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INTEGRAL EQUATION APPROACH TO COMPOSITE LAMINATE ANALYSIS

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ABSTRACT

A comprehensive integral equation based approach is presented to determine the elastic response of composite laminates under axial, bending, shear/bending and torsional loadings. The integral equations governing the laminate behavior are directly deduced from the reciprocity theorem for beam-type structures by employing the fundamental solution of generalized plain strain anisotropic problems. Taking into account the displacement and stress continuity along the interfaces and the external boundary conditions the formulation is numerically solved by the multidomain boundary element method. The resolving system of linear algebraic equations is solved to provide the solution of the problem in terms of displacements and tractions on the boundary of each ply within the laminate. Once this boundary elastic response is determined the displacements and stresses at any point of the laminate can be computed using the appropriate boundary integral representations. The approach, based on a pointwise formulation, makes it possible to analyze laminates with the widest generalities as regards the section shape and lay-up. Some applications are presented in order to demonstrate the accuracy and effectiveness of the method proposed.

I. INTRODUCTION

The analysis of the elastic response of multilayered, fiber-reinforced composite laminates is a growing concern in composite structural design, probably due to the wide spread of these structural members in lightweight technology. Due to the inherent anisotropy and mismatch in the elastic properties of the adjacent plies within the laminates a complex three-dimensional stress state arises showing high interlaminar stresses along the interfaces in the free edge region (Jones, 1975). These interlaminar stresses are responsible for the delamination phenomenon, which can lead to laminate failure at loads that are lower than those at which the structure would fail if only the classical failure mechanisms were involved. A survey of the literature shows that investigators have used various approaches to attempt to analyze the composite laminate elastic response. Starting from the pioneering work by Pipes and Pagano (1970), based on the finite difference technique, results have been obtained for different cases by using the finite element method (Wang and Crossman, 1977; Spilker and Chou, 1980; Raju and Crews, 1981; Wang and Yuan, 1983; Ye, 1990; Chan and Ochoa, 1987; 1990), boundary layer theories (Tang, 1975; Tang and Levy, 1975), perturbation techniques (Hsu and Herakovich, 1977), polynomial

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and Fourier series (Pipes and Pagauo, 1974; Wang and Dickson, 1978), Lekhnitskii's stress potentials (Wang and Choi, 1982a, b; Yin, 1994a, b), Reissner's variational principle (Pagano, 1978; 1978) and the force balance method coupled with minimization of complementary energy (Kassapoglou and Lagace, 1986; Lin *et al.*, 1995). These methods provide simple and sufficiently accurate tools for calculating interlaminar stresses. In many cases the attention was focused on axial and pure bending loads; only some authors (Chan and Ochoa, 1990; Ko and Lin, 1992; 1993) dealt with other important loading conditions, i.e. torsion and shear. Moreover the stress distributions obtained by using the above-mentioned approaches show good agreement among them for sites away from the free edge, whereas considerable disagreement exists for points near the free edge location among the various analytical and numerical solutions proposed. This is to be expected as a result of a priori assumptions or because the traction boundary conditions of the continuum problem have been transformed into generalized conditions through equivalent nodal forces. From this point of view, boundary integral equations (BIE) and the boundary element method (BEM) represent an interesting and effective tool for composite laminate structural analysis. We know that: (1) integral equations preserve the pointwise description of the continuum problem in the modelization; (2) boundary integral equations and the boundary element method are very well suited to treat interface problems like those involved in composite laminate analysis; (3) the boundary element method has meaningful computational advantages with respect to the more common field methods. On this basis, the integral equation approach has been used to achieve solutions for anisotropic elasticity problems. Many efforts have successfully been made for two-dimensional anisotropic linear elasticity where the integral equation kernels are available in closed form. However, only recently the authors have presented an alternative and original boundary integral formulation (Daví, 1996; Daví and Milazzo, 1997a,b; Daví, 1997; Daví and Milazzo, 1999) for the three-dimensional analysis of anisotropic and unhomogeneous beams by which a Saint Venant solution for general composite laminates is recovered, accounting for the basic 3D nature of the laminate elastic behavior. In the present paper a comprehensive formulation is presented to analyze composite laminates under axial, bending, torsion and shear/bending loadings. The method is founded on the integral equation theory, and the beam-type reciprocity theorem is used to obtain an exact boundary integral formulation. The anisotropic singular fundamental solution, due to a concentrated load uniformly distributed along a line, is explicitly

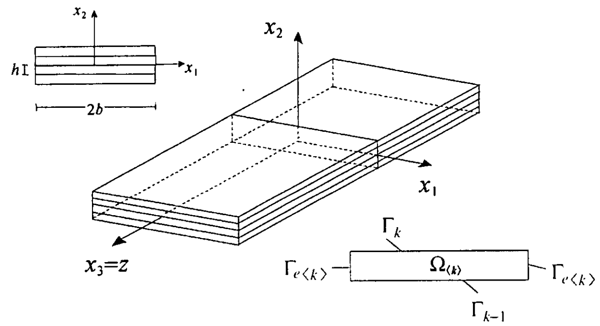

Fig. 1. Laminate configuration.

determined and used to infer the displacement and stress boundary integral representation. Once the integral equations for each ply of the laminate are written and the interface continuity conditions are taken into account, the laminate model is numerically solved through the multidomain boundary element method. It provides the solution in terms of displacements and tractions along the boundary of each ply within the laminate. The elastic response charac-teristics, i.e. displacements and stresses, at any point of the laminate can then be computed by means of their appropriate integral representations. This approach makes it possible, in the context of the hypothesis presented, to analyze composite laminates with the widest generality as regards the shape of the section and the lay-up. The method presents itself as a powerful, effective and sound tool of analysis which is able to provide an overview of the features of the stress fields in composite laminates under various loadings with the computational advantages of a boundary integral method.

II. NOTATION AND GOVERNING EQUATIONS

Let us consider a beam-type composite laminate referred to a coordinate system x_1, x_2, x_3 with the $x_3 \equiv z$ axis parallel to the generators of the beam lateral surface as shown in Fig. 1. The laminate consists of N anisotropic plies with general lay-up and perfectly bonded along the interfaces. Each individual ply has cross section $\Omega_{\langle k\rangle}$ with boundary $\Gamma_{\langle k\rangle}=\Gamma_{e\langle k\rangle}\cup\Gamma_k\cup\Gamma_{k-1}$ as shown in Fig. 1. The laminate is subjected to a combined load characterized by the uniform extension ε_0, the bending curvatures κ_1 and κ_2, the twisting curvature ϑ and the shear/bending loading parameters γ_1 and γ_2. Assuming Saint Venant's principle to be satisfied, sufficiently far from the laminate ends, the displacement field s can be expressed as (Lekhnitskii, 1963)

$$s = \boldsymbol{u} + z\boldsymbol{v} + z\boldsymbol{X}_1\boldsymbol{k} - \frac{1}{2}z^2\boldsymbol{X}_2\boldsymbol{k} + \frac{1}{2}z^2\boldsymbol{X}_3\boldsymbol{\gamma} - \frac{1}{6}z^3\boldsymbol{X}_4\boldsymbol{\gamma} \qquad (1)$$

where the rigid motion terms have been dropped and

$$u=\{u_1(x_1, x_2),\ u_2(x_1, x_2),\ u_3(x_1, x_2)\}^T \qquad (2)$$

$$v=\{v_1(x_1, x_2),\ v_2(x_1, x_2),\ v_3(x_1, x_2)\}^T \qquad (3)$$

$$X_1=\begin{bmatrix} 0 & 0 & 0 & -x_2 \\ 0 & 0 & 0 & x_1 \\ 1 & x_1 & x_2 & 0 \end{bmatrix} \qquad (4)$$

$$X_2=\begin{bmatrix} 0\,1\,0\,0 \\ 0\,0\,1\,0 \\ 0\,0\,0\,0 \end{bmatrix} \qquad (5)$$

$$X_3=\begin{bmatrix} 0 & 0 & x_1 \\ 0 & 0 & x_2 \end{bmatrix}^T \qquad (6)$$

$$X_4=\begin{bmatrix} 1\,0\,0 \\ 0\,1\,0 \end{bmatrix}^T \qquad (7)$$

Introducing the strain operators

$$\mathcal{D}=\begin{bmatrix} \partial/\partial x_1 & 0 & \partial/\partial x_2 & 0 & 0 \\ 0 & \partial/\partial x_2 & \partial/\partial x_1 & 0 & 0 \\ 0 & 0 & 0 & \partial/\partial x_1 & \partial/\partial x_2 \end{bmatrix}^T \qquad (8)$$

$$\mathcal{D}_z=\begin{bmatrix} 0 & 0 & 0 & \partial/\partial z & 0 \\ 0 & 0 & 0 & 0 & \partial/\partial z \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \qquad (9)$$

