A NEW BOUNDARY INTEGRAL EQUATION METHOD FOR ANALYSIS OF CRACKED LINEAR ELASTIC BODIES

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ABSTRACT

A novel integral equation method is developed in this paper for the analysis of two-dimensional general anisotropic elastic bodies with cracks. In contrast to the conventional boundary integral methods based on reciprocal work theorem, the present method is derived from Stroh's formalism for anisotropic elasticity in conjunction with Cauchy's integral formula. The proposed boundary integral equations contain boundary displacement gradients and tractions on the non-crack boundary and the dislocations on the crack lines. In cases where only the crack faces are subjected to tractions, the integrals on the non-crack boundary are non-singular. The boundary integral equations can be solved using Gaussian-type integration formulas directly without dividing the boundary into discrete elements. Numerical examples of stress intensity factors are given to illustrate the effectiveness and accuracy of the present method.

Key Words:

I. INTRODUCTION

It is well known that the stress fields near the tip of a crack in a linear elastic material exhibit a square-root singularity. The amplitudes of the singular stress fields are characterized by the stress intensity factors (SIFs). In linear elastic fracture mechanics, the SIFs are usually used as parameters in criteria governing crack propagation or fatigue crack growth. It is therefore an important task to determine the SIF's in the analyses of cracked elastic bodies.

As analytical solutions of SIFs are limited to a few idealized cases, for practical problems involving finite geometries and complex loadings, numerical methods must be employed to compute the SIFs. Several boundary integral equation methods have been proposed to handle the crack-tip singularity. One method is to use a special Green's function (Snyder and Cruse, 1975), which satisfies the traction-free conditions on the crack faces. In this method only non-crack boundaries need to be treated. However, Such Green's functions are only available for simple crack geometries. Another method is the so-called dual boundary element method, where the displacement integral equation is applied to the non-crack boundary and the traction integral equation is used on the crack faces (Hong and Chen, 1988) or on one side of the crack surfaces (Pan, 1997). In this formulation hyper-singularity is present in the traction integral equation. Another approach is to use singular integral equations of Cauchy's type defined only on the crack lines. The singular integral equations can be solved numerically using Gauss-Chebyshev integration formulas (Erdogan and Gupta, 1972). This approach was applied by Delale and Erdogan (1977) for internal cracks or edge cracks in an orthotropic strip.

In this paper a new integral equation method is presented. The integral equations are based on the well-known Cauchy's integral formula for analytic functions in conjunction with the stroh formalism for anisotropic elasticity. The proposed integral equations are of Cauchy's type for either the non-crack boundary or the crack lines. Furthermore, in cases where only the crack faces are subjected to tractions, the integrals on the non-crack boundary are non-singular. The boundary integral equations can be solved using Gaussian-type integration formulas directly without dividing the boundary into discrete

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elements. Numerical examples of SIFs are given to illustrate the effectiveness and accuracy of the present method.

II. FORMULATION

The Stroh formalism is an elegant and powerful method for two-dimensional anisotropic elasticity. In the Stroh formalism the displacement $\boldsymbol{u}$ and the stress function $\boldsymbol{\phi}$ are associated with analytic function $\boldsymbol{f}(\boldsymbol{z})$ as (Yeh *et al*., 1993):

$$\boldsymbol{A}^T\boldsymbol{\phi}+\boldsymbol{B}^T\boldsymbol{u}=\boldsymbol{f}(\boldsymbol{z}) \tag{1}$$

where $\boldsymbol{A}=[\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3]$, $\boldsymbol{B}=[\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3]$, $\boldsymbol{f}(\boldsymbol{z})=[f_1(z_1), f_2(z_2), f_3(z_3)]^T$, $z_k=x_1+p_kx_2$, p_k is a complex constant with positive imaginary part, k=1,2,3. $\boldsymbol{a}_k$, $\boldsymbol{b}_k$ and p_k are determined by the following eigenvalue problem (Ting, 1996):

