# THE STRESS-INTENSITY PARAMETERS: A STRENGTH-OF-MATERIALS APPROACH

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## ABSTRACT

This paper presents strength-of-materials approach to estimate the stress intensity parameters of elastic and plastic solids. Based on knowledge of the local stress field and using structural mechanics principles, estimates are made of the stress-intensity factor and the J-integral for plane and three-dimensional cracked components.

***Key Words:*** cracks, k-factor, j-integral.

## I. INTRODUCTION

This contribution owes much to George C Sih's early publications and his role in developing Fracture Mechanics into an independent discipline (Sih, 1973). The purpose is to point out that strength-of-materials approach can be used to estimate the stress intensity parameters, namely, the stress-intensity factor and the J-integral of cracked components. Such an approach provides an easy way to generate approximate new results and a useful tool to check the formulas available in many handbooks (Paris and Sih, 1965; Sih, 1973; Tada *et al*., 2000; Murakami, 1987). The tensile behavior of the material obeys the standard Ramberg-Osgood relation, which includes the elastic material as special case. Knowing the stress field at the crack tip, it is shown that elementary structural mechanics techniques yield good estimates of the stress-intensity factor and the J-integral for plane and three-dimensional components. In the cited examples, the accuracy of the results is acceptable for engineering applications.

## II. PLANE COMPONENTS

Let us explain the method by considering an infinite plane sheet containing a central crack of length $2a$. The crack is subject to normal stress, $p$, acting at remote distance from the crack plane. It is known that the leading term of the asymptotic expansion of the stress field along the crack axis is (Rice and Rosengren, 1968; Hutchinson, 1968)

$$\sigma_y = Ar^{-\frac{1}{n+1}} + \ldots\ldots\ldots \quad (1)$$

where $A$ is the amplitude of the singular term and $r$ is a small distance from the crack tip.

Since the value of $A$ gives the magnitude of the stress elevation at the crack tip, it is related to the stress intensity parameters, namely, the stress-intensity factor and the J-integral. In fact, when $n$=1, $A$ is equivalent to $\frac{k}{\sqrt{2\pi}}$, with $k$ as the mode $I$ stress-intensity factor while for $n$>1, the constant $A$ gives the magnitude of the J-integral.

In order to determine the value of $A$, it is observed that at a small horizontal distance from the crack tip, say $L$, the induced stress is equal to $p$, and Eq. (1) gives

$$L = \left(\frac{A}{p}\right)^{n+1} \quad (2)$$

The equilibrium of a free body diagram of the half plane, $y \geq 0$, assuming unit thickness, is maintained by requiring

$$pa = \iint \sigma_y dA = \int_0^L Ar^{-\frac{1}{n+1}} dr \quad (3)$$

which gives

$$pa = \left(\frac{n+1}{n}\right) AL^{\frac{n}{n+1}} \quad (4)$$

Solving Eqs. (2) and (4) yields

$$L = \left(\frac{n}{n+1}\right) a \quad (5a)$$

and

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$$A = \left(\frac{na}{n+1}\right)^{\frac{1}{n+1}} p \quad (5b)$$

It is observed that for $n$=1, $A=\sqrt{\frac{a}{2}}p$, $L=\frac{a}{2}$, and $k=p\sqrt{\pi a}$ which is the exact value of the stress-intensity factor for this geometry.

If the tensile behavior of the material is characterized by the Ramberg-Osgood relation

$$\varepsilon/\varepsilon_o=\alpha(\sigma/\sigma_o)^n \quad (6)$$

where $\sigma_o$ and $\varepsilon_o$ are reference stress and strain, $\alpha$ is a constant and $n$ is the hardening exponent, the stress field at the crack tip is (Rice and Rosengren, 1968; Hutchinson, 1968)

$$\sigma_{ij} = \sigma_o\left(\frac{J}{\alpha\sigma_o\varepsilon_o r}\right)^{\frac{1}{n+1}}\lambda(n, \theta) \quad (7)$$