the strain field associated with the displacement system in Eq. (1) is given by

$$\varepsilon=\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{23} \end{Bmatrix} = \mathcal{D}s + \mathcal{D}_z s = \mathcal{D}u + I_z v + X_1'k + z\mathcal{D}v = \varepsilon_u + z\varepsilon_v \qquad (10)$$

$$\varepsilon_{33} = v_3 + X_2'k + zX_3'\gamma = \varepsilon_{33u} + z\varepsilon_{33v} \qquad (11)$$

where

$$I_z=\begin{bmatrix} 0\,0\,0\,1\,0 \\ 0\,0\,0\,0\,1 \\ 0\,0\,0\,0\,0 \end{bmatrix}^T \qquad (12)$$

$$X_1'=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2 & x_1 \end{bmatrix}^T \qquad (13)$$

$$X_2'=[1\ \ x_1\ \ x_2\ \ 0] \qquad (14)$$

$$X_3'=[x_1\ \ x_2] \qquad (15)$$

In the previous formulas the subscripts u and v refer to components of the elastic response which are constant and linearly variable along the z-axis, respectively. The stress field in each ply of the laminate is expressed by using the generalized Hooke's law which governs ply's material behavior. Accounting for the form of the strain field, the constitutive equations are appropriately written as

$$\begin{Bmatrix} \sigma \\ \hdashline \sigma_{33} \end{Bmatrix} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \\ \sigma_{31} \\ \sigma_{32} \\ \hdashline \sigma_{33} \end{Bmatrix} = \left[\begin{array}{ccccc:c} E_{11} & E_{12} & E_{13} & E_{14} & E_{15} & E_{16} \\ E_{12} & E_{22} & E_{23} & E_{24} & E_{25} & E_{26} \\ E_{13} & E_{23} & E_{33} & E_{34} & E_{35} & E_{36} \\ E_{14} & E_{24} & E_{34} & E_{44} & E_{45} & E_{46} \\ E_{15} & E_{25} & E_{35} & E_{45} & E_{55} & E_{56} \\ \hdashline E_{16} & E_{26} & E_{36} & E_{46} & E_{56} & E_{66} \end{array}\right] \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \\ \varepsilon_{31} \\ \varepsilon_{32} \\ \hdashline \varepsilon_{33} \end{Bmatrix}$$

$$= \left[\begin{array}{c:c} E_{11} & E_{12} \\ \hdashline E_{12}^T & E_{66} \end{array}\right] \begin{Bmatrix} \varepsilon \\ \hdashline \varepsilon_{33} \end{Bmatrix} = \begin{Bmatrix} \sigma_u \\ \hdashline \sigma_{33u} \end{Bmatrix} + z\begin{Bmatrix} \sigma_v \\ \hdashline \sigma_{33v} \end{Bmatrix} \qquad (16)$$

Again, let us denote

$$\tau=\begin{Bmatrix} \sigma_{31} \\ \sigma_{32} \\ \sigma_{33} \end{Bmatrix} = \left[\begin{array}{ccccc:c} E_{14} & E_{24} & E_{34} & E_{44} & E_{45} & E_{46} \\ E_{15} & E_{25} & E_{35} & E_{45} & E_{55} & E_{56} \\ E_{16} & E_{26} & E_{36} & E_{46} & E_{56} & E_{66} \end{array}\right] \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \\ \varepsilon_{31} \\ \varepsilon_{32} \\ \hdashline \varepsilon_{33} \end{Bmatrix}$$

$$= [Q_1 \,\vdots\, Q_2]\begin{Bmatrix} \varepsilon \\ \hdashline \varepsilon_{33} \end{Bmatrix} = \tau_u + z\tau_v \qquad (17)$$

With the notation introduced above, the ply equilibrium equations in $\Omega_{<k>}$ are given by

$$\mathcal{D}^T\sigma + \frac{\partial \tau}{\partial z} = \mathbf{0} \qquad (18)$$

Upon substituting for σ and τ from Eqs. (16) and (17) and noting that

$$E_{11}I_z v + E_{12}v_3 = Q_1^T v \qquad (19)$$

Eq. (18) gives

$$\mathcal{D}^T E_{11}\mathcal{D}u + (\mathcal{D}^T Q_1^T + Q_1\mathcal{D})v + \mathcal{D}^T(E_{11}X_1' + E_{12}X_2')k$$
$$+ Q_2 X_3'\gamma + z(\mathcal{D}^T E_{11}\mathcal{D}v + \mathcal{D}^T E_{12}X_3'\gamma) = \mathbf{0} \qquad (20)$$

Equation (20) is verified to make the following expressions fulfilled simultaneously

$$\mathcal{D}^T\boldsymbol{E}_{11}\mathcal{D}\boldsymbol{v} + \mathcal{D}^T\boldsymbol{E}_{12}\boldsymbol{X}_3'\gamma = \boldsymbol{0} \tag{21}$$

$$\mathcal{D}^T\boldsymbol{E}_{11}\mathcal{D}\boldsymbol{u} + (\mathcal{D}^T\boldsymbol{Q}_1^T + \boldsymbol{Q}_1\mathcal{D})\boldsymbol{v} + \mathcal{D}^T(\boldsymbol{E}_{11}\boldsymbol{X}_1' + \boldsymbol{E}_{12}\boldsymbol{X}_2')\boldsymbol{k}$$

$$+\boldsymbol{Q}_2\boldsymbol{X}_3'\gamma = \boldsymbol{0} \tag{22}$$

Equations (21) and (22) constitute an uncoupled system of partial differential equations which governs the ply elastic response with the appropriate boundary conditions. According to the mathematical structure evidenced by the elastic response, the boundary tractions on $\Gamma_{\langle k\rangle}$ are given by

$$\boldsymbol{t} = \mathcal{D}_n\sigma = \boldsymbol{t}_u + z\boldsymbol{t}_v \tag{23}$$

where the boundary traction operator $\mathcal{D}_n$ is defined as

$$\mathcal{D}_n = \begin{bmatrix} \alpha_1 & 0 & \alpha_2 & 0 & 0 \\ 0 & \alpha_2 & \alpha_1 & 0 & 0 \\ 0 & 0 & 0 & \alpha_1 & \alpha_2 \end{bmatrix} \tag{24}$$

In Eq. (24) α_1 and α_2 are the direction cosines of the outer normal to the ply section boundary. The mechanical boundary conditions for Eqs. (21) and (22) are therefore supplied in terms of assigned values of the traction functions $\boldsymbol{t}_v$ and $\boldsymbol{t}_u$ respectively, whereas the kinematical boundary conditions are given in terms of prescribed displacement functions $\boldsymbol{v}$ and $\boldsymbol{u}$.

III. INTEGRAL EQUATION REPRESENTATION

For each individual ply within the laminate, let us consider the elasticity problem governed by the following equilibrium equation

$$\mathcal{D}^T\sigma_j + \boldsymbol{f}_j = \boldsymbol{0} \tag{25}$$

where $\boldsymbol{f}_j = \boldsymbol{f}_j(x_1, x_2)$ is a fictitious system of body forces applied to the ply. Let $\boldsymbol{u}_j = \boldsymbol{u}_j(x_1, x_2)$ be the displacement field characterizing an elastic solution of Eq. (25) which, according to the notation introduced, satisfies the relation

$$\mathcal{D}^T\boldsymbol{E}_{11}\mathcal{D}\boldsymbol{u}_j + \boldsymbol{f}_j = \boldsymbol{0} \tag{26}$$

Let again ε_j and $\boldsymbol{t}_j$ be the strains and boundary tractions of this solution. The external elementary work done by the tractions $\boldsymbol{t}$ and τ through the displacements $\boldsymbol{u}_j$ is

$$dL_{12} = (\int_{\Gamma_{\langle k\rangle}} \boldsymbol{u}_j^T \boldsymbol{t}\, d\Gamma + \frac{\partial}{\partial z}\int_{\Omega_{\langle k\rangle}} \boldsymbol{u}_j^T \tau\, d\Omega)dz \tag{27}$$

Analogously, the external elementary work done by the body forces $\boldsymbol{f}_j$ and tractions $\boldsymbol{t}_j$ and τ_j through the displacements $\boldsymbol{s}$ is

$$dL_{21} = (\int_{\Gamma_{\langle k\rangle}} \boldsymbol{t}_j^T \boldsymbol{s}\, d\Gamma + \int_{\Omega_{\langle k\rangle}} \boldsymbol{f}_j^T \boldsymbol{s}\, d\Omega + \frac{\partial}{\partial z}\int_{\Omega_{\langle k\rangle}} \tau_j^T \boldsymbol{s}\, d\Omega)dz \tag{28}$$

Since from the reciprocal work theorem $dL_{12} = dL_{21}$, from Eqs. (27) and (28) we have

$$\int_{\Gamma_{\langle k\rangle}} (\boldsymbol{t}_j^T \boldsymbol{s} - \boldsymbol{u}_j^T \boldsymbol{t})d\Gamma + \int_{\Omega_{\langle k\rangle}} \boldsymbol{f}_j^T \boldsymbol{s}\, d\Omega = \int_{\Omega_{\langle k\rangle}} \frac{\partial}{\partial z}(\boldsymbol{u}_j^T \tau - \tau_j^T \boldsymbol{s})d\Omega \tag{29}$$