$$\begin{bmatrix} \boldsymbol{N}_1 & \boldsymbol{N}_2 \\ \boldsymbol{N}_3 & \boldsymbol{N}_1^T \end{bmatrix}\begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{bmatrix} = \boldsymbol{p}\begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{bmatrix} \tag{2}$$

where, $\boldsymbol{N}_1=-\boldsymbol{T}^{-1}\boldsymbol{R}^T$, $\boldsymbol{N}_2=\boldsymbol{T}^{-1}$, $\boldsymbol{N}_3=\boldsymbol{R}\boldsymbol{T}^{-1}\boldsymbol{R}^T-\boldsymbol{Q}$ and $Q_{ik}=C_{i1k1}$, $R_{ik}=C_{i1k2}$, $T_{ik}=C_{i2k2}$, i, k=1,2,3, $C_{ijk\ell}$ are the elastic constants. The matrix products, $\boldsymbol{AB}^T$, $\boldsymbol{AA}^T$ and $\boldsymbol{BB}^T$ can be expressed as

$$\boldsymbol{AB}^T=\frac{1}{2}(\boldsymbol{I}-i\boldsymbol{S}),\quad \boldsymbol{AA}^T=-\frac{i}{2}\boldsymbol{H},\quad \boldsymbol{BB}^T=\frac{i}{2}\boldsymbol{L} \tag{3}$$

where $\boldsymbol{S}$, $\boldsymbol{H}$, and $\boldsymbol{L}$ are real matrices, $\boldsymbol{I}$ is the identity matrix and $i=\sqrt{-1}$.

In the theory of complex analytic functions it is well known that a function $f(z)$ of $z=x_1+ix_2$, analytic in a closed region R can be expressed in terms of its values on the boundary Γ of R by Cauchy's integral formula as

$$f(z)=\frac{1}{2\pi i}\int_{\Gamma}\frac{f(\zeta)}{\zeta-z}d\zeta \tag{4}$$

If we regard the complex variables $z_k=x_1+p_kx_2$ as $z_k=y_1+iy_2$, where $y_1=x_1+Re[p]x_2$ and $y_2=Im[p]x_2$, (4) remains valid for $f_k(z_k)$ or its derivative $f_k'(z_k)$. The generalized form of (4) for $f_k'(z_k)$ is expressed in matrix form as

$$\boldsymbol{f}'(\boldsymbol{z})=\frac{1}{2\pi i}\int_{\Gamma}\left\langle\frac{d\zeta}{\zeta-z}\right\rangle\boldsymbol{f}'(\zeta) \tag{5}$$

where $\boldsymbol{f}'(\boldsymbol{z})=[f_1'(z_1), f_2'(z_2), f_3'(z_3)]^T$, $\left\langle\frac{d\zeta}{z-\zeta}\right\rangle$ stands for a diagonal matrix with the k-th diagonal element given by $\frac{d\zeta_k}{z_k-\zeta_k}$. Substitution of (1) into (5) leads to

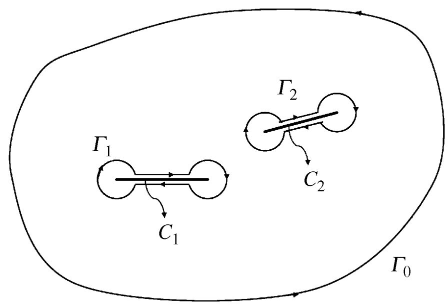

Fig. 1 A body bounded by an external boundary Γ_0 containing N=2 crack lines

$$(-\boldsymbol{A}^T\boldsymbol{t}+\boldsymbol{B}^T\boldsymbol{d})\Big|_{\boldsymbol{x}}$$

$$=\frac{1}{2\pi i}\int_{\Gamma}\left\langle\frac{\hat{z}}{\zeta-z}\right\rangle(-\boldsymbol{A}^T\overline{\boldsymbol{t}}+\boldsymbol{B}^T\overline{\boldsymbol{d}})\Big|_{\xi}d\Gamma(\boldsymbol{\xi}) \tag{6}$$

where $\boldsymbol{x}$ represents a generic point in the body, $\boldsymbol{d}$ and $\boldsymbol{t}$ are the displacement gradient and the traction, respectively, along the contour s at $\boldsymbol{x}$, $\hat{z}=\partial z/\partial s$, $\boldsymbol{\xi}$ denotes a boundary point and $\overline{\boldsymbol{d}}$ and $\overline{\boldsymbol{t}}$, respectively, are the displacement gradient and the traction along the boundary Γ at $\boldsymbol{\xi}$.

