Journal of the Chinese Institute of Engineers, Vol. 22, No. 6, pp. 709-720 (1999)

COMPLEX VARIABLES BIE AND BEM FOR A PLANE DOUBLY PERIODIC SYSTEM OF FLAWS

Alexander M. Linkov* and Vadim F. Koshelev
Institute for Problems of Mechanical Engineering
Russian Academy of Sciences
61, Bolshoy Prosp., St. Petersburg, 199178, Russia

Key Words: complex variable boundary equations, doubly periodic flaws, elasticity.

ABSTRACT

New complex variable, singular and hypersingular, boundary integral equations (CV-BIE) are derived for doubly periodic plane elasticity problems. They refer to systems of blocks (grains), inclusions, holes and cracks. Their forms, convenient when adjusting conventional programs for non-periodic systems to periodic problems, are suggested. Simple formulae are presented to calculate effective compliances in complex variables.

Numerical implementation of the derived complex variable hypersingular (CVH) BIE in the mentioned forms is carried out by appropriately adjusting a program of CVH-BEM, previously developed for non-periodic problems. It is used to check accuracy and to obtain new results for doubly periodic systems of cracks. Stress intensity factors (SIFs) and effective compliances are calculated for straight cracks in square and triangular lattices to compare them with published results. They show agreement within the accuracy reached by other authors. New data on SIFs and effective compliances for doubly periodic systems of angular and curvilinear, strongly interacting cracks, illustrate abilities of the method.

I. INTRODUCTION

Doubly periodic elasticity problems (Fig. 1), being of interest for material science, fracture and rock mechanics, have been already tackled by using complex variables (Filshtinski, 1974; Grigoliuk and Filshtinski, 1970; Ioakimidis and Theocaris, 1978; Koiter, 1960; Linkov, 1976; Panasiuk et al., 1976). Koiter (Koiter, 1960) was the first to derive complex variable (CV) boundary integral equations (BIE) for such problems. He started from his theorems (Koiter, 1959) and followed the way suggested by Muskhelishvili (1934; 1975) for non-periodic problems. Koiter's theorems and CV-BIE referred to *closed* contours; they were of Friedholm's type. Equations for *open* arcs were first obtained for a particular case of *straight* cracks along a period (Filshtinski, 1974). The author used integral representation of holomorphic functions by Kolosov-Muskhelishvili (K-M). This approach was modified to include straight cracks not parallel to periods in (Panasiuk *et al.*, 1976). CV-BIE for a general case of *arbitrary* doubly periodic systems of cracks were derived and studied in (Linkov, 1976). The author proved a holomorphicity theorem for doubly periodic open arcs and followed the way previously used by

*Correspondence addressee

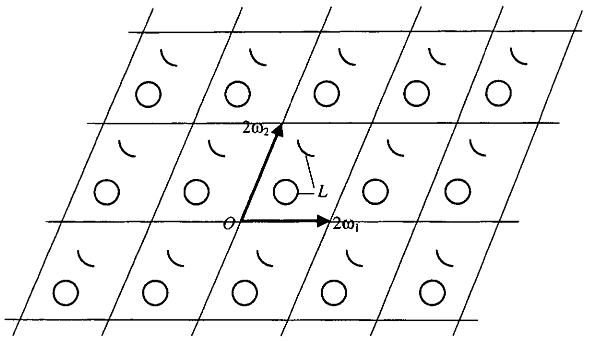

Fig. 1. Doubly periodic, with the periods $2\omega_1$, $2\omega_2$, system of cracks and holes.

him to derive general CV-BIE for non-periodic arbitrary cracks (Linkov, 1974).