Equation (29) is the expression of Betti's reciprocity theorem inherent to beam-type structures. Bearing in mind Eq. (1), the expression of the reciprocity theorem provides the following set of equations which have to be fulfilled simultaneously

$$\int_{\Omega_{\langle k\rangle}} \boldsymbol{f}_j^T \boldsymbol{X}_4 d\Omega\gamma + \int_{\Gamma_{\langle k\rangle}} \boldsymbol{t}_j^T \boldsymbol{X}_4 d\Gamma\gamma = 0 \tag{30}$$

$$[\int_{\Omega_{\langle k\rangle}} \boldsymbol{f}_j^T \boldsymbol{X}_2 d\Omega + \int_{\Gamma_{\langle k\rangle}} \boldsymbol{t}_j^T \boldsymbol{X}_2 d\Gamma]\boldsymbol{k} - [\int_{\Omega_{\langle k\rangle}} \boldsymbol{f}_j^T \boldsymbol{X}_3 d\Omega$$

$$+ \int_{\Gamma_{\langle k\rangle}} \boldsymbol{t}_j^T \boldsymbol{X}_3 d\Gamma - \int_{\Omega_{\langle k\rangle}} \tau_j^T \boldsymbol{X}_4 d\Omega]\gamma = 0 \tag{31}$$

$$\int_{\Gamma_{\langle k\rangle}} (\boldsymbol{t}_j^T \boldsymbol{v} - \boldsymbol{u}_j^T \boldsymbol{t}_v)d\Gamma + \int_{\Omega_{\langle k\rangle}} \boldsymbol{f}_j^T \boldsymbol{v}\, d\Omega + \int_{\Omega_{\langle k\rangle}} \tau_j^T \boldsymbol{X}_3 \gamma\, d\Omega = 0 \tag{32}$$

$$\int_{\Gamma_{\langle k\rangle}} (\boldsymbol{t}_j^T \boldsymbol{u} - \boldsymbol{u}_j^T \boldsymbol{t}_u)d\Gamma + \int_{\Omega_{\langle k\rangle}} \boldsymbol{f}_j^T \boldsymbol{u}\, d\Omega + \int_{\Omega_{\langle k\rangle}} (\tau_j^T \boldsymbol{v} - \boldsymbol{u}_j^T \tau_v)d\Omega$$

$$+ \int_{\Omega_{\langle k\rangle}} \tau_j^T \boldsymbol{X}_1 \boldsymbol{k}\, d\Omega = 0 \tag{33}$$

By using the divergence theorem one recognizes that Eqs. (30) and (31) are identically satisfied. Therefore the reciprocity theorem for the ply within the laminate reduces to the two integral relations (32) and (33) only. These integral relations constitute the basis to directly derive the integral equation representation employed in the present method. Let us consider the body forces $\boldsymbol{f}_j$ to be a concentrated load uniformly distributed along a line parallel to the

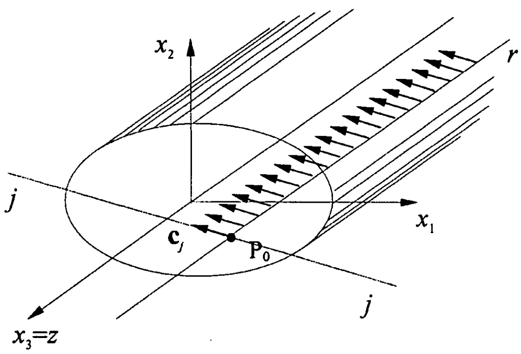

Fig. 2. Fundamental solution load configuration.

longitudinal axis z and applied in the j direction as shown in Fig. 2. Indicating, with P_0, the load application point in the ply section, the mathematical representation of f_j is given by

$$f_j=c_j\delta(P-P_0) \tag{34}$$

where δ denotes the Dirac function and c_j is the vector containing the components of the concentrated load. In this case the solution of Eq. (26) represents the singular fundamental solution of the problem. Therefore, taking the constitutive relations into account, through a suitable limit procedure and the use of the divergence theorem, Eqs. (32) and (33) become

$$c_j^T v(P_0)+\int_{\Gamma\langle k\rangle}(t_j^T v-u_j^T t_v)d\Gamma+\int_{\Omega\langle k\rangle}\varepsilon_j^T Q_1^T X_3\gamma d\Omega=0 \tag{35}$$

$$c_j^T u(P_0)+\int_{\Gamma\langle k\rangle}(t_j^T u-u_j^T t_u)d\Gamma-\int_{\Omega\langle k\rangle}u_j^T(\mathcal{D}^T Q_1^T+Q_1\mathcal{D})v d\Omega$$

$$+\int_{\Gamma\langle k\rangle}u_j^T\mathcal{D}_n Q_1^T v d\Gamma+\int_{\Omega\langle k\rangle}\varepsilon_j^T Q_1^T X_1 k d\Omega$$

$$-\int_{\Omega\langle k\rangle}u_j^T Q_2 X_3'\gamma d\Omega=0 \tag{36}$$

Equations (35) and (36) can be regarded as the form of the beam-type Somigliana identity for the ply within the laminate. They provide a direct link between the displacement functions v and u at the field point P_0 and the characteristics of the elastic response, displacements and tractions on the boundary. Therefore, Eqs. (35) and (36) give a boundary integral representation of the displacement field of the ply. Writing the boundary integral representation given by Eqs. (35) and (36) for three independent fundamental solutions, related to three independent load conditions, one obtains the matrix form of the Somigliana identity for beam-type structures, which gives the three displacement components at P_0. One has

$$c^*v(P_0)=\int_{\Gamma\langle k\rangle}(u^*t_v-t^*v)d\Gamma-\int_{\Omega\langle k\rangle}\varepsilon^*E_{12}X_3'\gamma d\Omega \tag{37}$$

$$c^*u(P_0)=\int_{\Gamma\langle k\rangle}(u^*t_u-t^*u)d\Gamma+\int_{\Omega\langle k\rangle}u^*(\mathcal{D}^T Q_1^T+Q_1\mathcal{D})v d\Omega$$

$$-\int_{\Gamma\langle k\rangle}u^*\mathcal{D}_n Q_1^T v d\Gamma-\int_{\Omega\langle k\rangle}\varepsilon^* Q_1^T X_1 k d\Omega$$

$$+\int_{\Omega\langle k\rangle}u^* Q_2 X_3'\gamma d\Omega \tag{38}$$

where

$$u^*=[u_{ij}]^T \tag{39}$$

$$t^*=[t_{ij}]^T \tag{40}$$

$$\varepsilon^*=[\mathcal{D}(u^{*T})]^T \tag{41}$$

In the previous relations u_{ij} and t_{ij} indicate the i-th component of displacements and tractions of the fundamental solution associated with the load applied along the direction j. The matrix of the coefficients c^* arising from the limit procedure which provide the ply Somigliana identity is defined as (Daví, 1996)

$$c^*=[c_{ij}]^T=-\int_{\Gamma\langle k\rangle}t^*d\Gamma \tag{42}$$

Equations (37) and (38) are valid for each point P_0 and thus, setting P_0 on the boundary, they provide a system of integral equations whose solution with appropriate boundary conditions gives the displacements and tractions on the boundary of the ply. Also, starting from the boundary integral representation given by Eqs. (37) and (38), one can deduce the boundary integral representation for the stress field in terms of the boundary displacements and tractions. Applying the strain operator $\mathcal{D}$ to Eqs. (37) and (38) and taking into account the constitutive equations in the form of Eq. (16) the integral equation representation for the stresses is given by

$$\sigma_v(P_0)=\int_{\Gamma\langle k\rangle}(T_{11}^*v-U_{11}^*t_v)d\Gamma$$

$$+\int_{\Omega\langle k\rangle}(\mathcal{E}_{11}^*E_{12}X_3'\gamma d\Omega+E_{12}X_3'(P_0)\gamma \tag{43}$$

$$\sigma_u(P_0)=\int_{\Gamma\langle k\rangle}(T^*_{11}u-U^*_{11}t_u)d\Gamma+\int_{\Gamma\langle k\rangle}U^*_{11}\mathcal{D}_n Q_1^T v d\Gamma$$

$$-\int_{\Omega\langle k\rangle}U^*_{11}(\mathcal{D}^T Q_1^T+Q_1\mathcal{D})v d\Omega-\int_{\Omega\langle k\rangle}U^*_{11}Q_2 X'_3\gamma d\Omega$$

$$+\int_{\Omega\langle k\rangle}\mathcal{E}^*_{11}Q_1^T X_1 k d\Omega+Q_1^T v(P_0)$$