Let a body bounded by an external boundary Γ_0 contain N crack lines denoted by C_k, k=1, ⋯, N. In the presence of cracks, (6) becomes

$$(-\boldsymbol{A}^T\boldsymbol{t}+\boldsymbol{B}^T\boldsymbol{d})\Big|_{\boldsymbol{x}}$$

$$=\frac{1}{2\pi i}\sum_{k=0}^{N}\int_{\Gamma_k}\left\langle\frac{\hat{z}}{\zeta-z}\right\rangle(-\boldsymbol{A}^T\overline{\boldsymbol{t}}+\boldsymbol{B}^T\overline{\boldsymbol{d}})\Big|_{\xi}d\Gamma(\boldsymbol{\xi}) \tag{7}$$

where Γ_k, k=1, ⋯, N is the contour enclosing C_k as shown in Fig. 1. Consider the contour integral around Γ_k, k=1, ⋯, N, then

$$\int_{\Gamma_k}\left\langle\frac{\hat{z}}{\zeta-z}\right\rangle(-\boldsymbol{A}^T\overline{\boldsymbol{t}}+\boldsymbol{B}^T\overline{\boldsymbol{d}})d\Gamma$$

$$=\int_{C_k}\left\langle\frac{\hat{z}}{\zeta-z}\right\rangle[-\boldsymbol{A}^T(\overline{\boldsymbol{t}}^++\overline{\boldsymbol{t}}^-)+\boldsymbol{B}^T\boldsymbol{\alpha})d\Gamma(\boldsymbol{\xi}) \tag{8}$$

where $\overline{\boldsymbol{t}}^+$ and $\overline{\boldsymbol{t}}^-$, respectively, are the tractions on the upper and lower crack faces, $\boldsymbol{\alpha}=\partial\Delta\boldsymbol{u}/\partial s$ is the dislocation density $\Delta\boldsymbol{u}$ is the relative crack face displacement. The contribution from the contours encircling the crack tips vanishes as $\overline{\boldsymbol{t}}$ and $\overline{\boldsymbol{d}}$ are square-root singular at the crack tips. If the tractions on the crack faces are self-equilibrium, (7) is reduced to

$$(-\boldsymbol{A}^T\boldsymbol{t}+\boldsymbol{B}^T\boldsymbol{d})\Big|_{\boldsymbol{x}}$$

$$=\frac{1}{2\pi i}\int_{\Gamma_0}\left\langle\frac{\hat{z}}{\zeta-z}\right\rangle(-\boldsymbol{A}^T\overline{\boldsymbol{t}}+\boldsymbol{B}^T\overline{\boldsymbol{d}})\Big|_{\xi}d\Gamma(\xi)$$

$$+\int_C\left\langle\frac{\hat{z}}{\zeta-z}\right\rangle\boldsymbol{B}^T\boldsymbol{\alpha}(\xi)d\Gamma(\xi) \qquad (9)$$

where C is the collection of the crack lines. Since the tractions on Γ_0 can always be converted to proper tractions at the crack faces, as far as the stress intensity factors are concerned, we assume that only the crack faces are subjected to tractions. In this case as $\boldsymbol{x}\rightarrow\Gamma_0$ (7) yields

$$\overline{\boldsymbol{d}}(\boldsymbol{x})=\int_{\Gamma_0}\boldsymbol{G}(\boldsymbol{x};\,\xi)\overline{\boldsymbol{d}}(\xi)d\Gamma(\xi)$$

$$+\int_C\boldsymbol{G}(\boldsymbol{x};\,\xi)\boldsymbol{\alpha}(\xi)d\Gamma(\xi) \qquad (10)$$

where

$$\boldsymbol{G}(\boldsymbol{x};\,\xi)=\frac{1}{\pi}\mathrm{Im}\left((\boldsymbol{B}^T)^{-1}\left\langle\frac{\hat{z}}{\zeta-z}\right\rangle\boldsymbol{B}^T\right) \qquad (11)$$