The complex Eqs. of the papers (Grigoliuk and Filshtinski, 1970; Ioakimidis and Theocaris, 1978; Koiter, 1960; Linkov, 1976; Panasiuk *et al.*, 1976) were singular. In essence, they contained either derivative of displacement discontinuity and tractions (Filshtinski, 1974; Ioakimidis and Theocaris, 1978; Panasiuk *et al.*, 1976) or displacement discontinuity itself and the resultant force (Linkov, 1976). For numerical calculations the authors of (Filshtinski, 1974; Panasiuk *et al.*, 1976) used mechanical quadrature formulae based on Chebyshev's polynomials. Meanwhile, recently it has been stated (Linkov and Mogilevskaya, 1991; Linkov *et al.*, 1994; Linkov and Mogilevskaya, 1994; 1998; Mogilevskaya, 1996) that complex variables hypersingular (CVH) equations and *complex variable BEM* suggest significant virtues. Consequently it seems reasonable to extend these techniques to doubly periodic problems. We have made the first steps in this direction in our brief papers (Koshelev and Linkov, 1998; 1999). This paper aims to present a comprehensive study.

II. CV-BIE FOR DOUBLY PERIODIC SYSTEMS OF CRACKS AND/OR HOLES

1. Prerequisites

Consider a doubly periodic system of cracks and/or holes with the contour L in the main cell (Fig. 1). The contour L has Helder's continuous derivative. It consists of p open arcs (cracks) L_j ($j=1, \ldots, p$) and m closed contours L_j ($j=p+1, \ldots, p+m$). Contours in other cells are congruent to the contour L. Boundary conditions there doubly periodically reproduce those at L. The origin $z=0$ is located in the main cell outside the holes, if they are present; it does not belong to L. The periods $2\omega_1$, $2\omega_2$ are not collinear, that is, Im $(\overline{\omega}_1\omega_2)\neq 0$. For certainty, we assume that the direction of the period $2\omega_2$ is obtained from the direction $2\omega_1$ by rotating the latter counter-clockwise with the angle less than π. In this case, $\mathrm{Im}(\overline{\omega}_1\omega_2)=S/4$, where S is the area of a cell. Points are given by complex coordinates $z=x+iy$; i is the imaginary unit.

We assume that each individual contour L_j is loaded by a self-balanced load. This implies

$$\int_{L_j} \Delta\sigma d\tau = 0 \qquad (j=1, \ldots, p+m) \tag{1}$$

where $\Delta\sigma=\sigma^+-\sigma^-$; for certainty the sign "plus" ("minus") marks the limit from the left (right) of the traveling path; for holes, we assume that their contours are traveled clockwise; consequently, for them $\sigma^+=\sigma$, $\sigma^-=0$; $\sigma(z)=\sigma_n+i\sigma_t$ is the traction vector in the local coordinates (n,t), with the normal $\boldsymbol{n}$ directed to the right of the traveling path and the tangent $\boldsymbol{t}$ directed along the path.

In a case when the contour consists only of cracks, the CV-BIE and conditions for cyclic constants were derived and studied in (Linkov, 1976), where it was also noted that the results were easily extended to the case of holes. For the problem considered we have

$$K_t\Delta\varphi+G_t\Delta f-(t\cdot 2\mathrm{Re}C_\alpha+\overline{t}\,C_\beta)=\frac{1}{2}(f^++f^-) \tag{2}$$

$$\alpha_2\omega_1-\alpha_1\omega_2=\frac{1}{4}\int_L \Delta\varphi d\tau \tag{3}$$

$$2i\mathrm{Im}(\alpha_2\overline{\omega}_1-\alpha_1\overline{\omega}_2)$$

$$=\frac{1}{4}\int_L \overline{\Delta f}d\tau+\frac{1}{2i}\mathrm{Im}\int_L \Delta\varphi d\tau+\overline{\gamma}_1\omega_2-\overline{\gamma}_2\omega_1 \tag{4}$$

where

$$K_t\Delta\varphi\equiv\frac{1}{2\pi i}\int_L \{2\Delta\varphi[\varsigma(\tau-t)-\varsigma(\tau)]d\tau$$
$$-\Delta\varphi dk_{d1}(\tau,t)-\Delta\overline{\varphi}(\tau,t)dk_{d2}\} \tag{5}$$