Here, $r$ and $\theta$ are polar coordinates centered at the crack tip and $\lambda(n, \theta)$ is a parameter with known values for $n$ and $\theta$. It follows that for $\theta$=0, the constant $A$ stands for

$$A = \sigma_o\left(\frac{J}{\alpha\sigma_o\varepsilon_o}\right)^{\frac{1}{n+1}}\lambda(n, 0) \quad (8)$$

Substituting the value of $A$ found in Eq. (5b) into Eq. (8) results in an estimate of the magnitude of the J-integral as

$$\frac{J}{\alpha\sigma_o\varepsilon_o\left(\frac{p}{\sigma_o}\right)^{n+1}a} = \left(\frac{n}{n+1}\right)[\lambda(n, 0)]^{-(n+1)} \quad (9)$$

In the next section, several examples are used to estimate the stress intensity parameters by strength of materials approach.

## 1. Stress-Intensity Factor

In the particular case of $n$=1, values of the constant $A$ gives the stress-intensity factor. The procedure is illustrated by considering few examples.

*(i) Griffith Crack under Concentrated Load F*

Suppose that a uniform stress is acting over a region $2b$ ($b<a$) of the crack surfaces then using $L=a/2$, equilibrium of the half-plane requires

$$pb = \int_0^L \frac{A}{\sqrt{r}}dr = 2A\sqrt{\frac{a}{2}}$$

Hence, $k=\frac{b}{a}p\sqrt{\pi a}$. Taking the limit as $b$ approaches zero and $F=2bp$, it follows that

$$k = 1.57\frac{F}{\sqrt{\pi a}} \quad (10)$$

while the exact result is (Sih, 1973; Tada *et al*., 2000)

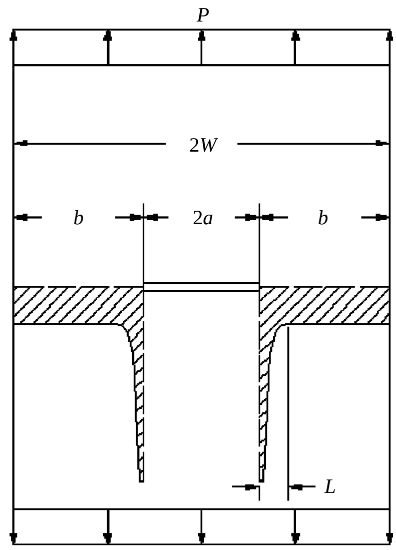


Fig. 1 A centrally cracked panel subjected to tension

$$k = \sqrt{2}\frac{F}{\sqrt{\pi a}}$$

*(ii) Finite Width Panel Containing a Central Crack*

A panel of width $2w$ contains a central crack of length $2a$, $w=a+b$, is subject to stress $p$ as shown in Fig. 1. To find the influence of the boundary on the stress-intensity factor, it is observed that when b is large, the geometry corresponds to that of the infinite sheet and $L=a/2$. Thus if $b>a/2$, i.e., $W/a>1.5$ , then $k=p\sqrt{\pi a}$. If $W/a<1.5$, equilibrium of a free-body diagram of the half-panel requires

$$pa = \int_0^b \frac{A}{\sqrt{r}}dr$$

which yields the approximate value of the stress-intensity factor

$$\frac{k}{p\sqrt{\pi a}} = \frac{0.707}{\sqrt{\frac{w}{a}-1}}\ ,\ 1<w/a<1.5 \quad (11)$$

while the exact result is (Tada *et al*., 2000)

$$\frac{k}{p\sqrt{\pi a}} = [1-0.025(a/w)^2 + 0.06(a/w)^4\sqrt{\sec(\pi a/2w)}$$

Numerical values of the two results are given in Table 1. The accuracy is acceptable for many engineering applications.