$$+[E_{11}X'_1(P_0)+E_{12}X'_2(P_0)]k \tag{44}$$

$$\sigma_{33v}(P_0)=\int_{\Gamma\langle k\rangle}(T^*_{12}v-U^*_{12}t_v)d\Gamma+\int_{\Omega\langle k\rangle}\mathcal{E}^*_{12}E_{12}X'_3\gamma d\Omega$$

$$+E_{66}X'_3(P_0)\gamma \tag{45}$$

$$\sigma_{33u}(P_0)=\int_{\Gamma\langle k\rangle}(T^*_{12}u-U^*_{12}t_u)d\Gamma+\int_{\Gamma\langle k\rangle}U^*_{12}\mathcal{D}_n Q_1^T v d\Gamma$$

$$-\int_{\Omega\langle k\rangle}U^*_{12}(\mathcal{D}^T Q_1^T+Q_1\mathcal{D})v d\Omega$$

$$-\int_{\Omega\langle k\rangle}U^*_{12}Q_2 X'_3\gamma d\Omega+\int_{\Omega\langle k\rangle}\mathcal{E}^*_{12}Q_1^T X_1 k d\Omega$$

$$+Q_2^T v(P_0)+[E^T_{12}X'_1(P_0)+E_{66}X'_2(P_0)]k \tag{46}$$

where we have put

$$U^*_{1r}=E^T_{1r}\mathcal{D}c^{*-1}u^* \tag{47}$$

$$T^*_{1r}=E^T_{1r}\mathcal{D}c^{*-1}t^* \tag{48}$$

$$\mathcal{E}^*_{1r}=E^T_{1r}\mathcal{D}c^{*-1}\varepsilon^* \tag{49}$$

IV. FUNDAMENTAL SOLUTIONS

No general closed form is available for the fundamental solution of three-dimensional anisotropic problems (Schclar, 1994) whereas in the field of two-dimensional anisotropic linear elasticity problems the integral equation kernels are directly available for computations and thus BEM solutions have been presented for 2D elasticity problems (Banerjee and Butterfield 1981; Schclar, 1994). For the present approach the formulation of the integral equation representation requires the knowledge of a singular solution of Eq. (26) in the unbounded domain, i.e. the fundamental solution of the problem due to a uniform distribution of a concentrated load along a line. The characteristics of the fundamental solutions for general anisotropic elasticity due to the load condition described above were studied by Kayupov and Kuriyagawa (1996) and Mantic and Paris (Mamtic & Paris, 1997). They provided an expression of the fundamental solution which is rather cumbersome to implement and involves complex mathematics. For generalized orthotropic media Daví (1996) has proposed a suitable form of the fundamental solution for the generalized plane strain problem defined by Eq. (26). The anisotropic fundamental solutions of the problem are expressly obtained by integrating Eq. (26) on the basis of the Lekhnitskii stress potential theory (1963) and in the following they are explicitly given for a suitable use in computations. The anisotropic fundamental solutions depend on the roots $\mu_k=\xi_k+i\zeta_k$ of the equation

$$a\mu^6+b\mu^5+c\mu^4+d\mu^3+e\mu^2+f\mu+g=0 \tag{50}$$

where

$$a=\beta_{11}\beta_{44}-\beta_{14}^2 \tag{51}$$

$$b=-2(\beta_{11}\beta_{54}+\beta_{13}\beta_{44}-\beta_{14}\beta_{15}-\beta_{14}\beta_{43}) \tag{52}$$

$$c=\beta_{11}\beta_{55}+2\beta_{12}\beta_{44}+4\beta_{13}\beta_{54}-2\beta_{14}\beta_{24}-2\beta_{14}\beta_{53}-\beta_{15}^2$$
$$-2\beta_{15}\beta_{43}+\beta_{33}\beta_{44}-\beta_{43}^2 \tag{53}$$

$$d=-2(2\beta_{12}\beta_{54}+\beta_{13}\beta_{55}-\beta_{14}\beta_{25}-\beta_{15}\beta_{24}-\beta_{15}\beta_{53}+\beta_{23}\beta_{44}$$
$$-\beta_{24}\beta_{43}+\beta_{33}\beta_{54}-\beta_{43}\beta_{53}) \tag{54}$$

$$e=2\beta_{12}\beta_{55}-2\beta_{15}\beta_{25}+\beta_{22}\beta_{44}+4\beta_{23}\beta_{54}-\beta_{24}^2-2\beta_{24}\beta_{53}$$
$$-2\beta_{25}\beta_{43}+\beta_{33}\beta_{55}-\beta_{53}^2 \tag{55}$$

$$f=-2(\beta_{22}\beta_{54}+\beta_{23}\beta_{55}-\beta_{25}\beta_{24}-\beta_{25}\beta_{53}) \tag{56}$$

$$g=\beta_{22}\beta_{55}-\beta_{25}^2 \tag{57}$$

and

$$\beta_{ij}=S_{ij}-\frac{S_{i6}S_{j6}}{S_{66}} \tag{58}$$

The quantities S_{ij} are the material compliances. The fundamental displacement field associated with a load directed along the j direction is

$$u_j(P,P_0)=2\sum_{k=1}^{3}\Re(Q_{kj}w_k)\ln R_k-\Im(Q_{kj}w_k)\tan^{-1}\frac{Y_k}{X_k} \tag{59}$$

where

$$\boldsymbol{w}_k=\begin{Bmatrix} w_{1k} \\ w_{2k} \\ w_{3k} \end{Bmatrix}=\begin{Bmatrix} \beta_{11}\mu_k^2+\beta_{12}-\beta_{13}\mu_k+\beta_{14}\eta_k\mu_k-\beta_{15}\eta_k \\ \beta_{12}\mu_k+\beta_{22}/\mu_k-\beta_{23}+\beta_{24}\eta_k-\beta_{25}\eta_k/\mu_k \\ \beta_{15}\mu_k+\beta_{25}/\mu_k-\beta_{35}+\beta_{45}\eta_k-\beta_{55}\eta_k/\mu_k \end{Bmatrix} \tag{60}$$

$$\eta_k=-\frac{\beta_{14}\mu_k^3-(\beta_{15}+\beta_{43})\mu_k^2+(\beta_{24}+\beta_{53})\mu_k-\beta_{25}}{\beta_{44}\mu_k^2-2\beta_{45}\mu_k+\beta_{55}} \tag{61}$$

$$X_k=[x_1(P)-x_1(P_0)]+\xi_k[x_2(P)-x_2(P_0)] \tag{62}$$

$$Y_k=\zeta_k[x_2(P)-x_2(P_0)] \tag{63}$$

$$R_k=\sqrt{X_k^2+Y_k^2} \tag{64}$$

In the previous expressions P and P_0 denote the observed and the source point respectively and the symbols $\Re$ and $\Im$ indicate the real and imaginary part respectively. The coefficients $Q_{kj}=Q_{kj}^{\Re}+iQ_{kj}^{\Im}$ are determined in such a way that the fundamental displacement field matches the congruence and equilibrium conditions. This leads to a system of algebraic equations from which the coefficients Q_{kj} are calculated

$$\boldsymbol{AQ}=\boldsymbol{F} \tag{65}$$

where

$$\boldsymbol{Q}=\{Q_{1j}^{\Re}\ Q_{2j}^{\Re}\ Q_{3j}^{\Re}\ Q_{1j}^{\Im}\ Q_{2j}^{\Im}\ Q_{3j}^{\Im}\}^T \tag{66}$$

$$\boldsymbol{F}=\{0\ \ 0\ \ 0\ \ c_{1j}\ \ c_{2j}\ \ c_{3j}\}^T \tag{67}$$

The coefficients of the system are (i, k=1, 2, 3)

$$A_{ik}=\Im(w_{ik}) \tag{68}$$

$$A_{i(k+3)}=\Re(w_{ik}) \tag{69}$$

$$A_{4k}=M_{1k}\Re(\mu_k^2)+M_{2k}\Im(\mu_k^2)-N_{1k}\Re(\mu_k)-N_{2k}\Im(\mu_k) \tag{70}$$

$$A_{4(k+3)}=-M_{1k}\Im(\mu_k^2)+M_{2k}\Re(\mu_k^2)+N_{1k}\Im(\mu_k)-N_{2k}\Re(\mu_k) \tag{71}$$

$$A_{5k}=-M_{1k}\Re(\mu_k)-M_{2k}\Im(\mu_k)+N_{1k} \tag{72}$$

$$A_{5(k+3)}=M_{1k}\Im(\mu_k)-M_{2k}\Re(\mu_k)+N_{2k} \tag{73}$$

$$A_{6k}=M_{1k}\Re(\eta_k\mu_k)+M_{2k}\Im(\eta_k\mu_k)-N_{1k}\Re(\eta_k)-N_{2k}\Im(\eta_k) \tag{74}$$

$$A_{6(k+3)}=-M_{1k}\Im(\eta_k\mu_k)+M_{2k}\Re(\eta_k\mu_k)+N_{1k}\Im(\eta_k) -N_{2k}\Re(\eta_k) \tag{75}$$

where

$$M_{1k}=-2\pi\frac{Sgn(\zeta_k)[\xi_k^2-\zeta_k^2+1]Sgn(\zeta_k)+\zeta_k(\xi_k^2+\zeta_k^2-1)]}{\xi_k^4+2\xi_k^2(\zeta_k^2+1)+\zeta_k^4-2\zeta_k^2+1} \tag{76}$$

$$M_{2k}=-2\pi\frac{\xi_k Sgn(\zeta_k)[2|\zeta_k|-\xi_k^2-\zeta_k^2-1]}{\xi_k^4+2\xi_k^2(\zeta_k^2+1)+\zeta_k^4-2\zeta_k^2+1} \tag{77}$$

$$N_{1k}=-2\pi\frac{\xi_k Sgn(\zeta_k)[(\xi_k^2+\zeta_k^2+1)Sgn(\zeta_k)-2\zeta_k]}{\xi_k^4+2\xi_k^2(\zeta_k^2+1)+\zeta_k^4-2\zeta_k^2+1} \tag{78}$$

$$N_{2k}=-2\pi\frac{Sgn(\zeta_k)[(\xi_k^2+\zeta_k^2-1)|\zeta_k|+\xi_k^2-\zeta_k^2+1]}{\xi_k^4+2\xi_k^2(\zeta_k^2+1)+\zeta_k^4-2\zeta_k^2+1} \tag{79}$$