$$G_t\Delta f\equiv\frac{1}{2\pi i}\int_L \{-\Delta f[\varsigma(\tau-t)-\varsigma(\tau)]d\tau+\Delta f dk_{d1}(\tau,t)\} \tag{6}$$

$$dk_{d1}(\tau,t)\equiv[\varsigma(\tau-t)-\varsigma(\tau)]d\tau-[\overline{\varsigma(\tau-t)}-\overline{\varsigma(\tau)}]d\overline{\tau} \tag{7}$$

$$dk_{d2}(\tau,t)\equiv d[(\tau-t)\overline{\varsigma(\tau-t)}-\tau\overline{\varsigma(\tau)}]+\overline{Q(\tau-t)}d\overline{\tau} \tag{8}$$

$\zeta(z)$ is the dzeta-function by Weierstrasse; $Q(z)$ is the function by Natanzon (1935); these functions are defined by the series

$$\varsigma(z)=\frac{1}{z}+\sum{}'\left(\frac{1}{z-w}+\frac{1}{w}+\frac{z}{w^2}\right),$$

$$Q(z)=\sum{}'\left[\frac{\overline{w}}{(z-w)^2}-\frac{\overline{w}}{w^2}-\frac{2z\overline{w}}{w^3}\right] \tag{9}$$

$w=j_1\cdot 2\omega_1+j_2\cdot 2\omega_2$; herein and further a sum marked with a prime is taken over all integer j_1 and j_2 (positive, negative and zero) but $j_1=j_2=0$; the overbar denotes complex conjugation; the function $\zeta(z)$ has cyclic constants $2\eta_k$ along the periods $2\omega_k$; the function $Q(z)$ has constants $2\lambda_k$ (k=1, 2); these constants are given by formulae $2\eta_k=2\varsigma(\omega_k)$, $2\lambda_k=2\overline{\omega}_k\varsigma'(\omega_k)+2Q(\omega_k)$; $\Delta\varphi=\varphi^+-\varphi^-$ is the discontinuity of the K-M function $\varphi(z)$; $\Delta f=f^+-f^-$ is the discontinuity of the function $f(z)$ which is proportional with the multiplier i to the resultant force (for brevity we will term $f(z)$ itself the resultant force); for holes we have $\varphi^+=\varphi$, $\varphi^-=0$, $f^+=f$, $f^-=0$,

$$f^{\pm}(t)=\int_{a_j}^{t}\sigma^{\pm}d\tau+C_j^{\pm}(t)\quad (j=1,\ ...,\ p+m)$$

a_j is start point at the contour L_j; for holes these points are fixed arbitrarily; $C^{\pm}(t)=C^{\pm}_j$ is a piece-wise constant function; for cracks we have $C_j^+=C_j^-=C_j$ (j=1, ..., p); for holes $C_j^+=C_j$, $C_j^-=0$; the constant C_j presents the value of the resultant force $f(z)$ at start point a_j (j=1, ..., $p+m$);

$$C_\alpha=\frac{2}{\pi i}(\alpha_1\eta_2-\alpha_2\eta_1),\quad C_\beta=\frac{2}{\pi i}(\beta_1\eta_2-\beta_2\eta_1) \qquad (10)$$

$2\alpha_k$ is the cyclic constant of the function $\varphi(z)$ along the period $2\omega_k$;

$$2\beta_k=-\frac{4}{\pi i}\lambda_k(\alpha_1\omega_2-\alpha_2\omega_1)+2(\overline{\gamma}_k-\overline{\alpha}_k)+2\overline{\omega}_kC_\alpha \quad (k=1,\ 2) \qquad (11)$$

$2\gamma_k$ is the cyclic constant of the resultant force $f(z)$ along the period $2\omega_k$; it is expressed through three prescribed average stresses s_{xx}, s_{yy}, s_{xy} in the plane by formulae (Koiter, 1960)

$$2\gamma_k=(s_{xx}+s_{yy})\omega_k+(s_{yy}-s_{xx}-2is_{xy})\overline{\omega}_k \qquad (12)$$

given the total momentum applied to L is zero, that is Muskhelishvili (1975)

$$\mathrm{Re}\int_L\Delta f d\overline{\tau}=0 \qquad (13)$$

We will assume that (13) holds. Then (4) implies $\mathrm{Re}(\gamma_1\overline{\omega}_2-\gamma_2\overline{\omega}_1)=0$.

