisotropic elasticity, recently we (Hwu, 2003a) developed a Stroh-like formalism for the coupled stretching-bending analysis of composite laminates. Like the extension of Stroh formalism to anisotropic piezoelectric materials, the Stroh-like formalism was also extended to coupled mechanical-electrical-magnetical analysis for magneto-electro-elastic composite laminates (Hsieh and Hwu, 2003). Due to the resemblance of Stroh formalism and Stroh-like formalism, it has been observed through these two formalisms the solutions for the corresponding two-dimensional problems and coupled stretching-bending problems are really very alike (Hwu, 2003b). The only difference between these formalisms is that the symbols they use have different dimensions and

C. Hwu is with the Institute of Aeronautics and Astronautics, National Cheng Kung University, Tainan, Taiwan 701, R.O.C. (Fax: 886-6-2389940; Email:chwu@mail.ncku.edu.tw)

different contents for different types of problems. Therefore, it is expected a unified formalism for several different coupling analyses of composites can be established. To show the usefulness and powerfulness of this unified formalism, a typical example of crack problems is presented in this paper.

II. TWO-DIMENSIONAL DEFORMATION

1. Inplane-Antiplane Coupling

In a fixed rectangular coordinate system x_i, i=1, 2, 3, let u_i, σ_{ij}, ε_{ij} be, respectively, the displacement, stress and strain. The strain-displacement equations, the stress-strain laws, and the equations of equilibrium for anisotropic elasticity are (Ting, 1996).

$$\varepsilon_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}),$$

$$\sigma_{ij}=C_{ijkl}\varepsilon_{kl},$$

$$\sigma_{ij,j}=0, \qquad i, j, k, l=1, 2, 3 \tag{1}$$

where repeated indices imply summation, a comma stands for differentiation and C_{ijks} are the elastic constants which are assumed to be fully symmetric and positive definite. By substituting $(1)_1$ into $(1)_2$ and then into $(1)_3$, the governing equations can be written as

$$C_{ijkl}u_{k,lj}=0, \qquad i, j, k, l=1, 2, 3 \tag{2}$$

Since the governing Eq. (2) are a set of homogeneous partial differential equations, for two-dimensional deformation in which u_i, i=1, 2, 3, depend on x_1 and x_2 only the final solution can be expressed in terms of functions with arguments

$$z=x_1+\mu_k x_2 \tag{3}$$

For *generalized plane strain*, ε_{33}=0 and the stress-strain laws $(1)_2$ can be written in contracted notation as

$$\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{Bmatrix}=\begin{bmatrix}C_{11} & C_{12} & C_{14} & C_{15} & C_{16}\\ C_{12} & C_{22} & C_{24} & C_{25} & C_{26}\\ C_{14} & C_{24} & C_{44} & C_{45} & C_{46}\\ C_{15} & C_{25} & C_{45} & C_{55} & C_{56}\\ C_{16} & C_{26} & C_{46} & C_{56} & C_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_4\\ \varepsilon_5\\ \varepsilon_6\end{Bmatrix} \tag{4}$$

If the anisotropic material has one plane of elastic symmetry at x_3=0, it is one kind of *monoclinic material*. Due to the material symmetry with respect to the plane x_3=0, it can easily be proved that parts of the elastic constants will be identical to zero, i.e.,

$$C_{14}=C_{15}=C_{24}=C_{25}=C_{34}=C_{35}=C_{46}=C_{56}=0 \tag{5}$$

Substituting (5) into (4), the stress-strain laws for the problems of generalized plane strain can then be split into two parts, i.e.,

$$\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_6\end{Bmatrix}=\begin{bmatrix}C_{11} & C_{12} & C_{16}\\ C_{12} & C_{22} & C_{26}\\ C_{16} & C_{26} & C_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_6\end{Bmatrix},$$

$$\begin{Bmatrix}\sigma_4\\ \sigma_5\end{Bmatrix}=\begin{bmatrix}C_{44} & C_{45}\\ C_{45} & C_{55}\end{bmatrix}\begin{Bmatrix}\varepsilon_4\\ \varepsilon_5\end{Bmatrix} \tag{6}$$

which means that the inplane and antiplane problems will be decoupled for monoclinic materials with symmetry plane x_3=0. The former (6) corresponds to the *inplane problems* defined by

$$u_1=u_1(x_1, x_2),\ u_2=u_2(x_1, x_2),\ u_3=0 \tag{7}$$

whereas the latter corresponds to *antiplane problems* defined by

$$u_1=u_2=0,\ u_3=u(x_1, x_2) \tag{8}$$

For the problems of *generalized plane stress*, similar relations can be written in terms of $\hat{C}_{pq}$ instead of C_{pq} where

$$\hat{C}_{pq}=C_{pq}-\frac{C_{p3}C_{3q}}{C_{33}}=\hat{S}_{qp}, \qquad p, q=1, 2, 4, 5, 6 \tag{9}$$

For general anisotropic materials, no material symmetry has been assumed. Hence, applying in-plane forces, out-of-plane components of displacements and stresses may be induced, and applying anti-plane forces, in-plane components of displacements and stresses may be induced. That is, the in-plane and anti-plane deformations may couple each other for general anisotropic materials subjected to inplane and/or antiplane forces.

2. Electro-Elastic Coupling

It is well known that piezoelectric materials have the ability to convert energy from one form (mechanical to electric) to another. In other words, these materials can produce an electric field when deformed and undergo deformation when subjected to an electric field. Therefore, instead of the constitutive relation shown in $(1)_2$, the constitutive law for an anisotropic linear piezoelectric medium can be written as (Rogacheva, 1993)

$$\sigma_{ij}=C^E_{ijkl}\varepsilon_{kl}-e_{kij}E_k,$$

$$D_j=e_{jkl}\varepsilon_{kl}+\omega^S_{jk}E_k, \qquad i, j, k, l=1, 2, 3 \tag{10}$$

where σ_{ij}, ε_{kl}, D_j and E_k are, respectively, the stress, strain, electric displacement and electric field; C^E_{ijkl}, e_{kij} and ω^S_{jk} are the elastic stiffness tensor at constant electric field, piezoelectric stress tensor and dielectric permittivity tensor at constant strain, respectively, which are all assumed to be fully symmetric.

Unlike the constitutive laws, the strain-displacement relations and equilibrium equations shown in (1) remain the same for anisotropic piezoelasticity. In addition to these relations, one more relation that should be considered is the electrostatic condition expressed by

$$D_{i,i}=0 \tag{11}$$

By letting

$$\begin{aligned}
&E_l=-u_{4,l},\ D_j=\sigma_{4j}, && j, l=1, 2, 3,\\
&C_{ijkl}=C^E_{ijkl}, && i, j, k, l=1, 2, 3,\\
&C_{ij4l}=e_{lij}, && i, j, l=1, 2, 3,\\
&C_{4jkl}=e_{jkl}, && j, k, l=1, 2, 3,\\
&C_{4j4l}=-\omega^S_{jl}, && j, l=1, 2, 3
\end{aligned} \tag{12}$$

the strain-displacement relations $(1)_1$, constitutive laws (10), equilibrium Eq. $(1)_3$ and electrostatic Eq. (11) can be rewritten in an *expanded tensor notation* as

$$\begin{aligned}
&\varepsilon_{qj}=\frac{1}{2}(u_{q,j}+u_{j,q}),\\
&\sigma_{pj}=C_{pjql}\varepsilon_{ql},\\
&\sigma_{pj,j}=0, \quad p, q=1, 2, 3, 4, \quad j, l=1, 2, 3
\end{aligned} \tag{13}$$

Like the coupled inplane-antiplane problems, by combining all the equations of (13) the governing equation can be written as

$$C_{pjql}u_{q,lj}=0, \quad j, l=1, 2, 3, \quad p, q=1, 2, 3, 4 \tag{14}$$

For two-dimensional problems, the displacements and electric fields are assumed to depend on x_1 and x_2 only, i.e.

$$\begin{aligned}
&u_1=u_1(x_1, x_2),\ u_2=u_2(x_1, x_2),\ u_3=u_3(x_1, x_2),\\
&E_1=E_1(x_1, x_2),\ E_2=E_2(x_1, x_2)
\end{aligned} \tag{15}$$