It can be shown that the limiting form of $\boldsymbol{G}(\boldsymbol{x};\,\xi)$ as $\xi\rightarrow\boldsymbol{x}$ is given by

$$\boldsymbol{G}(\boldsymbol{x};\,\xi)=-\frac{1}{2\pi}\mathrm{Im}\left((\boldsymbol{B}^T)^{-1}\left\langle\frac{\partial^2z/\partial s^2}{\partial z/\partial s}\right\rangle\boldsymbol{B}^T\right) \qquad (12)$$

where the contour is assumed to be parametrized by the arc length s. Thus $\boldsymbol{G}(\boldsymbol{x};\,\xi)$ is not singular. Similarly as $\boldsymbol{x}\rightarrow C$ (7) gives

$$\boldsymbol{t}(\boldsymbol{x})=\int_{\Gamma_0}\boldsymbol{C}(\boldsymbol{x};\,\xi)\overline{\boldsymbol{d}}(\xi)d\Gamma(\xi)+\int_C\boldsymbol{C}(\boldsymbol{x};\,\xi)\boldsymbol{\alpha}(\xi)d\Gamma(\xi) \qquad (13)$$

where

$$\boldsymbol{C}(\boldsymbol{x};\,\xi)=-\frac{1}{\pi}\mathrm{Im}\left(\boldsymbol{B}\left\langle\frac{\hat{z}}{\zeta-z}\right\rangle\boldsymbol{B}^T\right) \qquad (14)$$

and (3) has been used. Eqs. (10) and (13) are the main result in this section.

III. NUMERICAL INTEGRATION

Equations (10) and (13) can be solved for $\overline{\boldsymbol{d}}$ and $\boldsymbol{\alpha}$ with given crack-face tractions using integration quadratures. As mentioned previously, the integral on the non-crack boundary is regular and can be easily handled by Gauss-Legendre quadrature. The integral on the crack lines, however, requires further treatment.

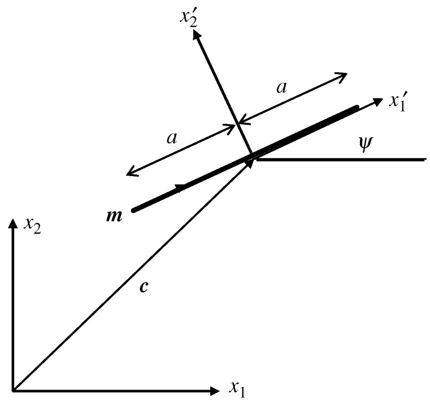

Fig. 2 A typical crack line

Consider a typical crack line C_k as shown in Fig. 2, where (x_1', x_2') is the local coordinate system such that the crack tips are located at $x_1'=\pm a$, $x_2'=0$. Near the crack tips, the relative crack face displacements are given by (Wu, 1989)

$$\lim_{x_1'\rightarrow\pm a}\Delta\boldsymbol{u}'=2\sqrt{\frac{2(a\mp x_1')}{\pi}}(\boldsymbol{L}')^{-1}\boldsymbol{K}_{\pm}' \qquad (15)$$

where $\boldsymbol{u}'$ and $\boldsymbol{L}'$, respectively, are the displacement and $\boldsymbol{L}$ matrix defined in (3) with respect to the local system; $\boldsymbol{K}_{\pm}'=[K_{II}^{\pm},\ K_{I}^{\pm},\ K_{III}^{\pm}]^T$ and K_I, K_{II}, and K_{III}, respectively, are the SIFs for Mode I, II and III. Since $\boldsymbol{u}$ and $\boldsymbol{L}$ are, respectively, tensors or rank one and two (Ting, 1996), we have