The fundamental solution tractions are given by

$$t_j(P,P_0)=2\sum_{k=1}^{3}\{[\Re(Q_{kj}\boldsymbol{m})\alpha_1-\Re(Q_{kj}\frac{\boldsymbol{m}}{\mu_k})\alpha_2]\frac{X_k}{R_k^2} + [\Im(Q_{kj}\boldsymbol{m})\alpha_1-\Im(Q_{kj}\frac{\boldsymbol{m}}{\mu_k})\alpha_2]\frac{Y_k}{R_k^2}\} \tag{80}$$

where

$$\boldsymbol{m}=\{\mu_k^2,\mu_k,\mu_k\eta_k\}^T \tag{81}$$

The three independent fundamental solutions employed for the outlining of the formulation are obtained by setting in Eq. (67)

$$c_{ij}=\delta_{ij} \tag{82}$$

where δ_{ij} is the Kronecker delta. The kernels needed for the calculation of the internal stress field can be found by using Eqs. (47), (48) and (49) respectively. Their expressions are not given here for the sake of conciseness. In the common case of plies characterized by a generalized orthotropic law the expressions of the fundamental solutions simplify and reduce to the form given in References (Daví, 1996; Daví and Milazzo, 1997a, b; Daví, 1997; Daví and Milazzo, 1999).

V. BOUNDARY ELEMENT MODEL

Following the classical approach for solving integral equation models the solution of the formulation proposed is obtained by using the multidomain boundary element method (Banerjee and Butterfield, 1981). The numerical solution of the system given by the integral Eqs. (37) and (38) is obtained by discretizing the boundary $\Gamma_{\langle k\rangle}$ of each ply into n boundary elements and the domain $\Omega_{\langle k\rangle}$ into m internal cells with domain Ω_r (see Fig. 3). On each boundary element Γ_j the displacement function $\boldsymbol{v}$ and the

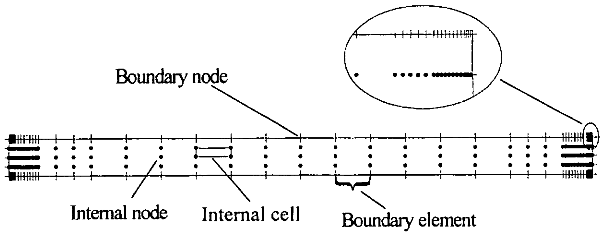

Fig. 3. Discretization scheme.

tractions $\boldsymbol{t}_v$ are expressed by using their nodal values $\delta_v^{(j)}$ and $\boldsymbol{p}_v^{(j)}$ through suitable shape functions $\mathcal{F}(s)$

$$\boldsymbol{v} = \mathcal{F}\delta_v^{(j)} \tag{83}$$

$$\boldsymbol{t}_v = \mathcal{F}\boldsymbol{p}_v^{(j)} \tag{84}$$

where the superscript (j) denotes quantities referred to the j-th element. With these assumptions the discretized version of Eq. (37) for any point P_i is

$$\boldsymbol{c}^*\boldsymbol{v}(P_i) + \sum_{j=1}^{n} \hat{\boldsymbol{H}}_{ij}\delta_v^{(j)} + \sum_{j=1}^{n} \boldsymbol{G}_{ij}\boldsymbol{p}_v^{(j)} = \sum_{r=1}^{m} \boldsymbol{B}_{ir} \tag{85}$$

where the influence matrices and the right-hand-side are defined by

$$\hat{\boldsymbol{H}}_{ij} = \int_{\Gamma_j} \boldsymbol{t}^*(P, P_i)\mathcal{F}(P)d\Gamma \tag{86}$$

$$\boldsymbol{G}_{ij} = -\int_{\Gamma_j} \boldsymbol{u}^*(P, P_i)\mathcal{F}(P)d\Gamma \tag{87}$$

$$\boldsymbol{B}_{ir} = -\int_{\Omega_r} \varepsilon^*(P, P_i)\boldsymbol{E}_{12}\boldsymbol{X}_3'\gamma d\Omega \tag{88}$$

Analogously, the displacement functions $\boldsymbol{u}$ and the tractions $\boldsymbol{t}_u$ on the boundary element Γ_j are given in terms of their boundary nodal values $\delta_u^{(j)}$ and $\boldsymbol{p}_u^{(j)}$ and one has

$$\boldsymbol{u} = \mathcal{F}\delta_u^{(j)} \tag{89}$$

$$\boldsymbol{t}_u = \mathcal{F}\boldsymbol{p}_u^{(j)} \tag{90}$$

Moreover let us assume that on the r-th cell the displacement function $\boldsymbol{v}$ can be interpolated as

$$\boldsymbol{v} = \mathcal{G}\boldsymbol{d}_v^{(r)} \tag{91}$$

where $\boldsymbol{d}_v^{(r)}$ is the vector collecting the cell nodal values of $\boldsymbol{v}$ and $\mathcal{G}(s_1, s_2)$ is a suitable matrix of shape functions. The discretized version of Eq. (38) is

$$\boldsymbol{c}^*\boldsymbol{u}(P_i) + \sum_{j=1}^{n} \hat{\boldsymbol{H}}_{ij}\delta_u^{(j)} + \sum_{j=1}^{n} \boldsymbol{G}_{ij}\boldsymbol{p}_u^{(j)}$$

$$= \sum_{j=1}^{n} \boldsymbol{J}_{ij}\delta_v^{(j)} + \sum_{r=1}^{m} \boldsymbol{W}_{ir}\boldsymbol{d}_v^{(r)} + \sum_{r=1}^{m} \boldsymbol{Y}_{ir} \tag{92}$$

where

$$\boldsymbol{J}_{ij} = -\int_{\Gamma_j} \boldsymbol{u}^*(P, P_i)\mathcal{D}_n\boldsymbol{Q}_1^T\mathcal{F}(P)d\Gamma \tag{93}$$

$$\boldsymbol{W}_{ir} = \int_{\Omega_r} \boldsymbol{u}^*(P, P_i)[\mathcal{D}^T\boldsymbol{Q}_1^T + \boldsymbol{Q}_1\mathcal{D}]\mathcal{G}(P)d\Omega \tag{94}$$

$$\boldsymbol{Y}_{ir} = \int_{\Omega_r} \boldsymbol{u}^*(P, P_i)\boldsymbol{Q}_2\boldsymbol{X}_3'\gamma d\Omega - \int_{\Omega_r} \varepsilon^*(P, P_i)\boldsymbol{Q}_1^T\boldsymbol{X}_1 k d\Omega \tag{95}$$