3. Magneto-Electro-Elastic Coupling

In addition to electro-elastic coupling, in some cases the additional coupling of a magnetic field should be considered. For linear anisotropic magneto-electro-elastic solids, the strain-displacement relations, the constitutive laws and the equilibrium equations including the balance of the body force and electric charge and current can be written as (Hsieh and Hwu, 2003)

$$\begin{aligned}
&\varepsilon_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}),\\
&\sigma_{ij}=C_{ijks}\varepsilon_{ks}-e_{kij}E_k-q_{kij}H_k,\\
&D_i=e_{iks}\varepsilon_{ks}+\omega_{ik}E_k+d_{ik}H_k,\\
&B_i=q_{iks}\varepsilon_{ks}+d_{ik}E_k+\chi_{ik}H_k,\\
&\sigma_{ij,j}+f_i=0,\ D_{j,j}-f_e=0,\ B_{j,j}-f_m=0,\ i, j, k, s=1, 2, 3
\end{aligned} \tag{16}$$

in which the additional notations H_k, B_i, q_{kij},d_{ik} and χ_{ik} are, respectively, the 6 magnetic field, magnetic induction, piezomagnetic coefficients, magnetoelectric and magnetic permeability coefficients. f_i, f_e and f_m are the body force, electric charge density, and magnetic charge density, respectively. Similar to the augmentation introduced in (12), if we let

$$\begin{aligned}
&E_k=-u_{4,k},\ H_k=-u_{5,k},\\
&D_k=\sigma_{4k},\ B_k=\sigma_{5k},
\end{aligned} \tag{17}$$

$$\begin{aligned}
&C_{ijkl}=C_{ijkl}, && i, j, k, l=1, 2, 3,\\
&C_{ij4l}=e_{sij}, && i, j, l=1, 2, 3,\\
&C_{ij5l}=q_{sij}, && i, j, l=1, 2, 3,\\
&C_{4jkl}=e_{jkl}, && j, k, l=1, 2, 3,\\
&C_{5jkl}=q_{jks}, && j, k, l=1, 2, 3,\\
&C_{4j4l}=-\omega_{jl}, && j, l=1, 2, 3,\\
&C_{4j5l}=-d_{js}, && j, l=1, 2, 3,\\
&C_{5j5l}=-\chi_{js}, && j, l=1, 2, 3
\end{aligned} \tag{18}$$

the basic Eq. (16) can be rewritten into the expanded tensor notation stated in (13) with f_i, f_e and f_m neglected. The only difference is the range of subscripts p and q, which is now expanded to 5 to include the magnetic field. Thus, the governing equation of the coupled magneto-electro-elastic problems can also be written as that shown in (14) by only expanding the range of the subscripts p and q.

III. EXTENSION-FLEXURE DEFORMATION

1. Stretching-Bending Coupling

To describe the overall properties and macromechanical behavior of a laminate, the most popular way is the classical lamination theory. According to the observation of actual mechanical behavior of laminates, Kirchhoff's assumptions are usually made for the displacement fields, i.e., the laminate displacements U_1 and U_2 in the x_1 and x_2 directions are assumed to be

$$U_i(x_1, x_2, x_3)=u_i(x_1, x_2)-x_3w_{,i}(x_1, x_2),\ i=1, 2 \tag{19}$$

where u_1, u_2 and w are the middle surface displacements in the x_1, x_2 and x_3 directions. Based upon this assumption, the kinematic relations, the constitutive laws, and the equilibrium equations can be written in tensor notation as (Hwu, 2003)

$$\begin{aligned}&\varepsilon_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}),\quad \kappa_{ij}=\frac{1}{2}(\beta_{i,j}+\beta_{j,i}),\\&N_{ij}=A_{ijkl}\varepsilon_{kl}+B_{ijkl}\kappa_{kl},\ M_{ij}=B_{ijkl}\varepsilon_{kl}+D_{ijkl}\kappa_{kl},\\&N_{ij,j}=0,\ M_{ij,ij}+q=0,\ Q_i=M_{ij,j},\ i, j, k, l=1, 2\end{aligned} \tag{20}$$

where $\beta_1=-w_{,1}$, $\beta_2=-w_{,2}$ are the negative of the slope of the middle surface in the x_1 and x_2 directions; ε_{ij} and κ_{ij} denote the mid-plane strain and plate curvature; N_{ij}, M_{ij} and Q_i denote the resultant forces, bending moments and shear forces; A_{ijkl}, B_{ijkl} and D_{ijkl} are, respectively, the extensional, coupling and bending stiffness tensors; q is the lateral distributed load applied on the laminates. Again, combining all the basic Eq. (20) and considering free lateral loading condition (q=0), the governing equations for the coupled extension-bending problems can be written as

$$\begin{aligned}&A_{ijkl}u_{k,lj}+B_{ijkl}\beta_{k,lj}=0,\\&B_{ijkl}u_{k,lij}+D_{ijkl}\beta_{k,lij}=0,\quad i, j, k, l=1, 2\end{aligned} \tag{21}$$

which are also a set of homogeneous partial differential equations.

In the previous section we know that the inplane and antiplane problems can be uncoupled if the anisotropic materials have one plane of symmetry at x_3=0. A similar situation occurs for the uncoupling of the inplane and bending problems if the composite laminates are symmetric with respect to the mid-plane because the coupling stiffness will be identical to zero for the symmetric laminates. The displacement fields defined in (19) can then be spilt into two parts. One is the inplane stretching problem defined by

$$\begin{aligned}&U_i(x_1, x_2, x_3)=u_i(x_1, x_2),\ i=1, 2,\\&U_3(x_1, x_2, x_3)=0\end{aligned} \tag{22}$$

The other is the plate bending problem defined by

$$\begin{aligned}&U_i(x_1, x_2, x_3)=-x_3w_{,i}(x_1, x_2),\ i=1, 2,\\&U_3(x_1, x_2, x_3)=w(x_1, x_2)\end{aligned} \tag{23}$$

2. Electro-Elastic Coupling

Like the two-dimensional problems, to consider the piezoelectric coupling the material properties of each lamina may be expressed by the constitutive laws given in (10). Based upon Kirchhoff's assumptions, the displacement and electric fields can be 8 written as

$$\begin{aligned}&U_i(x_1, x_2, x_3)=u_i(x_1, x_2)-x_3w_{,i}(x_1, x_2),\\&E_i(x_1, x_2, x_3)=E_i^{(0)}(x_1, x_2)+x_3E_i^{(1)}(x_1, x_2),\\&\qquad i=1, 2\end{aligned} \tag{24}$$

where U_i, E_i, i=1, 2 are the displacements and electric fields in x_i-direction;$E_i^{(0)}$ are the mid-plane electric fields and $E_i^{(1)}$ are the rate change of the electric fields in the thickness direction.

Based upon the assumptions (24) and the basic equations for the piezoelectric materials described in Section II, the generalized constitutive law and equilibrium equations for the extension-flexure deformation can be expressed in tensor notation as (Hwu and Hsieh, 2004)

$$\begin{aligned}&N_{pq}+A_{pqrs}u_{r,s}+B_{pqrs}\beta_{r,s},\\&M_{pq}=B_{pqrs}u_{r,s}+D_{pqrs}\beta_{r,s},\\&N_{pj,j}=0,\ M_{ij,ij}=0,\ M_{4j,j}=0,\\&p, q, r, s=1, 2, 4;\ i, j=1, 2\end{aligned} \tag{25}$$

Combination of all the basic Eq. (25) now leads to

$$\begin{aligned}&A_{pjrl}u_{r,lj}+B_{pjrl}\beta_{r,lj}=0,\\&B_{ijrl}u_{r,lij}+D_{ijrl}\beta_{r,lij}=0,\\&B_{4jrl}u_{r,lj}+D_{4jrl}\beta_{r,lj}=0,\\&p, r=1, 2, 4;\ i, j, l=1, 2\end{aligned} \tag{26}$$

In (25) we only consider the homogeneous case: no lateral load or electric charge is applied on the laminates. To include the electric field and electric displacement in the generalized expressions (25), some symbols have been augmented such as

$$N_{4j} = \tilde{D}_j = \int_{-h/2}^{h/2} D_j dx_3 \,,$$

$$M_{4j} = \tilde{D}_j^* = \int_{-h/2}^{h/2} D_j x_3 dx_3 \,, \quad j=1,\, 2 \tag{27}$$

and the contracted notations of A_{ijkl}, B_{ijkl}, and D_{ijkl} are defined by

$$A_{\alpha\beta} = \sum_{k=1}^{n} (C_{\alpha\beta})_k (h_k - h_{k-1}) \,,$$

$$B_{\alpha\beta} = \frac{1}{2}\sum_{k=1}^{n} (C_{\alpha\beta})_k (h_k^2 - h_{k-1}^2) \,,$$

$$D_{\alpha\beta} = \frac{1}{3}\sum_{k=1}^{n} (C_{\alpha\beta})_k (h_k^3 - h_{k-1}^3) \tag{28a}$$

where

$$\begin{aligned} C_{\alpha\beta} &= C_{\alpha\beta}^E, && \alpha,\, \beta=1,\, 2,\, 6, \\ &= e_{1\alpha}, && \alpha=1,\, 2,\, 6,\, \beta=7, \\ &= e_{2\alpha}, && \alpha=1,\, 2,\, 6,\, \beta=8, \\ &= -\omega_{(\alpha-6)(\beta-6)}^S, && \alpha,\, \beta=7,\, 8 \end{aligned} \tag{28b}$$

and h_k and h_{k-1} denote, respectively, the location of the bottom and top surfaces of the kth lamina. Detailed explanations of these symbols can be found in (Hwu and Hsieh, 2004).

3. Magneto-Electro-Elastic Coupling

Like the magento-electro-elastic coupling discussed in two-dimensional deformation, by following the same approach the final governing equation of the present case can also be written into Eq. (26) by only expanding the range of the subscripts p and r from 1,2,4 to 1,2,4,5. One can refer to (Hsieh and Hwu, 2003) for detailed derivation.