*(iii) Bending of a Cracked Beam*

A rectangular beam of width $B$, depth $2c$ has an edge crack of length a as shown in Fig. 2. At a distant $L$ ahead of the crack tip

$$\frac{k}{\sqrt{2\pi L}} = \frac{M}{I}(c-a) \quad (12)$$

**Table 1 Stress intensity factors for a finite width panel**

| $\frac{w}{a}$ | $\frac{k_{est}}{p\sqrt{\pi a}} = \frac{0.707}{\sqrt{\frac{w}{a}-1}}$ | $\frac{k}{p\sqrt{\pi a}} = [1-0.025(a/w)^2+0.06(a/w)^4]\sqrt{\sec(\pi a/2w)}$ |
|---|---|---|
| 1.05 | 3.162 | 3.753 |
| 1.10 | 2.236 | 2.702 |
| 1.15 | 1.826 | 2.251 |
| 1.20 | 1.581 | 1.985 |
| 1.25 | 1.414 | 1.788 |
| 1.30 | 1.291 | 1.697 |

where $M$ is the applied moment and $I$ is the moment of inertia of the cross section of the beam.

Equilibrium of the horizontal forces in the beam, i.e., $\iint \sigma_x dA=0$, implies

$$\int_{c=a}^{c} \frac{My}{I} dy = \int_{c=a-L}^{c=a} \frac{k dy}{\sqrt{2\pi(c-a-y)}}$$

Evaluating the integrals, the result is

$$\frac{k}{\sqrt{2\pi}} L^{1/2} = \frac{Ma}{4I}(2c-a) \tag{13}$$

Eliminating $L$ from Eqs. (12) and (13) yields

$$k = \frac{M}{I} a^{3/2} [\frac{\pi}{2}(\frac{c}{a}-1)(2\frac{c}{a}-1)]^{1/2} \tag{14}$$

while the exact result is

$$k = \frac{M}{I} a^{3/2} \frac{4(\frac{c}{a})^3}{(2\frac{c}{a}-1)^{3/2}} g(\frac{a}{2c}) \tag{15}$$

The function $g(\frac{a}{2c})$ is a tabulated numerical factor (Tada *et al.*, 2000; Barsom and Rolfe, 1987). For $a/c$=0.1, Eq. (14) gives $k_{est}=16.4\frac{M}{I}a^{1.5}$ while the exact result from Eq. (15) is $k=17.4\frac{M}{I}a^{1.5}$. The accuracy of the expression in Eq. (14) decreases as the ratio a/c increases.

## 2. Estimates of J-Integral

Let us illustrate the procedure for the finite width panel shown in Fig. 1. When the tensile behavior of the material is characterized by the usual Ramberg-Osgood relation, the strength of materials estimate of the J-integral for the fully plastic panel is given in Eq. (9). Assuming $L=(\frac{n}{n+1})a$, the expression in Eq. (9) is valid as long as

$$w \geq a+L, \text{ i.e., } \frac{w}{a} \geq \frac{2n+1}{n+1}.$$

However, if $\frac{w}{a} \geq \frac{2n+1}{n+1}$, equilibrium of the half panel, assuming plane stress with unit thickness, implies

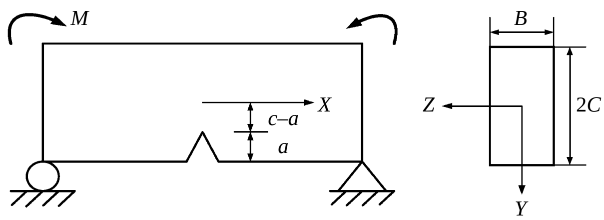


Fig. 2 Bending of a cracked beam

$$pa = \int_0^b A r^{\frac{-1}{n+1}} dr$$

giving

$$A = (\frac{n}{n+1}) \frac{p}{(\frac{w}{a}-1)^{n/n+1}}$$

Making use of Eq. (8), it follows that an estimate of the J-integral is

$$\frac{J}{\alpha\sigma_0\varepsilon_0(\frac{p}{\sigma_0})^{n+1} a} = (w/a-1)^{-n}[\frac{n}{(n+1)\lambda(n,0)}]^{n+1},$$

$$\frac{w}{a} \leq \frac{2n+1}{n+1} \tag{16}$$

For $n$=3, $a/w$=0.25, and using the plane stress values of $\lambda(3, 0)$, Eq. (9) yields

$$\frac{J}{\alpha\sigma_0\varepsilon_0(\frac{p}{\sigma_0})^4 a} = 1.985$$

while the stress analysis value is (Anderson, 1995).