The discretized model governing the behavior of the k-th ply within the laminate is obtained by collocating Eqs. (85) and (92) at the boundary nodes and taking into account the relation between the cell nodal values of $\boldsymbol{v}$ and the boundary nodal values of $\boldsymbol{v}$ and $\boldsymbol{t}_v$, i.e., Eq. (85). By so doing the set of linear equations is obtained in the form

$$\boldsymbol{H}_{\langle k\rangle}\delta_{v\langle k\rangle} + \boldsymbol{G}_{\langle k\rangle}\boldsymbol{p}_{v\langle k\rangle} = \boldsymbol{B}_{\langle k\rangle} \tag{96}$$

$$\boldsymbol{H}_{\langle k\rangle}\delta_{u\langle k\rangle} + \boldsymbol{G}_{\langle k\rangle}\boldsymbol{p}_{u\langle k\rangle} = \boldsymbol{J}_{\langle k\rangle}\delta_{v\langle k\rangle} + \boldsymbol{W}_{\langle k\rangle}\boldsymbol{d}_{v\langle k\rangle} + \boldsymbol{Y}_{\langle k\rangle} \tag{97}$$

$$d_{v\langle k\rangle} = \overline{\boldsymbol{H}}_{\langle k\rangle}\delta_{v\langle k\rangle} + \overline{\boldsymbol{G}}_{\langle k\rangle}\boldsymbol{p}_{v\langle k\rangle} + \overline{\boldsymbol{B}}_{\langle k\rangle} \tag{98}$$

where $\delta_{v\langle k\rangle}$ and $\delta_{u\langle k\rangle}$ contain all the boundary nodal values of the displacements $\boldsymbol{v}$ and $\boldsymbol{u}$,$\boldsymbol{p}_{v\langle k\rangle}$ and $\boldsymbol{p}_{u\langle k\rangle}$ contain all the nodal values of the boundary tractions $\boldsymbol{t}_v$ and $\boldsymbol{t}_u$ and $\boldsymbol{d}_v$ is the vector of the nodal displacements $\boldsymbol{v}$. Once again the notation $\langle k\rangle$ denotes quantities referred to the k-th ply. Now the boundary element model for the whole laminate is deduced starting from the model obtained for the individual ply. The resolving system for the laminate problem is obtained by writing the integral equation model for all of the N plies of the laminate and imposing the interface continuity conditions and the external boundary conditions. The solution strategy of the discretized model is based on the uncoupling of the equations constituting the resolving system. Indeed, upon substituting Eq. (98) into Eq. (96), one obtains the laminate resolving system in the following form

$$\boldsymbol{H}_{\langle k\rangle}\delta_{v\langle k\rangle} + \boldsymbol{G}_{\langle k\rangle}\boldsymbol{p}_{v\langle k\rangle} = \boldsymbol{B}_{\langle k\rangle} \quad (k=1, 2, ..., N) \tag{99}$$

$$\boldsymbol{H}_{\langle k\rangle}\delta_{u\langle k\rangle} + \boldsymbol{G}_{\langle k\rangle}\boldsymbol{p}_{u\langle k\rangle}$$

$$= (\boldsymbol{J}_{\langle k\rangle} + \boldsymbol{W}_{\langle k\rangle}\overline{\boldsymbol{H}}_{\langle k\rangle})\delta_{v\langle k\rangle} + \boldsymbol{W}_{\langle k\rangle}\overline{\boldsymbol{G}}_{\langle k\rangle}\boldsymbol{p}_{v\langle k\rangle} + \boldsymbol{W}_{\langle k\rangle}\overline{\boldsymbol{B}}_{\langle k\rangle} + \boldsymbol{Y}_{\langle k\rangle}$$

$$(k=1, 2, ..., N) \tag{100}$$

Let us adopt a partition in Eqs. (99) and (100) in such a way that the generic vector $\boldsymbol{y}_{\langle k\rangle}$ can be written as

$$y^T_{\langle k\rangle} = \{y^{\Gamma_{e(k)}}_{\langle k\rangle}\ y^{\Gamma_{k-1}}_{\langle k\rangle}\ y^{\Gamma_k}_{\langle k\rangle}\} \quad (101)$$

where the vectors $y^{\Gamma_{k-1}}_{\langle k\rangle}$ and $y^{\Gamma_k}_{\langle k\rangle}$ collect the components of $y_{\langle k\rangle}$ associated with the nodes belonging to the interfaces Γ_{k-1} and Γ_k and the vector $y^{\Gamma_{e(k)}}_{\langle k\rangle}$ contains the components of $y_{\langle k\rangle}$ associated with the nodes lying on the external boundary $\Gamma_{e\langle k\rangle}$ (see Fig. 1 for the definition of ply interfaces and external boundary). As a consequence of this partition the interface continuity conditions in the discretized model are given by

$$\delta^{\Gamma_k}_{v\langle k\rangle} = \delta^{\Gamma_k}_{v\langle k+1\rangle} \quad (k=1, 2, ..., N-1) \quad (102)$$

$$\delta^{\Gamma_k}_{u\langle k\rangle} = \delta^{\Gamma_k}_{u\langle k+1\rangle} \quad (k=1, 2, ..., N-1) \quad (103)$$

$$p^{\Gamma_k}_{v\langle k\rangle} = -p^{\Gamma_k}_{v\langle k+1\rangle} \quad (k=1, 2, ..., N-1) \quad (104)$$

$$p^{\Gamma_k}_{u\langle k\rangle} = -p^{\Gamma_k}_{u\langle k+1\rangle} \quad (k=1, 2, ..., N-1) \quad (105)$$

and the external boundary conditions are expressed by

$$p^{\Gamma_{e(k)}}_{v\langle k\rangle} = \mathbf{0} \quad (k=1, 2, ..., N) \quad (106)$$

$$p^{\Gamma_{e(k)}}_{u\langle k\rangle} = \mathbf{0} \quad (k=1, 2, ..., N) \quad (107)$$

The system of Eqs. (99) and (100), together with the interface continuity conditions, Eqs. (102)-(105), and external boundary conditions, Eqs. (106) and (107), evidences the boundary nature of the model due to the involvement of boundary unknowns only. Moreover the mathematical structure of the resolving system allows one to uncouple the two Eqs. (99) and (100) with their relative interface continuity and boundary conditions. Therefore the model solution is obtained by solving first for v and p_v and then, upon substituting in Eq. (100), by solving for u and p_u. Once the boundary solution is known in terms of displacement functions v and u and tractions p_v and p_u the stress field is calculated in a pointwise fashion by introducing the discretization in Eqs.(43), (44), (45), and (46).

VI. NUMERICAL REMARKS AND APPLICATIONS

The boundary integral equation approach developed in this paper was implemented in the computer code "DM-COMP" to test the soundness of the formulation and to perform laminate analyses. In the computations straight boundary elements with linear interpolation of the unknown data were employed. Four-node quadrilateral isoparametric elements were used to discretize the domain. The influence coefficients were computed through the gaussian quadrature formulas and an adaptive integration scheme was used to account for the kernel singular behavior and set correctly the order of the integration formula employed (Daví, 1989). The code handled laminates with general lay-up and section geometry exploiting possible symmetries of the system with respect to the coordinate axes. Moreover it allowed the calculation of displacements and stresses at internal points through the discretized form of their integral representations. Also, in this case, gaussian quadrature formulas were employed in the calculation of the relative influence coefficients. Some applications are presented to demonstrate the accuracy, the effectiveness and the robustness of the method. The results presented in this paper focus on the interlaminar stresses whose realistic prediction is a primary concern in the design and analysis of composite la-minates. Again, many results relative to interlaminar stress distributions are given in the literature and can be used, for comparison, to point out the features and the advantages of the present method with respect to other approaches. The first analysis was performed on a symmetric laminate with the [45/-45/0/90]$_S$ lay up subjected to axial extension. Due to the structural symmetries only a quarter of the laminate was considered in the computations. Each lamina has the material properties given in the first column (material # 1) of Table 1 where the subscripts L, T and S refer to the along fiber, thickness and width directions. Each individual ply was discretized by using 44 boundary elements. In the case of axial, bending and twisting loading the right-hand-side of Eq. (85) can be calculated after the transformation of the domain integrals into boundary integrals according to the particular solution tecnique discussed in References (Daví, 1996; Daví and Milazzo, 1997; 1999) and not presented here for the sake of conciseness. The results of this analysis are given in Fig. 4 and Fig. 5 where a comparison with the finite element results of Wang and Crossmann (1977) is shown. The good agreement between the present solution and the finite element solution is highlighted. Moreover, we should note the ability of the present solution to couple the accurate description of the high

Table 1. Ply properties.

Properties	Graphite/Epoxy (material #1)	IM6/3501-6 (material #2)
E_{LL} [GPa]	137.9	170.8
E_{TT}=E_{SS} [GPa]	14.5	9.6
G_{LT}=G_{LS}=G_{TS} [GPa]	5.9	6.2
ν_{LT}=ν_{LS}=ν_{TS}	0.21	0.329
Ply thickness [mm]	12.5	0.125
Ply width [mm]	200	17.3

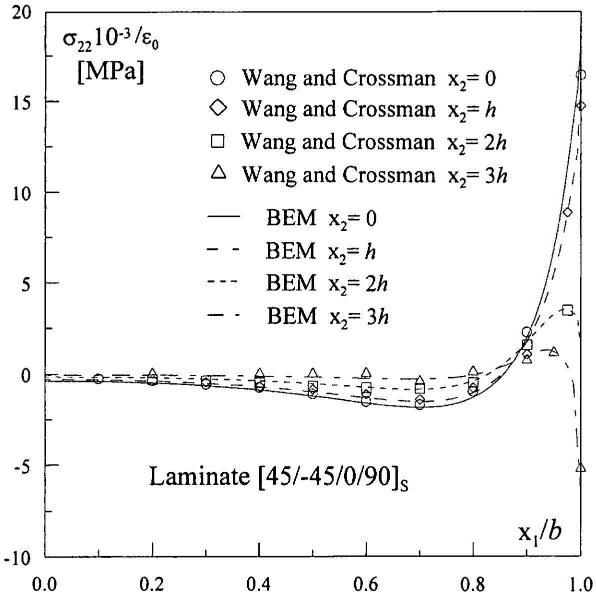

Fig. 4. Interlaminar stress σ_{22} for the $[45/-45/0/90]_S$ laminate under axial extension.