IV. STROH FORMALISM

To find the general solutions satisfying all the basic equations of each type of problem stated in Sections II and III, several ideas have been put forward in the literature. Since these problems are different in their nature and their mathematical form, it is hard to imagine that their solutions can all be expressed in a unified formalism. Through several researchers' studies in the literature, one common feature was obtained for these different coupling problems, i.e., their governing Eqs. (2), (14), (21) and (26) are all sets of homogeneous partial differential equations and hence their final solutions can all be expressed in terms of functions with arguments given in (3). Detailed comparison of the basic equations and their associated governing equations are shown in Table 1. From the governing equations shown in Table 1, we believe that it is possible to have a unified formalism for all these different coupling problems.

Because μ_k in (3) have been proved to be complex variables and will appear in pairs of complex conjugate, the formulations starting from the governing equations with sets of homogeneous partial differential equations are usually called *complex variable formulations*. In the literature, different complex variable formulations have been proposed for different types of problems. Among them, we have expended a lot of effort (Hwu, 1990; Liang and Hwu, 1996; Hwu, 2003a; Hwu and Hsieh, 2004) establishing a unified formalism extending from *Stroh formalism* for two-dimensional problems (Stroh, 1958; Ting, 1996). Here, we would like to state that this formalism can express all the general solutions of different types of problems discussed in Sections II and III in one form. By this formalism, all the general solutions for different kinds of problems can be expressed as

$$\boldsymbol{u}=2\mathrm{Re}\{\boldsymbol{A}\boldsymbol{f}(z)\},\ \boldsymbol{\phi}=2\mathrm{Re}\{\boldsymbol{B}\boldsymbol{f}(z)\} \tag{29}$$

where $\boldsymbol{u}$ and $\boldsymbol{\phi}$ are the displacement vector and the stress function vector; Re stands for the real part ; $\boldsymbol{A}$ and $\boldsymbol{B}$ are the material eigenvector matrices composed of the material eigenvectors $\boldsymbol{a}_k$ and $\boldsymbol{b}_k$; $\boldsymbol{f}(z)$ f is a complex function vector composed of $\boldsymbol{f}_k(z_k)$ which are holomorphic functions of complex variables z_k and these variables are related to the material eigenvalues μ_k by

$$z_k=x_1+\mu_k x_2 \tag{30}$$

The material eigenvalues μ_k and the material eigenvectors $(\boldsymbol{a}_k, \boldsymbol{b}_k)$ a can be determined by the following eigen-relation:

$$\boldsymbol{N}\boldsymbol{\xi}=\mu\boldsymbol{\xi} \tag{31}$$

where $\boldsymbol{N}$ is the fundamental elasticity matrix and ξ is a column vector defined by

$$\boldsymbol{N} = \begin{bmatrix} \boldsymbol{N}_1 & \boldsymbol{N}_2 \\ \boldsymbol{N}_3 & \boldsymbol{N}_1^T \end{bmatrix}, \quad \boldsymbol{\xi} = \begin{Bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{Bmatrix} \tag{32}$$

Table 1 Basic equations and governing equations

	Two-dimensional deformation		Extension-flexure deformation	
	Inplane-antiplane coupling	Electro-elastic coupling	Extension-bending coupling	Electro-elastic coupling
Basic equations	$\varepsilon_{ij}=(u_{i,j}+u_{j,i})/2$, $\sigma_{ij}=C_{ijkl}\varepsilon_{kl}$, $\sigma_{ij,j}=0$, $i, j, k, l=1, 2, 3$, $u_i=u_i(x_1, x_2)$, $i=1, 2, 3$	$\varepsilon_{ij}=(u_{i,j}+u_{j,i})/2$, $\sigma_{ij}=C^E_{ijkl}\varepsilon_{kl}-e_{lij}E_l$, $D_j=e_{jkl}\varepsilon_{kl}+\omega^S_{jl}E_l$, $\sigma_{ij,j}=0$, $D_{j,j}=0$, $i, j, k, l=1, 2, 3$, $u_i=u_i(x_1, x_2)$, $E_i=E_i(x_1, x_2)$, $i=1, 2, 3$	$\varepsilon_{ij}=(u_{i,j}+u_{j,i})/2$, $\kappa_{ij}=(\beta_{i,j}+\beta_{j,i})/2$, $N_{ij}=A_{ijkl}\varepsilon_{kl}+B_{ijkl}\kappa_{kl}$, $M_{ij}=B_{ijkl}\varepsilon_{kl}+D_{ijkl}\kappa_{kl}$, $N_{ij,j}=0$, $M_{ij,ij}=0$, $Q_i=M_{ij,j}$, $i, j, k, l=1, 2$, $\beta_1=-w_{,1}$, $\beta_2=-w_{,2}$ $u_i=u_i(x_1, x_2)$, $\beta_i=\beta_i(x_1, x_2)$, $i=1, 2$	$\varepsilon_{ij}=(u_{i,j}+u_{j,i})/2$, $\kappa_{ij}=(\beta_{i,j}+\beta_{j,i})/2$, $N_{ij}=A_{ijkl}\varepsilon_{kl}+B_{ijkl}\kappa_{kl}-A^e_{lij}E^{(0)}_l-B^e_{lij}E^{(1)}_l$, $M_{ij}=B_{ijkl}\varepsilon_{kl}+D_{ijkl}\kappa_{kl}-B^e_{lij}E^{(0)}_l-D^e_{lij}E^{(1)}_l$, $\tilde{D}_j=A^e_{jkl}\varepsilon_{kl}+B^e_{jkl}\kappa_{kl}+A^\omega_{jl}E^{(0)}_l+B^\omega_{jl}E^{(1)}_l$, $\tilde{D}^*_j=B^e_{jkl}\varepsilon_{kl}+D^e_{jkl}\kappa_{kl}+B^\omega_{jl}E^{(0)}_l+D^\omega_{jl}E^{(0)}_l$, $N_{ij,j}=0$, $\tilde{D}_{j,j}=0$ $M_{ij,ij}=0$, $Q_i=M_{ij,j}$, $\tilde{D}^*_{j,j}=0$ $i, j, k, l=1, 2$, $\beta_1=-w_{,1}$, $\beta_2=-w_{,2}$ $u_i=u_i(x_1, x_2)$, $\beta_i=\beta_i(x_1, x_2)$, $E^{(0)}_i=E^{(0)}_i(x_1, x_2)$, $E^{(1)}_i=E^{(1)}_i(x_1, x_2)$, $i=1, 2$
Augmentation		$E_l=-u_{4,l}$, $D_j=\sigma_{4j}$, $j, l=1, 2, 3$ $C_{ijkl}=C^E_{ijkl}$, $i, j, k, l=1, 2, 3$, $C_{ij4l}=e_{lij}$, $i, j, l=1, 2, 3$, $C_{4jkl}=e_{jkl}$, $j, k, l=1, 2, 3$, $C_{4j4l}=-\omega^S_{jl}$, $j, l=1, 2, 3$		$E^{(0)}_j=-u_{4,j}$, $E^{(1)}_j=-\beta_{4,j}$ $\tilde{D}_i=N_{4i}$, $\tilde{D}^*_i=M_{4i}$, $i=1, 2$, $C_{ijkl}=C^E_{ijkl}$, $i, j, k, l=1, 2$, $C_{ij4l}=e_{lij}$, $i, j, l=1, 2$, $C_{4jkl}=e_{jkl}$, $j, k, l=1, 2$, $C_{4j4l}=-\omega^S_{jl}$, $j, l=1, 2$,
Governing equation	$C_{ijkl}u_{k,lj}=0$, $i, j, k, l=1, 2, 3$,	$C_{pjql}u_{q,lj}=0$, $j, l=1, 2, 3$, $p, q=1, 2, 3, 4$	$A_{ijkl}u_{k,lj}+B_{ijkl}\beta_{k,lj}=0$, $B_{ijkl}u_{k,lij}+D_{ijkl}\beta_{k,lij}=0$, $\beta_{1,2}=\beta_{2,1}$, $i, j, k, l=1, 2$	$A_{pjrl}u_{r,lj}+B_{pjrl}\beta_{r,lj}=0$, $B_{ijrl}u_{r,lij}+D_{ijrl}\beta_{r,lij}=0$, $B_{4jrl}u_{r,lj}+D_{4jrl}\beta_{r,lj}=0$, $\beta_{1,2}=\beta_{2,1}$, $p, r=1, 2, 4; i, j, l=1, 2$

The physical components of the displacement vector $\boldsymbol{u}$ and the stress function vector $\boldsymbol{\phi}$ depend on the problem type. Moreover, the contents of the fundamental elasticity matrix $\boldsymbol{N}$ and its related by-products μ_k and $(\boldsymbol{a}_k, \boldsymbol{b}_k)$ a all depend on the problem type. Detailed representations of these symbols for different types of problems are shown in Tables 2-4. From Tables 3-4 we see that the fundamental elasticity matrix $\boldsymbol{N}$ can be determined from the mechanical properties and has different contents and dimensions for different types of problems. After calculating the fundamental elasticity matrix $\boldsymbol{N}$ from the mechanical properties, the material eigenvalues μ_k and the material eigenvectors $(\boldsymbol{a}_k, \boldsymbol{b}_k)$ a are determined from the eigen-relation (31). After we know μ_k and $(\boldsymbol{a}_k, \boldsymbol{b}_k)$, the eigenvector matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ can be constructed by the definitions given in Table 2 for different types of problems. Now, the only unknown remaining to be determined is the function vector $\boldsymbol{f}(z)$ which should then be determined by satisfaction of the boundary conditions set for each particular problem. To have a clear picture about the matrices used in Stroh formalism for different types of problems, Table 5 shows the properties and dimensions of all these matrices.