$$\boldsymbol{u}'=\boldsymbol{\Omega}\boldsymbol{u},\ \boldsymbol{L}'=\boldsymbol{\Omega}\boldsymbol{L}\boldsymbol{\Omega}^T \qquad (16)$$

where $\boldsymbol{\Omega}$ is given by

$$\boldsymbol{\Omega}=\begin{pmatrix}\cos\psi & \sin\psi & 0\\ -\sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{pmatrix} \qquad (17)$$

where ψ is the angle between the crack and the x_1 axis. Eqs. (16) into (15) gives

$$\lim_{x_1'\rightarrow\pm a}\Delta\boldsymbol{u}=2\sqrt{\frac{2(a\mp x_1')}{\pi}}\boldsymbol{L}^{-1}\boldsymbol{K}_{\pm} \qquad (18)$$

where $\boldsymbol{K}_{\pm}=\boldsymbol{\Omega}^T\boldsymbol{K}_{\pm}'$. The corresponding α is given by

$$\lim_{x_1'\rightarrow\pm a}\boldsymbol{\alpha}=\mp\sqrt{\frac{2}{\pi(a\mp x_1')}}\boldsymbol{L}^{-1}\boldsymbol{K}_{\pm} \qquad (19)$$

with square-root singularities at the tips. To incorporate the singularity explicitly, let

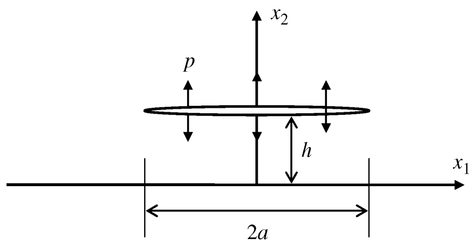

Fig. 3 An internal horizontal crack under a uniform pressure p in a half-plane

$$\boldsymbol{\alpha} = \frac{\boldsymbol{\beta}}{\sqrt{a^2 - x_1'^2}} \tag{20}$$

and the integral on this crack line becomes

$$\int_{C_k} \boldsymbol{C}(\boldsymbol{x};\, \boldsymbol{\xi})\boldsymbol{\alpha} d\Gamma(\boldsymbol{\xi}) = \int_{-1}^{1} \boldsymbol{C}(\boldsymbol{x};\, \boldsymbol{c} + at\boldsymbol{m})\frac{\boldsymbol{\beta}}{\sqrt{1-t^2}}dt \tag{21}$$

where $\boldsymbol{c}$ is the crack center, $\boldsymbol{m}$ is the unit vector in the direction of the crack, and $t=x_1'/a$. The integral on the right side of (21) can be evaluated using the Gauss-Chebyshev quadrature (Edogan and Gupta, 1972). If C_k is an internal crack, $\boldsymbol{\alpha}$ must satisfy the following single-valuedness conditions:

$$\int_{-a}^{a} \boldsymbol{\alpha} dx_1' = \int_{-1}^{1} \frac{\boldsymbol{\beta}}{\sqrt{1-t^2}}dt = \boldsymbol{0} \tag{22}$$

If C_k is an edge crack, $\boldsymbol{\alpha}$ at one of the crack tips is bounded. In this case either $\boldsymbol{\beta}(-a)=\boldsymbol{0}$ or $\boldsymbol{\beta}(a)=\boldsymbol{0}$ must be used in place of (22). Once $\boldsymbol{\beta}$ is determined, the SIFs can be calculated from (19) and (20) as

$$\boldsymbol{K}'_{\pm} = \mp \frac{1}{2}\sqrt{\frac{\pi}{a}}\boldsymbol{\Omega L \beta}(\pm a) \tag{23}$$

IV. NUMERICAL EXAMPLES

Equations (10) and (13) have been numerically solved for the SIFs of several problems using integration quadratures discussed in the previous section. The configurations considered include a half-space, a strip, and a finite plate. The elastic constants will be specified in the specific examples. An isotropic material is regarded as a weakly anisotropic material. A plane stress condition is assumed.