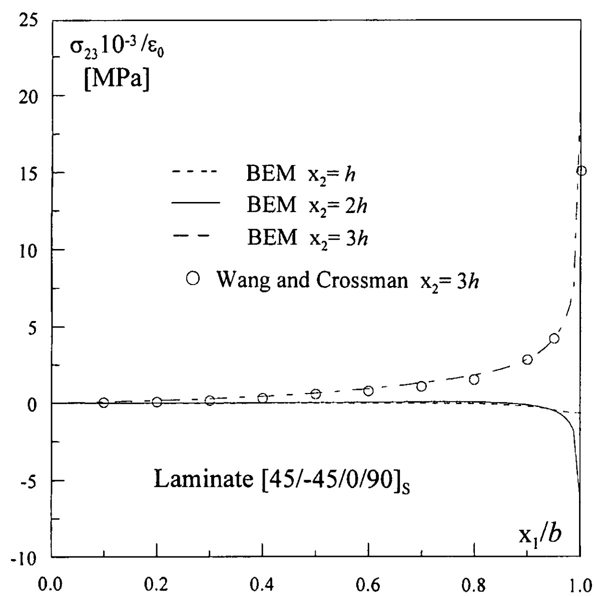

Fig. 5. Interlaminar stress σ_{23} for the $[45/-45/0/90]_S$ laminate under axial extension.

stress gradient of the interlaminar stresses in the free edge region and the computational advantages linked to the coarse boundary mesh needed for the solution. More complex laminate configurations are examined for bending and twisting loadings. For both these loading condition a symmetric $[+30/-30/+30/-30/90]_S$ laminate and an unsymmetric$[(+30)_2/(-30)_2/(+30)_2/(-30)_2/(90)_2]$ laminate are considered. No symmetry consideration was taken into account for this analysis and a 44-boundary element discretization was used. Once again, the particular solution technique

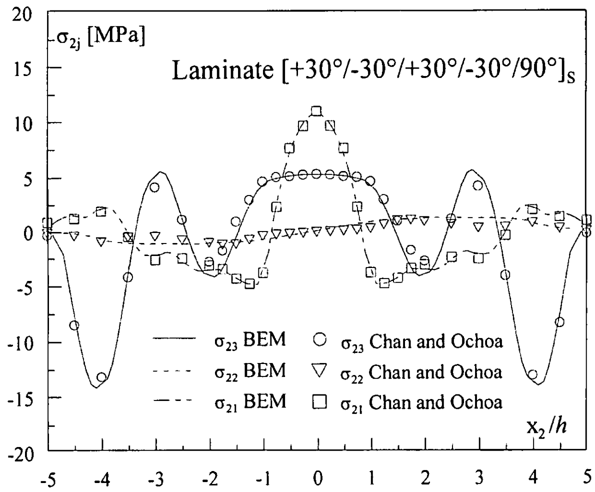

Fig. 6. Through-thickness interlaminar stress distributions at a distance h from the free edge for the $[+30/-30/+30/-30/90]_S$ laminate under bending.

was used to calculate the system right-hand-side through boundary integral computations. The material properties are given in the second column of Table 1 (material # 2). The interlaminar stress patterns along the laminate thickness at a distance from the free edge equal to the ply thickness are shown in Figs. 6, 7, 8 and 9. These interlaminar stresses were calculated starting from the boundary solution by means of the stress integral representation given by Eqs. (43) and (44). Again, the present results are compared with those obtained by using the finite element method (Chan and Ochoa, 1987; 1990) and the comparison demonstrates the effectiveness and the accuracy of the present method. We also evidence the soundness of the integral equation approach to deal with the analysis of composite laminates having complex lay-ups and various loading conditions. The results presented for the shear/bending are relative to the two classical $[0/90]_S$ and $[90/0]_S$ cross-ply configurations under a loading in the x_2x_3 plane. Fig. 10 and Fig. 11 show the interlaminar stress distributions along the semi-chord (x_1>0) at the top interface of the laminate. Due to the absence of the in-plane shear stress σ_{31}, the interlaminar stresses σ_{22} and σ_{23} exhibit a complete symmetry about the vertical middle plane x_2x_3, whereas the interlaminar stress σ_{21} has a complete antisymmetry. These results were obtained without exploiting any structural symmetry and they are compared with the solution of a boundary integral model for cross-ply laminates (Daví, 1997). The comparison highlights a good agreement between the two solutions and this confirms the features of the present method. This is also evidenced by the steep stress gradient near the free edge that suggests a singularity in the stress field at this point where such a behavior is expected. Figs. 12 and 13 show the through-thickness variations of the interlaminar stress

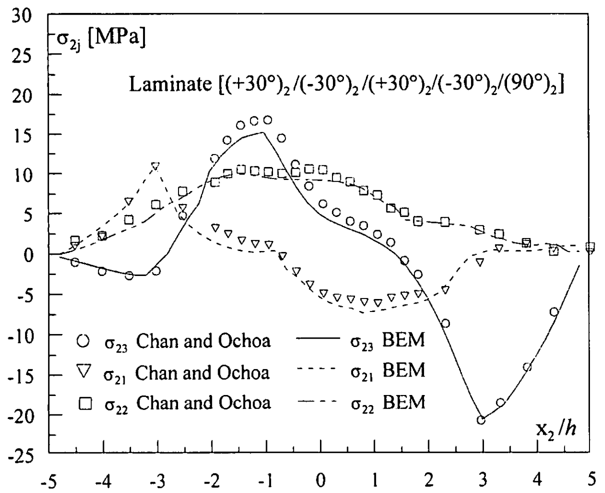

Fig. 7. Through-thickness interlaminar stress distributions at a distance h from the free edge for the [(+30)$_2$/(-30)$_2$/(+30)$_2$/(-30)$_2$/(90)$_2$] laminate under bending.

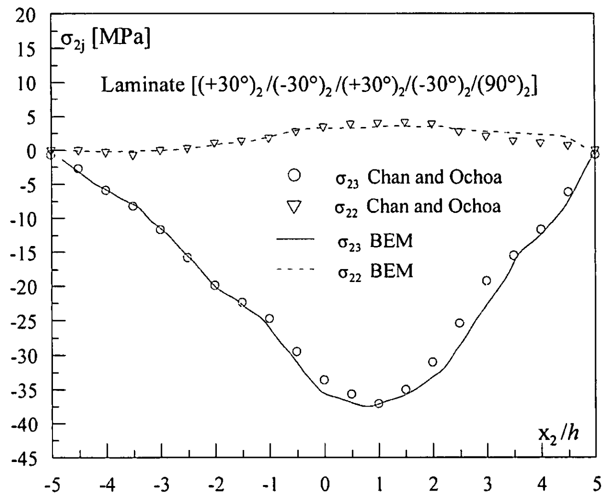

Fig. 9. Through-thickness interlaminar stress distributions at a distance h from the free edge for the [(+30)$_2$/(-30)$_2$/(+30)$_2$/(-30)$_2$/(90)$_2$] laminate under tonsion.

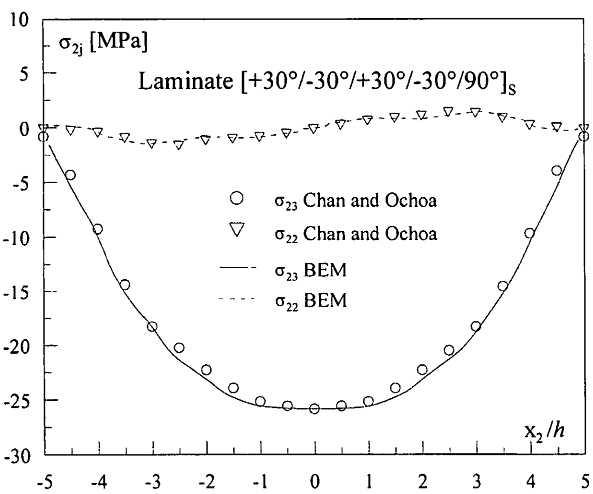

Fig. 8. Through-thickness interlaminar stress distributions at a distance h from the free edge for the [+30/-30/+30/-30/90]$_S$ laminate under torsion.