Note that in Tables 1-5 and the following examples, to save space in this paper we did not show the case of magneto-electro-elastic coupling because it is just a simple extension of the electro-elastic coupling.

V. CRACKS IN VARIOUS CONDITIONS

Consider an unbounded homogeneous plate containing a straight crack loaded at infinity. The crack is assumed to lie on the x_1-axis with its center located at the origin and the crack size is $2a$. If the crack surfaces are assumed to be free of traction and electric charge and the generalized loads at infinity are uniformly distributed, the boundary conditions for different types of problems can be written as follows

For two-dimensional (2D) deformation:

(i) Inplane-antiplane coupling,

$$\sigma_{12}=\sigma_{22}=\sigma_{32}=0, \text{ when } x_2=0, |x_1|<a,$$

$$\sigma_{ij}=\sigma_{ij}^{\infty}, \text{ when } x_1, x_2\to\infty \tag{33}$$

(ii) Electro-elastic coupling,

$$\sigma_{12}=\sigma_{22}=\sigma_{32}=D_2=0, \text{ when } x_2=0, |x_1|<a,$$

$$\sigma_{ij}=\sigma_{ij}^{\infty},\ D_i\to D_i^{\infty} \text{ when } x_1, x_2\to\infty \tag{34}$$

For extension-flexure (EF) deformation:

(i) Stretching-bending coupling,

$$N_{12}=N_{22}=M_{32}=V_2=0, \text{ when } x_2=0, |x_1|<a,$$

$$N_{ij}\to N_{ij}^{\infty},\ M_{ij}\to M_{ij}^{\infty}, \text{ when } x_1, x_2\to\infty \tag{35}$$

(ii) Electro-elastic coupling,

$$N_{12}=N_{22}=M_{22}=V_2=\tilde{D}_2=\tilde{D}_2^{*}=0,$$

$$\text{when } x_2=0, |x_1|<a,$$

$$N_{ij}\to N_{ij}^{\infty},\ M_{ij}\to M_{ij}^{\infty},\ \tilde{D}_i\to\tilde{D}_i^{\infty},\ \tilde{D}_i^{*}\to\tilde{D}_i^{*\infty},$$

$$\text{when } x_1, x_2\to\infty \tag{36}$$

From the relations shown in Table 2, we see that all four different boundary conditions, (33)-(36), can be written in terms of the stress function vector $\boldsymbol{\phi}$ and be expressed by the following unified expression

$$\boldsymbol{\phi}=0, \text{ when } x_2=0, |x_1|<a,$$

$$\boldsymbol{\phi}=x_1\boldsymbol{t}_2^{\infty}-x_2\boldsymbol{t}_1^{\infty}, \text{ when } x_1, x_2\to\infty \tag{37}$$

where $\boldsymbol{t}_1^{\infty}$, $\boldsymbol{t}_2^{\infty}$ and their associated strain vectors $\boldsymbol{\varepsilon}_1^{\infty}$, $\boldsymbol{\varepsilon}_2^{\infty}$ are defined by

$$\boldsymbol{t}_1^{\infty}=\begin{Bmatrix}\sigma_{11}^{\infty}\\ \sigma_{12}^{\infty}\\ \sigma_{13}^{\infty}\end{Bmatrix},\quad \boldsymbol{t}_2^{\infty}=\begin{Bmatrix}\sigma_{21}^{\infty}\\ \sigma_{22}^{\infty}\\ \sigma_{23}^{\infty}\end{Bmatrix},\quad \boldsymbol{\varepsilon}_1^{\infty}=\begin{Bmatrix}\varepsilon_{11}^{\infty}\\ \varepsilon_{12}^{\infty}\\ 2\varepsilon_{13}^{\infty}\end{Bmatrix},$$

$$\boldsymbol{\varepsilon}_2^{\infty}=\begin{Bmatrix}\varepsilon_{21}^{\infty}\\ \varepsilon_{22}^{\infty}\\ 2\varepsilon_{23}^{\infty}\end{Bmatrix}, \text{ 2D inplane-antiplane} \tag{38a}$$

$$\boldsymbol{t}_1^{\infty}=\begin{Bmatrix}\sigma_{11}^{\infty}\\ \sigma_{12}^{\infty}\\ \sigma_{13}^{\infty}\\ D_1^{\infty}\end{Bmatrix},\quad \boldsymbol{t}_2^{\infty}=\begin{Bmatrix}\sigma_{21}^{\infty}\\ \sigma_{22}^{\infty}\\ \sigma_{23}^{\infty}\\ D_2^{\infty}\end{Bmatrix},\quad \boldsymbol{\varepsilon}_1^{\infty}=\begin{Bmatrix}\varepsilon_{11}^{\infty}\\ \varepsilon_{12}^{\infty}\\ 2\varepsilon_{13}^{\infty}\\ -E_1^{\infty}\end{Bmatrix},$$

$$\boldsymbol{\varepsilon}_2^{\infty}=\begin{Bmatrix}\varepsilon_{21}^{\infty}\\ \varepsilon_{22}^{\infty}\\ 2\varepsilon_{23}^{\infty}\\ -E_2^{\infty}\end{Bmatrix}, \text{ 2D electro-elastic} \tag{38b}$$

$$\boldsymbol{t}_1^{\infty}=\begin{Bmatrix}N_{11}^{\infty}\\ N_{12}^{\infty}\\ M_{11}^{\infty}\\ M_{12}^{\infty}\end{Bmatrix},\quad \boldsymbol{t}_2^{\infty}=\begin{Bmatrix}N_{12}^{\infty}\\ N_{22}^{\infty}\\ M_{12}^{\infty}\\ M_{22}^{\infty}\end{Bmatrix},\quad \boldsymbol{\varepsilon}_1^{\infty}=\begin{Bmatrix}\varepsilon_{11}^{\infty}\\ \varepsilon_{12}^{\infty}\\ \kappa_{11}^{\infty}\\ \kappa_{12}^{\infty}\end{Bmatrix},$$

$$\boldsymbol{\varepsilon}_2^{\infty}=\begin{Bmatrix}\varepsilon_{21}^{\infty}\\ \varepsilon_{22}^{\infty}\\ \kappa_{12}^{\infty}\\ \kappa_{22}^{\infty}\end{Bmatrix}, \text{ EF stretching-bending} \tag{38c}$$

Table 2 General solution of Stroh formalism

The unified expression: $\boldsymbol{u}=2\mathrm{Re}\{\boldsymbol{A}\boldsymbol{f}(z)\}$, $\phi=2\mathrm{Re}\{\boldsymbol{B}\boldsymbol{f}(z)\}$, $z_k=x_1+\mu_k x_2$