1. A Crack in a Plane

Figure 3 shows an internal horizontal crack of length $2a$ under a uniform pressure p in an isotropic half-plane $x_2 \geq 0$. The distance from the crack to the free surface is h. The problem was solved by Ito (1994). The SIFs for several ratios of h/a are given in Table 1 and compared to those obtained by Ito (1994). It can be seen that our results are very close to those of Ito.

Table 1 SIFs for a horizontal crack I a half-plane under pressure

	Ito (1994)		Present	
h/a	$K_I/p\sqrt{\pi a}$	$K_{II}/p\sqrt{\pi a}$	$K_I/p\sqrt{\pi a}$	$K_{II}/p\sqrt{\pi a}$
2.0	1.1634	0.0367	1.1633	0.0367
1.0	1.5110	0.1849	1.5110	0.1849
0.4	2.9051	0.9939	2.9073	0.9934

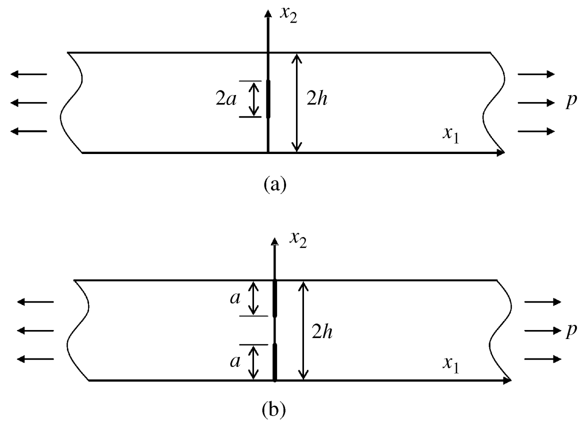

Fig. 4 (a) An internal crack, (b) double edge cracks in a strip under a uniaxial tension p

2. Cracks in a Strip

The configuration is shown in Fig. 4. The strip is assumed to be bounded by S_1: $-\infty < x_1 < \infty$, $x_2=0$ and S_2: $-\infty < x_1 < \infty$, $x_2=2h$. The SIFs have been computed for an internal crack or double edge cracks in a strip under a uniaxial tension p as shown in Fig. 4. The material considered was a boron-epoxy material with E_{11}=170.65 GPa, E_{22}=55.6 GPa, G_{12}=4.83 GPa, and v_{12}=0.1114 in a state *pf* plane stress. The numerical results are shown in Table 2 together with those reported by Delale and Edogan (1977). The agreement of our results with those Delale and Edogan is excellent.

3. An Inclined Crack in a Finite Plate

Figure 5 depicts a finite rectangular plate of height h and width w containing a central crack of length $2a$ inclined at 45° about the x_1 axis and subjected to a uniaxial tension p. The material is a glass-epoxy composite with E_{11}=48.26 GPa, E_{22}=17.24

Table 2 SIFs for an internal crack and double edge cracks in a strip under tension

a/h	An internal crack $K_I/p\sqrt{\pi a}$		Double edge cracks $K_I/p\sqrt{\pi a}$	
	Delale and Edogan (1977)	Present	Delale and Edogan (1977)	Present
0.1	1.0044	1.0044	1.458	1.460
0.2	1.0182	1.0183	1.462	1.464
0.3	1.0428	1.0427	1.486	1.487
0.4	1.0811	10.811	1.531	1.529
0.5	1.1387	1.1389	1.600	1.608
0.6	1.2246	1.2264	1.710	1.713
0.7	1.3674	1.3675	1.887	1.890
0.8	1.6241	1.6253	2.208	2.211
0.9	2.2487	2.2658	2.978	2.990