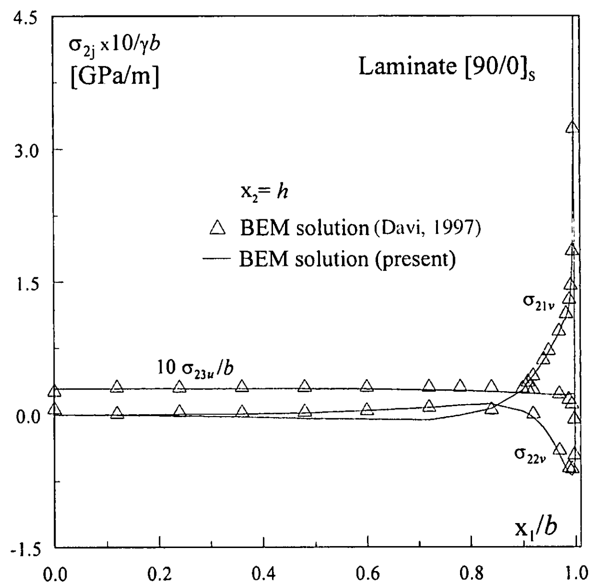

Fig. 10. Interlaminar stress distribution for the [90/0]$_S$ cross-ply laminate under shear/bending loading.

σ_{32} and of the in-plane stress σ_{33} for the two cross-ply laminates considered. The through-thickness stress distributions are plotted for two different values of the abscissa x_1. The stress patterns are practically coincident, as expected, due to the nature of the cross-ply laminate elastic response. In conclusion, for all of the configurations examined, the agreement of the present results with those available in the literature is excellent and the soundness, the reliability and the robustness of the proposed boundary integral equation method are therefore proved.

VII. CONCLUSIONS

An alternative and comprehensive analysis tool for composite laminates is presented. It is based on an original boundary integral representation from which the boundary integral equations governing the problem are deduced. The fundamental solutions employed are given in closed form and the laminate model is obtained by coupling the integral equations for each ply and the interface continuity conditions. The model is numerically solved through the multidomain boundary element method. The analyses performed and the results obtained show the features of the approach, which is very well suited for composite laminate structural analysis where the correct and accurate prediction of the interlaminar stresses is crucial in the assessment of laminate

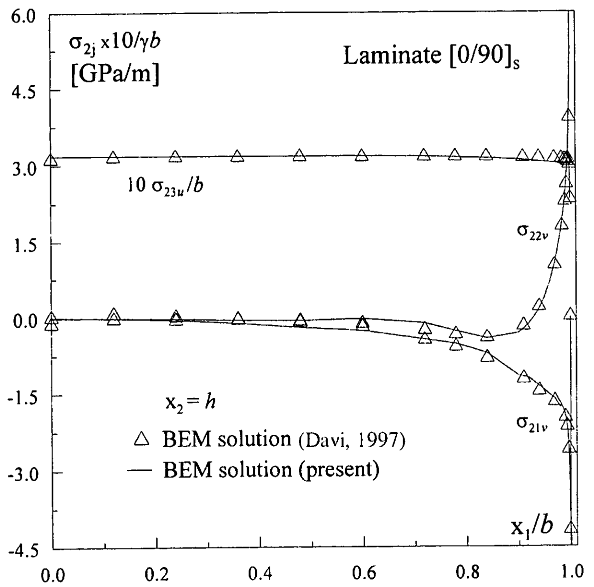

Fig. 11. Interlaminar stress distribution for the $[0/90]_S$ cross-ply laminate under shear/bending loading.

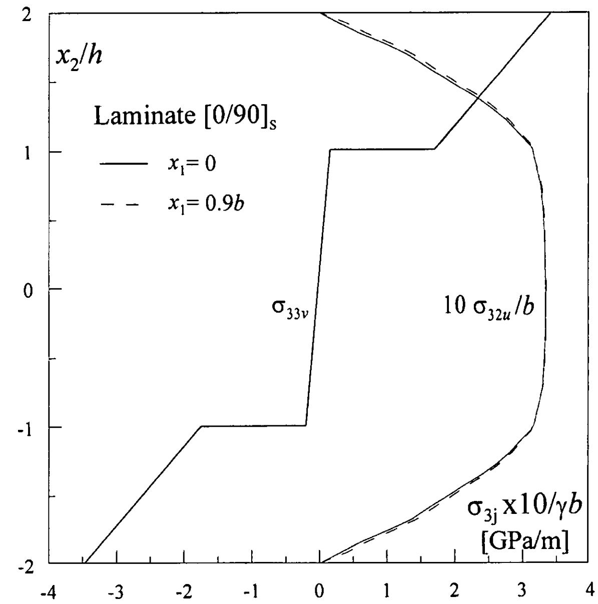

Fig. 13. Through-thickness σ_{32} and σ_{33} distribution for the $[0/90]_S$ cross-ply laminate under shear/bending loading (z=0).

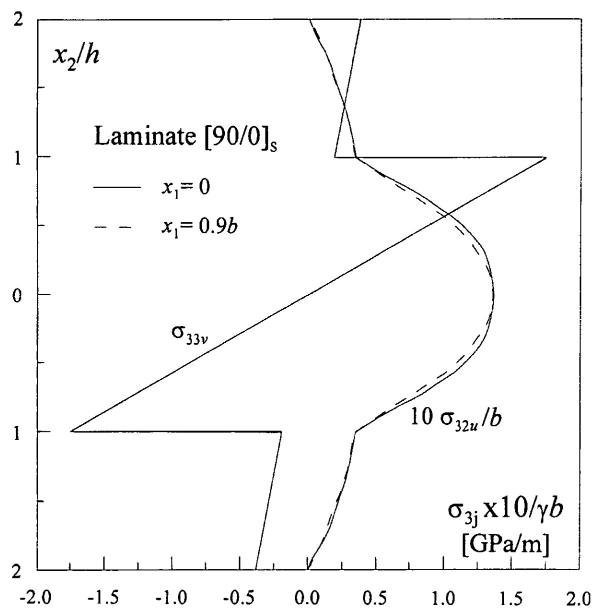

Fig. 12. Through-thickness σ_{32} and σ_{33} distribution for the $[90/0]_S$ cross-ply laminate under shear/bending loading (z=0).

failure. In fact, the integral equation method proposed allows a pointwise description of the elastic response and thus the boundary integral solution permits one to calculate, reliably and accurately, the steep stress gradients arising near the laminate free edges. Again, the boundary integral approach handles, naturally, interface problems due to its characteristic of involving boundary unknowns only. In the case of composite laminate analysis this is very important because one can implement different effective modelizations of the interface behavior without significant overload in the formulation. Finally, the boundary nature of the formulation provides a reduction in the problem dimensionality and therefore the present method evidences meaningful computational advantages with respect to the more common field methods. In conclusion, the boundary integral equation method here proposed gives accurate solutions with reduced computational effort. It therefore represents a valid and useful approach to composite laminate analysis and design.

NOMENCLATURE

$\mathcal{D}$, $\mathcal{D}_z$, $\boldsymbol{I}_z$	strain operators
$\mathcal{D}_n$	boundary traction operator
$\boldsymbol{E}_{1i}$, $\boldsymbol{Q}_i$	elasticity matrices
$\boldsymbol{E}_{ij}$	elasticity stiffness coefficients
f_j	fundamental solution body forces
$\mathcal{F}$, $\mathcal{G}$	shape function matrix
m, n	number of internal cells and boundary elements
N	number of laminate plies
$\boldsymbol{p}$	nodal tractions
$\boldsymbol{s}$	vector of displacements
$\boldsymbol{t}$	boundary tractions
$\boldsymbol{u}$, $\boldsymbol{v}$	vector of unknown displacement functions
u_j, t_j	fundamental solution displacements and tractions
x_1, x_2, $x_3 \equiv z$	coordinate system for the laminate
α_1, α_2	boundary normal direction cosines

β_{ij}	ply modified compliances
γ	shear/bending loading parameter
$\Gamma_{\langle k \rangle}$	ply section boundary
δ, d	nodal displacements
ε	vector of strains
ε_{ij}	strain components
ε_j, σ_j, τ_j	fundamental solution strain and stress vectors
$\Omega_{\langle k \rangle}$	ply section domain
σ, τ	stress vectors
σ_{ij}	stress components

Subscripts and Superscripts

u	along laminate axis constant components
v	along laminate axis linearly varying components
$\langle k \rangle$	k-th ply quantities
(j), (r)	j-th boundary element and r-th cell parameters

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複材積層板積分方程式分析法

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摘 要

本文提出以積分方程式爲基礎的方法，來決定複材積層板在承受軸力、彎矩、剪力／彎矩、扭矩時的彈性反應。這個行爲的積分方程是直接從互易定理推出，此互易定理是由梁形結構物廣義平面應變的異向性問題之基本解而來。在考慮位移與應力沿界面的連續性與外圍邊界條件後，可以用多領域邊界元素法以數值方法解出。所得線性代數式的分解系統可以提供以複材積層板中每一層邊界上的位移與曳引力來表示問題的解。一旦邊界的彈性反應可以確定，複材積層板中任一點的位移與應力就可以用適當的邊界積分表示式來加以計算。根據點適用的推導公式，可以推廣到斷面外形與不同組織的情形。我們提出一些應用以驗證此法的正確性與有效性。

關鍵詞：邊界積分方程式，複材積層板。