	Two-dimensional deformation		Extension-flexure deformation	
	Inplane-antiplane coupling	Electro-elastic coupling	Extension-bending coupling	Electro-elastic coupling
Components	$\boldsymbol{u}=\begin{Bmatrix}u_1\\u_2\\u_3\end{Bmatrix}$, $\boldsymbol{\phi}=\begin{Bmatrix}\phi_1\\\phi_2\\\phi_3\end{Bmatrix}$, $\boldsymbol{f}(z)=\begin{Bmatrix}f_1(z_1)\\f_2(z_2)\\f_3(z_3)\end{Bmatrix}$ $\boldsymbol{A}=[\boldsymbol{a}_1\ \boldsymbol{a}_2\ \boldsymbol{a}_3]$ $\boldsymbol{B}=[\boldsymbol{b}_1\ \boldsymbol{b}_2\ \boldsymbol{b}_3]$	$\boldsymbol{u}=\begin{Bmatrix}u_1\\u_2\\u_3\\u_4\end{Bmatrix}$, $\boldsymbol{\phi}=\begin{Bmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{Bmatrix}$, $\boldsymbol{f}(z)=\begin{Bmatrix}f_1(z_1)\\f_2(z_2)\\f_3(z_3)\\f_4(z_4)\end{Bmatrix}$ $\boldsymbol{A}=[\boldsymbol{a}_1\ \boldsymbol{a}_2\ \boldsymbol{a}_3\ \boldsymbol{a}_4]$ $\boldsymbol{B}=[\boldsymbol{b}_1\ \boldsymbol{b}_2\ \boldsymbol{b}_3\ \boldsymbol{b}_4]$	$\boldsymbol{u}=\begin{Bmatrix}u_1\\u_2\\\beta_1\\\beta_2\end{Bmatrix}$, $\boldsymbol{\phi}=\begin{Bmatrix}\phi_1\\\phi_2\\\psi_1\\\psi_2\end{Bmatrix}$, $\boldsymbol{f}(z)=\begin{Bmatrix}f_1(z_1)\\f_2(z_2)\\f_3(z_3)\\f_4(z_4)\end{Bmatrix}$ $\boldsymbol{A}=[\boldsymbol{a}_1\ \boldsymbol{a}_2\ \boldsymbol{a}_3\ \boldsymbol{a}_4]$ $\boldsymbol{B}=[\boldsymbol{b}_1\ \boldsymbol{b}_2\ \boldsymbol{b}_3\ \boldsymbol{b}_4]$	$\boldsymbol{u}=\begin{Bmatrix}u_1\\u_2\\u_4\\\beta_1\\\beta_2\\\beta_4\end{Bmatrix}$, $\boldsymbol{\phi}=\begin{Bmatrix}\phi_1\\\phi_2\\\phi_4\\\psi_1\\\psi_2\\\psi_4\end{Bmatrix}$, $\boldsymbol{f}(z)=\begin{Bmatrix}f_1(z_1)\\f_2(z_2)\\f_3(z_3)\\f_4(z_4)\\f_5(z_5)\\f_6(z_6)\end{Bmatrix}$ $\boldsymbol{A}=[\boldsymbol{a}_1\ \boldsymbol{a}_2\ \boldsymbol{a}_3\ \boldsymbol{a}_4\ \boldsymbol{a}_5\ \boldsymbol{a}_6]$ $\boldsymbol{B}=[\boldsymbol{b}_1\ \boldsymbol{b}_2\ \boldsymbol{b}_3\ \boldsymbol{b}_4\ \boldsymbol{b}_5\ \boldsymbol{b}_6]$
Relations	$\sigma_{i1}=-\phi_{i,2}$, $\sigma_{i2}=\phi_{i,1}$, $i=1, 2, 3$	$\sigma_{i1}=-\phi_{i,2}$, $\sigma_{i2}=\phi_{i,1}$, $i=1, 2, 3$ $D_1=-\phi_{4,2}$, $D_2=\phi_{4,1}$ $E_i=-u_{4,i}$, $i=1, 2$	$\beta_1=-w_{,1}$, $\beta_2=-w_{,2}$ $M_{i1}=-\psi_{i,2}-\frac{1}{2}\lambda_{i1}\psi_{k,k}$, $M_{i2}=-\psi_{i,1}-\frac{1}{2}\lambda_{i2}\psi_{k,k}$ $Q_1=-\frac{1}{2}\psi_{k,k2}$, $Q_2=-\frac{1}{2}\psi_{k,k1}$, $V_1=-\psi_{2,22}$, $V_2=\psi_{1,11}$, $N_{i1}=-\phi_{i,2}$, $N_{i2}=\phi_{i,1}$, $i, k=1, 2$ $\lambda_{11}=\lambda_{22}=0$, $\lambda_{12}=-\lambda_{21}=1$	$\beta_1=-w_{,1}$, $\beta_2=-w_{,2}$ $M_{i1}=-\psi_{i,2}-\frac{1}{2}\lambda_{i1}\psi_{k,k}$, $M_{i2}=\psi_{i,1}-\frac{1}{2}\lambda_{i2}\psi_{k,k}$ $Q_1=-\frac{1}{2}\psi_{k,k2}$, $Q_2=-\frac{1}{2}\psi_{k,k1}$, $V_1=-\psi_{2,22}$, $V_2=\psi_{1,11}$, $N_{i1}=-\phi_{i,2}$, $N_{i2}=\phi_{i,1}$, $i=1, 2, 4$, $k=1, 2$ $\lambda_{11}=\lambda_{22}=\lambda_{41}=\lambda_{42}=0$, $\lambda_{12}=-\lambda_{21}=1$ $N_{4j}=\tilde{D}_j=\int_{-h/2}^{h/2}D_j dx_3$, $M_{4j}=\tilde{D}_j^*=\int_{-h/2}^{h/2}D_j x_3 dx_3$, $j=1, 2$

Table 3 The eigen-relation

The unified expression: $\boldsymbol{N}\boldsymbol{\xi}=\mu\xi$, $\boldsymbol{N}=\begin{bmatrix}\boldsymbol{N}_1 & \boldsymbol{N}_2\\ \boldsymbol{N}_3 & \boldsymbol{N}_1^T\end{bmatrix}$, $\xi=\begin{Bmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{Bmatrix}$

Two-dimensional deformation		Extension-flexure deformation	
Inplane-antiplane coupling	Electro-elastic coupling	Extension-bending coupling	Electro-elastic coupling
$\boldsymbol{N}_1=-\boldsymbol{T}^{-1}\boldsymbol{R}^T$,	$\boldsymbol{N}_1=-\boldsymbol{T}^{-1}\boldsymbol{R}^T$,	$\boldsymbol{N}=\boldsymbol{I}_t\boldsymbol{N}_m\boldsymbol{I}_t$,	$\boldsymbol{N}=\boldsymbol{I}_t\boldsymbol{N}_m\boldsymbol{I}_t$,
$\boldsymbol{N}_2=\boldsymbol{T}^{-1}$,	$\boldsymbol{N}_2=\boldsymbol{T}^{-1}$,	$\boldsymbol{N}_m=\begin{bmatrix}(\boldsymbol{N}_m)_1 & (\boldsymbol{N}_m)_2\\ (\boldsymbol{N}_m)_3 & (\boldsymbol{N}_m)_1^T\end{bmatrix}$,	$\boldsymbol{N}_m=\begin{bmatrix}(\boldsymbol{N}_m)_1 & (\boldsymbol{N}_m)_2\\ (\boldsymbol{N}_m)_3 & (\boldsymbol{N}_m)_1^T\end{bmatrix}$,
$\boldsymbol{N}_3=\boldsymbol{R}\boldsymbol{T}^{-1}\boldsymbol{R}^T-\boldsymbol{Q}$	$\boldsymbol{N}_3=\boldsymbol{R}\boldsymbol{T}^{-1}\boldsymbol{R}^T-\boldsymbol{Q}$	$\boldsymbol{I}_t=\begin{bmatrix}\boldsymbol{I}_1 & \boldsymbol{I}_2\\ \boldsymbol{I}_2 & \boldsymbol{I}_1\end{bmatrix}$, $\boldsymbol{I}_1=\begin{bmatrix}\boldsymbol{I} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}$,	$\boldsymbol{I}_t=\begin{bmatrix}\boldsymbol{I}_1 & \boldsymbol{I}_2\\ \boldsymbol{I}_2 & \boldsymbol{I}_1\end{bmatrix}$, $\boldsymbol{I}_1=\begin{bmatrix}\boldsymbol{I} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}$,
$Q_{ik}=C_{i1k1}$,	$Q_{ik}=C_{i1k1}$,	$\boldsymbol{I}_1=\begin{bmatrix}\boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{I}\end{bmatrix}$	$\boldsymbol{I}_2=\begin{bmatrix}\boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{I}\end{bmatrix}$
$R_{ik}=C_{i1k2}$, $T_{ik}=C_{i2k2}$,	$R_{ik}=C_{i1k2}$, $T_{ik}=C_{i2k2}$,	$(\boldsymbol{N}_m)_1=-\boldsymbol{T}_m^{-1}\boldsymbol{R}_m^T$, $(\boldsymbol{N}_m)_2=\boldsymbol{T}_m^{-1}$, $(\boldsymbol{N}_m)_3=\boldsymbol{R}_m\boldsymbol{T}_m^{-1}\boldsymbol{R}_m^T-\boldsymbol{Q}_m$	$(\boldsymbol{N}_m)_1=-\boldsymbol{T}_m^{-1}\boldsymbol{R}_m^T$, $(\boldsymbol{N}_m)_2=\boldsymbol{T}_m^{-1}$, $(\boldsymbol{N}_m)_3=\boldsymbol{R}_m\boldsymbol{T}_m^{-1}\boldsymbol{R}_m^T-\boldsymbol{Q}_m$
i, k=1, 2, 3	i, k=1, 2, 3, 4	$\boldsymbol{Q}_m=\begin{bmatrix}\boldsymbol{Q}_{\tilde{A}} & \boldsymbol{R}_{\tilde{B}}\\ \boldsymbol{R}_{\tilde{B}}^T & -\boldsymbol{T}_{\tilde{D}}\end{bmatrix}$,	$\boldsymbol{Q}_m=\begin{bmatrix}\boldsymbol{Q}_{\tilde{A}} & \boldsymbol{R}_{\tilde{B}}\\ \boldsymbol{R}_{\tilde{B}}^T & -\boldsymbol{T}_{\tilde{D}}\end{bmatrix}$
C_{ijkl}: elastic tensor	C_{ijkl}: elastic tensor	$\boldsymbol{R}_m=\begin{bmatrix}\boldsymbol{R}_{\tilde{A}} & -\boldsymbol{Q}_{\tilde{B}}\\ \boldsymbol{T}_{\tilde{B}}^T & \boldsymbol{R}_{\tilde{D}}^T\end{bmatrix}$,	$\boldsymbol{R}_m=\begin{bmatrix}\boldsymbol{R}_{\tilde{A}} & -\boldsymbol{Q}_{\tilde{B}}\\ \boldsymbol{T}_{\tilde{B}}^T & \boldsymbol{R}_{\tilde{D}}^T\end{bmatrix}$,
		$\boldsymbol{T}_m=\begin{bmatrix}\boldsymbol{T}_{\tilde{A}} & -\tilde{\boldsymbol{R}}_{\tilde{B}}\\ -\tilde{\boldsymbol{R}}_{\tilde{B}}^T & -\boldsymbol{Q}_{\tilde{D}}\end{bmatrix}$,	$\boldsymbol{T}_m=\begin{bmatrix}\boldsymbol{T}_{\tilde{A}} & -\tilde{\boldsymbol{R}}_{\tilde{B}}\\ -\tilde{\boldsymbol{R}}_{\tilde{B}}^T & -\boldsymbol{Q}_{\tilde{D}}\end{bmatrix}$,
		$\boldsymbol{Q}_X=X_{i1k1}$, $\boldsymbol{R}_X=X_{i1k2}$, $\boldsymbol{T}_X=X_{i2k2}$, $X=\tilde{A}$, $\tilde{B}$ or $\tilde{D}$ $\tilde{\boldsymbol{R}}_{\tilde{B}}=\tilde{B}_{i2k1}$, i, k=1, 2, $\tilde{\boldsymbol{A}}=\boldsymbol{A}-\boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{B}$, $\tilde{\boldsymbol{B}}=\boldsymbol{B}\boldsymbol{D}^{-1}$, $\tilde{\boldsymbol{D}}=\boldsymbol{D}^{-1}$ A_{ijkl}, B_{ijkl}, D_{ijkl}: extensional, coupling, bending stiffness tensors	$\boldsymbol{Q}_X=X_{p1r1}$, $\boldsymbol{R}_X=X_{p1r2}$, $\boldsymbol{T}_X=X_{p2r2}$, $X=\tilde{A}$, $\tilde{B}$ or $\tilde{D}$ $\tilde{\boldsymbol{R}}_{\tilde{B}}=\tilde{B}_{p2r1}$, p, r=1, 2, 4, $\tilde{\boldsymbol{A}}=\boldsymbol{A}-\boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{B}$, $\tilde{\boldsymbol{B}}=\boldsymbol{B}\boldsymbol{D}^{-1}$, $\tilde{\boldsymbol{D}}=\boldsymbol{D}^{-1}$ A_{ijkl}, B_{ijkl}, D_{ijkl}: augmented extensional, coupling, bending stiffness tensors