$$\frac{J_{exact}}{\alpha\sigma_0\varepsilon_0(\frac{p}{\sigma_0})^4 a} = 2.36$$

Similarily, for $n$=3 and $a/w$=0.625, Eq. (16) gives

$$\frac{J}{\alpha\sigma_0\varepsilon_0(\frac{p}{\sigma_0})^4 a} = 1.01$$

while the exact value of the same quantity is 0.78.

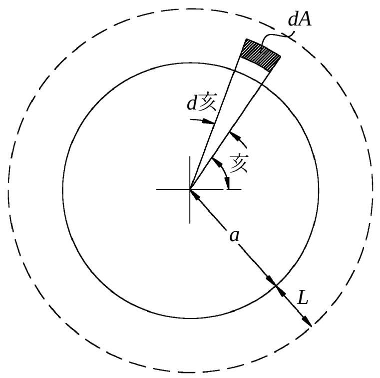


Fig. 3 A penny-shaped crack in an infinite solid subjected to uniform tension

## III. THREE DIMENSIONAL PROBLEMS

Consider a penny-shaped crack of radius = a subject to a remote stress $p$. The crack is situated in the $z$=0 plane with the $z$-axis normal to the crack plane as indicated in Fig. 3. The material behaves according to the power law of Eq. (6). In the $z$=0 plane the leading term of the asymptotic expansion of the stresses near the crack boundary is given by

$$\sigma_{ij} = Br^{-\frac{1}{n+1}} + \cdots \tag{17}$$

Here, $B$ measures the magnitude of the stress intensity at the crack boundary, and the procedure to estimate the values of $B$ is similar to the one used previously. At $r=L$

$$p = BL^{-\frac{1}{n+1}} \tag{18}$$

and equilibrium of the free body diagram of the half-space requires

$$\pi a^2 p = \iint_A Br^{-\frac{1}{n+1}} dA = 2\pi B \int_0^L (a + r) r^{-\frac{1}{n+1}} dr \tag{19}$$

where the integration is carried over an annular region surrounding the crack with internal radius = $a$ and external radius =$a$+$L$. Eq. (19) yields

$$a^2 p = 2(n + 1)BL^{\frac{n}{n+1}}\left(\frac{a}{n} + \frac{L}{2n + 1}\right) \tag{20}$$

Solving for $B$ and $L$ from Eqs. (18) and (20), it is found that

$$L = (2n + 1)\left(\frac{a}{2n}\right)\left[\sqrt{\frac{4n^2 + 3n + 1}{(2n + 1)(n + 1)}} - 1\right] \tag{21}$$

and

$$B = pL^{\frac{1}{n+1}} \tag{22}$$

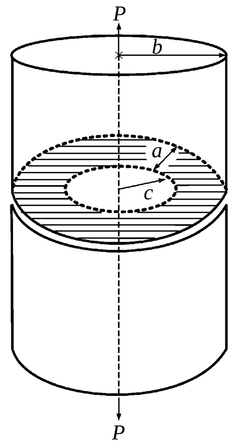


Fig. 4 A circumferentially cracked cylindrical bar

For $n$=1, Eqs. (21) and (22) yield $L$=0.232a, and $B=0.482p\sqrt{a}$. Since for $n$=1, the leading term of the stress field gives $B=\frac{k}{\sqrt{2\pi}}$, where $k$ is the stress-intensity factor of a penny-shaped crack in an infinite medium, it follows that in the proposed approach, $k=(1.069)(\frac{2}{\pi})p\sqrt{\pi a}$ while the result from the stress analysis is (Kassir and Sih, 1975) $k=\frac{2}{\pi}p\sqrt{\pi a}$, i.e., an error of 6.9%.

For solids obeying the power law given in Eq. (6), the singular stress field at the edge of a penny-shaped crack is identical to the plane strain field given in Eq. (7) except that now $r$ and $\theta$ are local coordinates in a plane perpendicular to the edge of the crack (He and Hutchinson, 1981).