For Helder continuous tractions $\sigma^{\pm}$ satisfying (1), (13), a solution of the singular BIE (2) under conditions (3), (4) exists (Linkov, 1976) in the Muskhelishvili class h_{2p} (Maskhelishvili, 1953). After the system (2)-(4) is solved, we can find the K-M functions $\varphi(z)$, $\psi(z)$ by using their integral representations (Koiter, 1960; Linkov, 1976):

$$\varphi(z)=\frac{1}{2\pi i}\int_L\Delta\varphi[\varsigma(\tau-z)-\varsigma(\tau)]d\tau-\frac{2}{\pi i}(\alpha_1\eta_2-\alpha_2\eta_1)z \qquad (14)$$

$$\psi(z)=\frac{1}{2\pi i}\int_L\{(\Delta\overline{f}-\Delta\overline{\varphi}-\overline{\tau}\Delta\varphi')[\varsigma(\tau-z)-\varsigma(\tau)]+\Delta\varphi Q(\tau-z)\}d\tau-\frac{2}{\pi i}(\beta_1\eta_2-\beta_2\eta_1)z \qquad (15)$$

Substitution of (14), (15) into K-M formulae

$$2\mu u(z)=\chi\varphi(z)-z\overline{\varphi'(z)}-\overline{\psi(z)},\quad f(z)=\varphi(z)+z\overline{\varphi'(z)}+\overline{\psi(z)} \qquad (16)$$

gives displacements $u(z)=u_x+iu_y$ and the resultant $f(z)$ force in a plane; stresses are found by using the formula

$$\sigma(z)=\sigma_n+i\sigma_t=df/dz \qquad (17)$$

where $dz=dt$ is taken along the area at which a traction is found; as usual the normal $\boldsymbol{n}$ is directed to the left of dt. In (16) μ is the shear modulus; χ is the Muskhelishvili's parameter: $\chi=3-4\nu$ for plane strain; $\chi=(3-\nu)/(1+\nu)$ for plane stress; ν is the Poisson's ratio.

2. CV-BIE with Satisfied Conditions for Cyclic Constants

We may use (3), (4) to express cyclic constants $2\alpha_k(k=1,2)$ through the values in the r.h.s. of (3), (4). Inserting the result into (10) and using (12), we may write the constants C_α and C_β entering (2) as

$$C_\alpha=\frac{2\mu}{\chi+1}A_\alpha-\frac{1}{4}(s_{xx}+s_{yy}) \qquad (18)$$

$$C_\beta=\frac{2\mu}{\chi+1}A_\beta-\frac{1}{2}(s_{yy}-s_{xx}+2is_{xy}) \qquad (19)$$

where

$$A_\alpha=\frac{\chi+1}{2\mu}\{-\frac{1}{2\pi i}\frac{\eta_1}{\omega_1}\int_L\Delta\varphi d\tau-\frac{1}{2S}[\mathrm{Im}\int_L(\Delta\varphi-\frac{1}{2}\Delta f)d\overline{\tau}+i\frac{\overline{\omega}_1}{\omega_1}\int_L\Delta\varphi d\tau]\} \qquad (20)$$

$$A_\beta=\frac{\chi+1}{2\mu}\{-\frac{1}{\pi}\frac{\eta_1}{\omega_1}[\mathrm{Im}\int_L\Delta\varphi d\overline{\tau}-\frac{i}{2}\int_L\Delta\overline{f}d\tau]+\frac{1}{2\pi i}\frac{\overline{\omega}_1}{\omega_1}(\frac{\eta_1}{\omega_1}-\frac{\lambda_1}{\overline{\omega}_1})\int_L\Delta\varphi d\tau+\frac{\overline{\omega}_1}{\omega_1}\frac{1}{S}\mathrm{Im}[\int_L(\Delta\varphi-\frac{1}{2}\Delta f)d\overline{\tau}-(\frac{\overline{\omega}_1}{|\omega_1|})^2\int_L\Delta\varphi d\tau]\} \qquad (21)$$