$$\boldsymbol{t}_1^\infty=\begin{Bmatrix}N_{11}^\infty\\ N_{12}^\infty\\ \tilde{D}_1^\infty\\ M_{11}^\infty\\ M_{12}^\infty\\ \tilde{D}_1^{*\infty}\end{Bmatrix},\quad \boldsymbol{t}_2^\infty=\begin{Bmatrix}N_{12}^\infty\\ N_{22}^\infty\\ \tilde{D}_2^\infty\\ M_{12}^\infty\\ M_{22}^\infty\\ \tilde{D}_2^{*\infty}\end{Bmatrix},\quad \boldsymbol{\varepsilon}_1^\infty=\begin{Bmatrix}\varepsilon_{11}^\infty\\ \varepsilon_{12}^\infty\\ -E_1^{(0)}\\ \kappa_{11}^\infty\\ \kappa_{12}^\infty\\ -E_1^{(1)}\end{Bmatrix},$$

$$\boldsymbol{\varepsilon}_2^\infty=\begin{Bmatrix}\varepsilon_{12}^\infty\\ \varepsilon_{22}^\infty\\ -E_2^{(0)}\\ \kappa_{12}^\infty\\ \kappa_{22}^\infty\\ -E_2^{(1)}\end{Bmatrix},\ \text{EF electro-elastic} \qquad (38d)$$

It is now clear that for cracks in these four different coupling conditions their general solutions and boundary conditions can all be expressed by the unified expressions (29) and (37). Therefore, it is reasonable to expect that their final explicit closed form solutions can also be expressed in one unified form. Actually, these different problems have been solved analytically by different approaches and there are several different versions of the explicit analytical solutions presented in the literature. Although they should all be identical, these solutions were written in several different expressions. By unifying the notations used in the literature and careful re-organization, the full field solution of the present