### Examples

Let us illustrate the procedure by considering the following examples:

*(i) Circumferentially Cracked Round Bar in Tension*

A cylindrical bar of radius $b$ contains an external crack outside a notched section of radius $c$, $b=c+a$, as shown in Fig. 4. The bar is stretched by uniform stress $p$. At a radial distant $L$ in the notched section

$$\frac{k}{\sqrt{2\pi L}} = p \tag{23}$$

Equilibrium of the region above the notched section gives

$$\pi(b^2 - c^2)p = \iint_{A_L} \frac{k}{\sqrt{2\pi r}} dA = 2\pi \int_{c=L}^{c} \frac{\xi d\xi}{\sqrt{2\pi r}}$$

Evaluating the integral

$$\pi(b^2 - c^2) = 2\sqrt{2\pi}\left(c - \frac{L}{3}\right)kL^{1/2} \tag{24}$$

**Table 2 Stress-intensity factor for circumferentially notched bar in tension**

| $b/c$ | $\frac{k}{p\sqrt{\pi a}} = \sqrt{c/a}[3 - \sqrt{12 - 3\frac{b^2}{c^2}}]^{1/2}$ | $\frac{k}{p\sqrt{\pi a}} = \frac{\sqrt{c/b}}{2}[1 + 0.5(\frac{c}{b}) + 0.375(\frac{c}{b})^2 - 0.363(\frac{c}{b})^3 + 0.731(\frac{c}{b})^4]$ |
|---|---|---|
| 1.10 | 1.03 | 0.95 |
| 1.15 | 1.05 | 0.89 |
| 1.20 | 1.07 | 0.83 |
| 1.25 | 1.08 | 0.75 |

Eliminating $L$ from Eqs. (23) and (24) and solving for $k$

$$\frac{k}{p\sqrt{\pi a}} = \sqrt{c/a}[3 - \sqrt{12 - 3\frac{b^2}{c^2}}]^{1/2}, \quad \text{valid for } 1<b/c<2 \tag{25}$$

The exact formula for the stress-intensity factor for the notched round bar is (Sih, 1973)

$$\frac{k}{p\sqrt{\pi a}} = \frac{1}{2}\sqrt{c/b}\,[1+0.5(c/b)+0.375(c/b)^2 - 0.363(c/b)^3+0.731(c/b)^4] \tag{26}$$

The formula (25) gives reasonable estimates of the $k$-factor for values of $b/c$ in the range $1<b/c<1.25$ as shown in Table 2.

*(ii) J-integral for a Penny Shaped Crack in an Infinite Medium*

The constant $B$ in Eq. (17) is related to the J-integral by

$$B = \sigma_0\left(\frac{J}{\alpha\sigma_0\varepsilon_0}\right)^{\frac{1}{n+1}}\lambda^*(n, 0) \tag{27}$$

where $\lambda^*(n, 0)$ is the normalizing factor. The strength of materials approach yields

$$\frac{J}{\alpha\sigma_0\varepsilon_0(\frac{p}{\sigma_0})^{n+1}a} = \frac{2n+1}{2n}\left[\sqrt{\frac{4n^2+3n+1}{(2n+1)(n+1)}} - 1\right][\lambda^*(n, 0)]^{-(n+1)} \tag{28}$$

while the stress analysis approach gives

$$\frac{J}{\alpha\sigma_0\varepsilon_0(\frac{p}{\sigma_0})^{n+1}a} = h_1(n, 0) \tag{29}$$

where $h_1$ is tabulated function (He and Hutchinson 1981). For $n$=3, the function $h_1$=1.333 while formula (28) gives

$$\frac{J}{\alpha\sigma_0\varepsilon_0(\frac{p}{\sigma_0})^4 a} = \frac{0.329}{[\lambda^*(3, 0)]^4}$$

Using the plane strain value for $\lambda^*(3, 0)$, it is found that the approximate value of $h_1$ is 1.40.

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