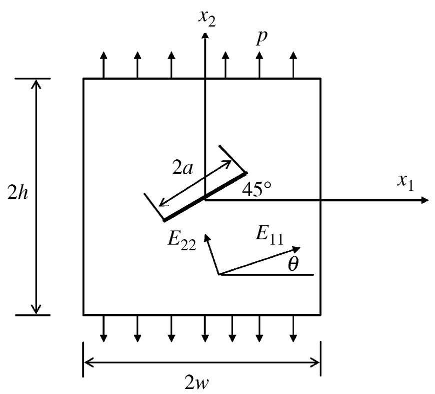

Fig. 5 An inclined crack in a finite plate under a uniform tension p

GPa, G_{12}=6.89 GPa, and ν_{12}=0.29. The SIFs with a/w=0.2, h/w=2 for various fiber orientations θ are shown in Table 3. The result obtained by Pan (1997) is also contained in Table 3. A comparison reveals that the present results and those of Pan are almost identical.

Table 3 SIFs for an incline crack in a finite plate under tension

θ	Pan (1997)		Present	
	$K_I/p\sqrt{\pi a}$	$K_{II}/p\sqrt{\pi a}$	$K_I/p\sqrt{\pi a}$	$K_{II}/p\sqrt{\pi a}$
0°	0.5228	0.5076	0.5232	0.5070
45°	0.5153	0.5048	0.5153	0.5048
90°	0.5133	0.5090	0.5132	0.5089
105°	0.5165	0.5107	0.5164	0.5107
120°	0.5240	0.5117	0.5239	0.5117
135°	0.5316	0.5111	0.5316	0.5111

V. CONCLUSIONS

A novel integral equation method has been derived. The method can be applied to general anisotropic elastic bodies containing an arbitrary number of cracks. The basic unknowns in the present method are displacement gradient on the non-crack boundary and the dislocation density representing the cracks in problems where tractions are specified. The integral equations can be conveniently solved using Gauss-type integration formulae. The SIFs can be obtained directly from the dislocation density. The method has been applied to several examples. In all examples the results are in excellent agreement with those in the literature. The fact indicates that the present method is effective, accurate and versatile.

REFERENCES

Delale, F., and Erdogan, F., 1977, "The Problem of Internal and Edge Cracks in an Orthotropic Strip," *Journal of Applied Mechanics*, Vol. 44, No. 2, pp. 237-242.

Erdogan, F., and Gupta, G. D., 1972, "On the Numerical Solution of Singular Integral Equations," *Quarterly of Applied Mathematics*, Vol. 30, pp. 525-534.

Hong, H. K., and Chen, J. T., 1988, "Derivations of Integral Equations of Elasticity," *Journal of Engineering Mechanics*, Vol. 114, pp. 1028-1044.

Ito, S., 1994, "Stress Intensity Factors around a Crack Parallel to a Free Surface of a Half-Plane," *International Journal of Fracture*, Vol. 67, No. 2, pp. 179-185.

Pan, E., 1997, "A General Boundary Element Analysis of 2-D Elastic Fracture Mechanics," *International Journal of Fracture*, Vol. 88, pp. 41-59.

Snyder, M. D., and Cruse, T. A., 1975, "Boundary-Integral Analysis of Anisotropic Cracked Plates," *International Journal of Fracture*, Vol. 11, No. 2, pp. 315-328.

Ting, T. C. T., 1996, *Anisotropic Elasticity – Theory and Applications*, Oxford University Press, Oxford, UK.

Wu, K. C., 1989, "On the Crack-Tip Fields and Energy Release Rate of a Dynamically Propagating Crack in an Anisotropic Elastic Solid," *International Journal of Fracture*, Vol. 41, No. 4, pp. 253-266.

Yeh, C. S., Shu, Y. C., and Wu, K. C., 1993, "Conservation Laws in Anisotropic Elasticity I. Basic Framework," *Proceedings of the Royal Society of London*, Vo. A443, pp. 139-151.

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