Table 4 Explicit expressions of *Q*, *R* and *T*

Two-dimensional deformation		Extension-flexure deformation	
Inplane-antiplane coupling	Electro-elastic coupling	Extension-bending coupling	Electro-elastic coupling
$\boldsymbol{Q}=\begin{bmatrix} C_{11} & C_{16} & C_{15} \\ C_{16} & C_{66} & C_{56} \\ C_{15} & C_{56} & C_{55} \end{bmatrix}$	$\boldsymbol{Q}=\begin{bmatrix} C_{11} & C_{16} & C_{15} & e_{11} \\ C_{16} & C_{66} & C_{56} & e_{16} \\ C_{15} & C_{56} & C_{55} & e_{15} \\ e_{11} & e_{16} & e_{15} & -\omega_{11} \end{bmatrix}$	$\boldsymbol{Q_m}=\begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{16} & \tilde{B}_{16}/2 & \tilde{B}_{12} \\ \tilde{A}_{16} & \tilde{A}_{66} & \tilde{B}_{66}/2 & \tilde{B}_{62} \\ \tilde{B}_{16}/2 & \tilde{B}_{66}/2 & -\tilde{D}_{66}/4 & -\tilde{D}_{26}/2 \\ \tilde{B}_{12} & \tilde{B}_{62} & -\tilde{D}_{26}/2 & -\tilde{D}_{22} \end{bmatrix}$	$\boldsymbol{Q_m}=\begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{16} & \tilde{A}_{17} & B_{16}/2 & \tilde{B}_{12} & \tilde{B}_{18}/2 \\ \tilde{A}_{16} & \tilde{A}_{66} & \tilde{A}_{67} & \tilde{B}_{66}/2 & \tilde{B}_{62} & \tilde{B}_{68}/2 \\ \tilde{A}_{17} & \tilde{A}_{67} & \tilde{A}_{77} & \tilde{B}_{67}/2 & \tilde{B}_{27} & \tilde{B}_{78}/2 \\ B_{16}/2 & \tilde{B}_{66}/2 & \tilde{B}_{67}/2 & -\tilde{D}_{66}/2 & -\tilde{D}_{26}/4 & -\tilde{D}_{68}/4 \\ \tilde{B}_{12} & \tilde{B}_{62} & \tilde{B}_{27} & -\tilde{D}_{26}/2 & -\tilde{D}_{22} & -\tilde{D}_{28}/2 \\ \tilde{B}_{18}/2 & \tilde{B}_{68}/2 & \tilde{B}_{78}/2 & -\tilde{D}_{68}/4 & -\tilde{D}_{28}/2 & -\tilde{D}_{88}/4 \end{bmatrix}$
$\boldsymbol{R}=\begin{bmatrix} C_{16} & C_{12} & C_{14} \\ C_{66} & C_{26} & C_{46} \\ C_{56} & C_{25} & C_{45} \end{bmatrix}$	$\boldsymbol{R}=\begin{bmatrix} C_{16} & C_{12} & C_{14} & e_{21} \\ C_{66} & C_{26} & C_{46} & e_{26} \\ C_{56} & C_{25} & C_{45} & e_{25} \\ e_{16} & e_{12} & e_{14} & -\omega_{12} \end{bmatrix}$	$\boldsymbol{R_m}=\begin{bmatrix} \tilde{A}_{16} & \tilde{A}_{12} & -\tilde{B}_{11} & -\tilde{B}_{16}/2 \\ \tilde{A}_{66} & \tilde{A}_{26} & -\tilde{B}_{61} & -\tilde{B}_{66}/2 \\ \tilde{B}_{66}/2 & \tilde{B}_{26}/2 & \tilde{D}_{16}/2 & \tilde{D}_{66}/4 \\ \tilde{B}_{62} & \tilde{B}_{22} & \tilde{D}_{12} & \tilde{D}_{26}/2 \end{bmatrix}$	$\boldsymbol{R_m}=\begin{bmatrix} \tilde{A}_{16} & \tilde{A}_{12} & \tilde{A}_{18} & -\tilde{B}_{11} & -\tilde{B}_{16}/2 & -\tilde{B}_{17}/2 \\ \tilde{A}_{66} & \tilde{A}_{26} & \tilde{A}_{68} & -\tilde{B}_{61} & -\tilde{B}_{66}/2 & -\tilde{B}_{67}/2 \\ \tilde{A}_{67} & \tilde{A}_{27} & \tilde{A}_{78} & -\tilde{B}_{17} & -\tilde{B}_{67}/2 & -\tilde{B}_{77}/2 \\ \tilde{B}_{66}/2 & \tilde{B}_{26}/2 & \tilde{B}_{68}/2 & \tilde{D}_{16}/2 & \tilde{D}_{66}/4 & \tilde{D}_{67}/4 \\ \tilde{B}_{62} & \tilde{B}_{22} & \tilde{B}_{28} & \tilde{D}_{12} & \tilde{D}_{26}/4 & \tilde{D}_{27}/2 \\ \tilde{B}_{68}/2 & \tilde{B}_{28}/2 & \tilde{B}_{88}/2 & \tilde{D}_{18}/2 & \tilde{D}_{68}/4 & \tilde{D}_{78}/4 \end{bmatrix}$
$\boldsymbol{T}=\begin{bmatrix} C_{66} & C_{26} & C_{46} \\ C_{26} & C_{22} & C_{24} \\ C_{46} & C_{24} & C_{44} \end{bmatrix}$	$\boldsymbol{T}=\begin{bmatrix} C_{66} & C_{26} & C_{46} & e_{26} \\ C_{26} & C_{22} & C_{24} & e_{22} \\ C_{46} & C_{24} & C_{44} & e_{24} \\ e_{26} & e_{22} & e_{24} & -\omega_{22} \end{bmatrix}$	$\boldsymbol{T_m}=\begin{bmatrix} \tilde{A}_{66} & \tilde{A}_{26} & -\tilde{B}_{61} & -\tilde{B}_{66}/2 \\ \tilde{A}_{26} & \tilde{A}_{22} & -\tilde{B}_{21} & -\tilde{B}_{26}/2 \\ -\tilde{B}_{61} & -\tilde{B}_{21} & -\tilde{D}_{11} & -\tilde{D}_{16}/2 \\ -\tilde{B}_{66}/2 & -\tilde{B}_{26}/2 & -\tilde{D}_{16}/2 & -\tilde{D}_{66}/4 \end{bmatrix}$	$\boldsymbol{T_m}=\begin{bmatrix} \tilde{A}_{66} & \tilde{A}_{26} & \tilde{A}_{68} & -\tilde{B}_{61} & -\tilde{B}_{66}/2 & -\tilde{B}_{67}/2 \\ \tilde{A}_{26} & \tilde{A}_{22} & \tilde{A}_{28} & -\tilde{B}_{21} & -\tilde{B}_{26}/2 & -\tilde{B}_{27}/2 \\ \tilde{A}_{68} & \tilde{A}_{28} & \tilde{A}_{88} & -\tilde{B}_{18} & -\tilde{B}_{68}/2 & -\tilde{B}_{87}/2 \\ -\tilde{B}_{61}/2 & -\tilde{B}_{21} & -\tilde{B}_{18} & -\tilde{D}_{11} & -\tilde{D}_{16}/2 & -\tilde{D}_{17}/2 \\ -\tilde{B}_{66}/2 & -\tilde{B}_{26}/2 & -\tilde{B}_{68}/2 & -\tilde{D}_{16}/2 & -\tilde{D}_{66}/4 & -\tilde{D}_{67}/4 \\ -\tilde{B}_{67}/2 & -\tilde{B}_{27}/2 & -\tilde{B}_{87}/2 & -\tilde{D}_{17}/2 & -\tilde{D}_{67}/4 & -\tilde{D}_{77}/4 \end{bmatrix}$
C_{ij}: elastic stiffness	C_{ij}: elastic stiffness e_{ij}: piezoelectric stress coefficient ω_{ij}: dielectric permittivity coefficient	$\tilde{\boldsymbol{A}}=\boldsymbol{A}-\boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{B}$, $\tilde{\boldsymbol{B}}=\boldsymbol{B}\boldsymbol{D}^{-1}$, $\tilde{\boldsymbol{D}}=\boldsymbol{D}^{-1}$ A_{ij}: extensional stiffness B_{ij}: coupling stiffness D_{ij}: bending stiffnes	$\tilde{\boldsymbol{A}}=\boldsymbol{A}-\boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{B}$, $\tilde{\boldsymbol{B}}=\boldsymbol{B}\boldsymbol{D}^{-1}$, $\tilde{\boldsymbol{D}}=\boldsymbol{D}^{-1}$ A_{ij}: augmented extensional stiffness B_{ij}: augmented coupling stiffness D_{ij}: augmented bending stiffnes $A_{\alpha 7}=A^{e}_{1\alpha}$, $A_{\alpha 8}=A^{e}_{2\alpha}$, $A_{78}=-A^{\omega}_{12}$, $B_{\alpha 7}=B^{e}_{1\alpha}$, $B_{\alpha 8}=B^{e}_{2\alpha}$, $B_{78}=-B^{\omega}_{12}$, $D_{\alpha 7}=D^{e}_{1\alpha}$, $D_{\alpha 8}=D^{e}_{2\alpha}$, $D_{78}=-D^{\omega}_{12}$, α=1, 2, 6

Table 5 Properties and dimensions of the matrices used in Stroh formalism

	Two-dimensional deformation				Extension-flexure deformation			
	Inplane	Antiplane	Inplane -antiplane coupling	Electro -elastic coupling	Extension	Bending	Extension -bending coupling	Electro -elastic coupling
$\boldsymbol{u}$, $\boldsymbol{\phi}$	2×1	scalar	3×1	4×1	2×1	2×1	4×1	6×1
$\boldsymbol{f}(z)$	2×1	function	3×1	4×1	2×1	2×1	4×1	6×1
$\boldsymbol{a}_k$, $\boldsymbol{b}_k$	2×1	scalar	3×1	4×1	2×1	2×1	4×1	6×1
$\boldsymbol{A}$, $\boldsymbol{B}$	2×2	scalar	3×3	4×4	2×2	2×2	4×4	6×6
$\boldsymbol{Q}$, $\boldsymbol{R}$, $\boldsymbol{T}$	2×2	C_{55}, C_{45}, C_{44}	3×3	4×4	2×2	2×2	4×4	6×6
$\boldsymbol{N}_i$	2×2	scalar	3×3	4×4	2×2	2×2	4×4	6×6
$\boldsymbol{N}$	2×4	2×2	6×6	8×8	4×4	4×4	8×8	12×12
μ_k	2 pairs	1 pairs	3 pairs	4 pairs	2 pairs	2 pairs	4 pairs	6 pairs

$\boldsymbol{u}$, $\boldsymbol{\phi}$: real vectors, physical quantities; $\boldsymbol{f}(z)$: complex function, problem dependent
$\boldsymbol{a}_k$, $\boldsymbol{b}_k$: complex vectors, material properties; $\boldsymbol{A}$, $\boldsymbol{B}$: complex matrices, material properties
$\boldsymbol{N}$, $\boldsymbol{N}_i$, $\boldsymbol{Q}$, $\boldsymbol{R}$, $\boldsymbol{T}$: real matrices, material properties; μ_k: complex numbers, material properties

problems can now be written in one unified form as (Hwu, 1992; Hwu, 2003b)

$$\boldsymbol{u} = x_1\boldsymbol{\varepsilon}_1^\infty + x_2\boldsymbol{\varepsilon}_2^\infty - a\text{Re}\{\boldsymbol{A} < \zeta_\alpha^{-1} > \boldsymbol{B}^{-1}\boldsymbol{t}_2^\infty\},$$

$$\boldsymbol{\phi} = x_1\boldsymbol{t}_2^\infty + x_2\boldsymbol{t}_1^\infty - a\text{Re}\{\boldsymbol{B} < \zeta_\alpha^{-1} > \boldsymbol{B}^{-1}\boldsymbol{t}_2^\infty\} \qquad (39a)$$

where

$$\zeta_\alpha = \frac{1}{a}\{z_\alpha + \sqrt{z_\alpha^2 - a^2}\} \qquad (39b)$$

The angular bracket stands for the diagonal matrix whose components vary according to its subscript α. From the solution (39) and the relations shown in Table 2 we see that the stresses are singular near the crack tip. To reflect the strength of the stress singularity, one usually defines the stress intensity factors by (Sih, 1973)

$$\boldsymbol{K} = \lim_{r\to 0}\sqrt{2\pi r}\,\boldsymbol{t}_2 \qquad (40)$$

where r is the distance ahead of the crack tip; $\boldsymbol{K}$ and $\boldsymbol{t}_2$ are the vectors of the stress intensity factors and the stresses near the crack tip, which are

$$\boldsymbol{K} = \begin{Bmatrix} K_{II} \\ K_I \\ K_{III} \end{Bmatrix}, \quad \boldsymbol{t}_2 = \begin{Bmatrix} \sigma_{12} \\ \sigma_{22} \\ \sigma_{32} \end{Bmatrix} = \phi_{,1},$$

for 2D inplane-antiplane coupling (41a)

$$\boldsymbol{K} = \begin{Bmatrix} K_{II} \\ K_I \\ K_{III} \\ K_D \end{Bmatrix}, \quad \boldsymbol{t}_2 = \begin{Bmatrix} \sigma_{12} \\ \sigma_{22} \\ \sigma_{32} \\ D_2 \end{Bmatrix} = \phi_{,1},$$

for 2D electro-elastic coupling (41b)

$$\boldsymbol{K} = \begin{Bmatrix} K_{II} \\ K_I \\ K_{IIB} \\ K_{IB} \end{Bmatrix}, \quad \boldsymbol{t}_2 = \begin{Bmatrix} N_{12} \\ N_{22} \\ M_{12} \\ M_{22} \end{Bmatrix} = \phi_{,1} - \eta\boldsymbol{i}_3,$$

for EF stretching-bending coupling (41c)

$$\boldsymbol{K} = \begin{Bmatrix} K_{II} \\ K_I \\ K_D \\ K_{IIB} \\ K_{IB} \\ K_{DB} \end{Bmatrix}, \quad \boldsymbol{t}_2 = \begin{Bmatrix} N_{12} \\ N_{22} \\ \tilde{D}_2 \\ M_{12} \\ M_{22} \\ \tilde{D}_2^* \end{Bmatrix} = \phi_{,1} - \eta\boldsymbol{i}_4,$$

for EF electro-elastic coupling (41d)

where

$$\eta(\psi_{1,1}+\psi_{2,2})/2,\ \boldsymbol{i}_3=(0\ 0\ 1\ 0)^T,\ \boldsymbol{i}_4=(0\ 0\ 0\ 1\ 0\ 0)^T \qquad (41e)$$

The last equalities of (41a-d) are obtained from the relations given in Table 2. By these equalities we know that the stresses ahead of the crack tip along the x_1-axis can be obtained from $\phi_{,1}$ with x_2=0, $|x_1|>a$, which is

$$\phi_{,1} = \frac{x_1}{\sqrt{x_1^2 - a^2}}\boldsymbol{t}_2^\infty \qquad (42)$$

Substituting (42) into (40), we can get a unified expression for the stress intensity factors defined in (40)-(41) except K_{IIB} as

$$\boldsymbol{K}=\sqrt{\pi a}\,\boldsymbol{t}_2^{\infty} \quad (43)$$

Similarly, by using $(39a)_1$ and setting $x_2=0^{\pm}$, $|x_1|<a$ where ± denotes the upper and lower surfaces of the crack, the generalized crack opening displacement $\Delta\boldsymbol{u}$ can be obtained as

$$\Delta\boldsymbol{u}=\boldsymbol{u}(x_1,0^{+})-\boldsymbol{u}(x_1,0^{-})=2\sqrt{a^2-x_1^2}\,\boldsymbol{L}^{-1}\boldsymbol{t}_2^{\infty} \quad (44)$$

in which $\boldsymbol{L}$ is the Barnett-Lothe tensor defined by $\boldsymbol{L}=-2i\boldsymbol{B}\boldsymbol{B}^T$. By applying the virtual crack closure method (Irwin, 1957), the total strain energy release rate G can be calculated as

$$G=\lim_{\Delta a\to 0}\frac{1}{2\Delta a}\int_0^{\Delta a}\Delta\boldsymbol{u}^T(s-\Delta a)\phi'(s)ds$$

$$=\frac{\pi a}{2}\boldsymbol{t}_2^{\infty}\frac{\pi a}{2}\boldsymbol{t}_2^{\infty T}\boldsymbol{L}^{-1}\boldsymbol{t}_2^{\infty}=\frac{1}{2}\boldsymbol{K}^T\boldsymbol{L}^{-1}\boldsymbol{K} \quad (45)$$

where s is the distance ahead of the crack tip.

Note that like the stress intensity factors obtained in (43) which are not valid for K_{IIB}, the equalities obtained in (45) for the total energy release rate are also not valid for the extension-flexure deformation in which the relation $\boldsymbol{t}_2=\boldsymbol{\phi}_{,1}=\boldsymbol{\phi}'$ should be modified. In (Hsieh and Hwu, 2002), the solution of K_{IIB} has been obtained explicitly for the pure bending case. For all the other cases of extension-flexure deformation, similar results can be obtained directly by using the relations given in (41c,d).

VI. CONCLUDING REMARKS

A unified general solution for several different types of coupling analyses is shown in (29). The notations used in the present formalism have different dimensions and different contents for different types of problems. If the boundary conditions in various coupling conditions can also be expressed in a unified form, the final solution should also have the same form for different types of problems. The crack problems discussed in this paper illustrate this feature. The simple forms of the full field 15 solution (39), the near tip stresses (42), the stress intensity factors (43), the crack opening displacements (44) and the energy release rate (45) are all valid for various coupling conditions discussed in this paper.

In addition to the coupling conditions discussed in this paper, there are several other coupling conditions in engineering applications. The most common one is the consideration of a hygrothermal environment. The appendix is the extension of Stroh formalism to uncoupled steady state thermoelastic problems in which the temperature and heat flux can be solved separately and their influence is treated as the external loading applied on the solids. To solve the general coupling condition, the coupled thermoelastic theory should be considered.

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APPENDIX: UNCOUPLED STEADY STATE THERMOELASTIC PROBLEMS

It is known that thermal strains develop in the body as a result of temperature changes. For anisotropic materials the coefficients of thermal expansion, like other properties, change with direction. Thus, the thermal changes result in unequal strains in different directions. Thermal strains do not produce stresses when the body is completely free to expand. The stresses are induced when constraints are placed on the deformation of elastic bodies. In a fixed rectangular coordinate system, let u_i, σ_{ij}, ε_{ij}, T and h_i be, respectively, the displacement, stress, strain, temperature and heat flux. The heat conduction, energy equation, strain-displacement relation, constitutive law and the equations of equilibrium for the uncoupled steady state thermoelastic problems can be written as (Nowacki, 1962)

$$h_i=-k_{ij}T_{,j},\ h_{i,i}=-k_{ij}T_{,ij}=0,$$
$$\varepsilon_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}),\ \sigma_{ij}=C_{ijks}\varepsilon_{ks}-\beta_{ij}T,\ \sigma_{ij,j}=0 \qquad \text{(A1)}$$

where k_{ij} and β_{ij} are heat conduction coefficients and thermal moduli, which are generally symmetric. Eq. (A1) constitutes 19 partial differential equations in terms of three coordinate variables x_i, i=1, 2, 3. If the deformations are considered to be dependent upon two coordinate variables x_1 and x_2 only, a general solution satisfying these 19 equations has been obtained as (Hwu, 1990)

$$T=2\mathrm{Re}\{g'(z_t)\},\ \boldsymbol{h}=-2\mathrm{Re}\{(\boldsymbol{k}_1+\tau\boldsymbol{k}_2)g''(z_t)\},$$
$$\boldsymbol{u}=2\mathrm{Re}\{\boldsymbol{A}\boldsymbol{f}(z)+\boldsymbol{c}g(z_t)\},\ \boldsymbol{\phi}=2\mathrm{Re}\{\boldsymbol{B}\boldsymbol{f}(z)+\boldsymbol{d}g(z_t)\} \qquad \text{(A2a)}$$

where

$$\boldsymbol{h}=\begin{Bmatrix}h_1\\h_2\\h_3\end{Bmatrix},\quad \boldsymbol{u}=\begin{Bmatrix}u_1\\u_2\\u_3\end{Bmatrix},\quad \boldsymbol{\phi}=\begin{Bmatrix}\phi_1\\\phi_2\\\phi_3\end{Bmatrix} \qquad \text{(A2b)}$$

and

$$\boldsymbol{k}_1=\begin{Bmatrix}k_{11}\\k_{21}\\k_{31}\end{Bmatrix},\quad \boldsymbol{k}_2=\begin{Bmatrix}k_{12}\\k_{22}\\k_{32}\end{Bmatrix},\quad \boldsymbol{f}(z)=\begin{Bmatrix}f_1(z_1)\\f_2(z_2)\\f_3(z_3)\end{Bmatrix},$$
$$\boldsymbol{A}=[\boldsymbol{a}_1\ \boldsymbol{a}_2\ \boldsymbol{a}_3],\ \boldsymbol{B}=[\boldsymbol{b}_1\ \boldsymbol{b}_2\ \boldsymbol{b}_3] \qquad \text{(A2c)}$$

and

$$z_k=x_1+\mu_k x_2,\ k=1,2,3,\quad z_t=x_1+\tau x_2 \qquad \text{(A2d)}$$

In the above τ and ($\boldsymbol{c}$, $\boldsymbol{d}$) are the thermal eigenvalues and eigenvectors, which can be determined by the following relations

$$k_{22}\tau^2+2k_{12}\tau+k_{11}=0 \qquad \text{(A3)}$$

and

$$\boldsymbol{N}\eta=\tau\eta+\gamma \qquad \text{(A4a)}$$

where

$$\gamma=-\begin{bmatrix}\boldsymbol{0} & \boldsymbol{N}\\ \boldsymbol{I} & \boldsymbol{N}_1^T\end{bmatrix}\begin{Bmatrix}\beta_1\\\beta_2\end{Bmatrix},\quad \eta=\begin{Bmatrix}\boldsymbol{c}\\\boldsymbol{d}\end{Bmatrix},$$
$$\beta_1=\begin{Bmatrix}\beta_1\\\beta_{21}\\\beta_{31}\end{Bmatrix},\quad \beta_2=\begin{Bmatrix}\beta_{12}\\\beta_{22}\\\beta_{32}\end{Bmatrix} \qquad \text{(A4b)}$$