These formulae are slightly simplified if one takes the x-axis along the period $2\omega_1$ (in this case $\omega_1=\overline{\omega}_1=|\omega_1|$).

Equation (2) with the constants C_α, C_β defined by (18), (19) does not need additional conditions (3), (4). The latter are satisfied; as shown below they serve to find effective properties of a plane with cracks and holes. In all the following, we will presume that (2) contains C_α, C_β defined as (18)-(21).

3. Singular CV-BIE with Respect to Displacement Discontinuities

From (16) it follows that the K-M function $\varphi(z)$ may be expressed in terms of displacements $u(z)=u_x+iu_y$ and the resultant force $f(z)$:

$$\varphi(z)=[u(z)+f(z)/(2\mu)]2\mu h \qquad (22)$$

where $h=1/(\chi+1)$.

At the contour L we have from (22)

$$\Delta\varphi=[\Delta u+\Delta f/(2\mu)]2\mu h \qquad (23)$$

where $\Delta u=u^+-u^-$ is the displacement discontinuity; for holes we assume $u^+=u$, $u^-=0$.

Inserting (23) into (2), (18), (19) and dividing by $2\mu h$, we obtain a singular equation with physically significant values Δu and Δf:

$$2\mu hK_t\Delta u + hK_t\Delta f + G_t\Delta f - (t\bullet 2\mathrm{Re}A_\alpha + \overline{t}\,\overline{A}_\beta)2\mu h + f^\infty(t)$$
$$=\frac{1}{2}(f^+ + f^-) \qquad (24)$$

where $f^\infty(z)$ is the resultant force corresponding to average stresses in a plane without cracks and holes:

$$f^\infty(z)=\frac{1}{2}[(s_{xx}+s_{yy})z+(s_{yy}-s_{xx}-is_{xy})\overline{z}] \qquad (25)$$

(note that, as it could be expected, for $\Delta u=0$, $\Delta f=0$, (24) yields $f^+(t)=f^-(t)=f^\infty(t)$); the operators K_t and G_t are defined by (5)-(8); the constants A_α, A_β are given by (20), (21) with $\Delta\varphi$ defined as (23).

Differentiation of (24) over t accompanied with integration by parts using (17) yields

$$2\mu hS_t\Delta u' + hS_t\Delta\sigma + R_t\Delta\sigma - (2\mathrm{Re}A_\alpha + \frac{\partial\overline{t}}{\partial t}\overline{A}_\beta)2\mu h + \sigma^\infty(t)$$
$$=\frac{1}{2}(\sigma^+ + \sigma^-) \qquad (26)$$

where $\Delta u'=d\Delta u/dt$; $\sigma^\infty(z)$ is traction generated by average stresses in a plane without cracks and holes:

$$\sigma^\infty(z)=\frac{1}{2}[(s_{xx}+s_{yy})+\frac{\partial\overline{z}}{\partial z}(s_{yy}-s_{xx}-2is_{xy})] \qquad (27)$$

the operators S_t and R_t are defined as

$$S_t\Delta u' \equiv \frac{1}{2\pi i}\int_L [2\Delta u'\varsigma(\tau-t)d\tau + \Delta u'\frac{\partial}{\partial t}k_{d1}(\tau,t)d\tau$$
$$+\overline{\Delta u'}\frac{\partial}{\partial t}k_{d2}(\tau,t)d\overline{\tau}] \qquad (28)$$

$$R_t\Delta\sigma \equiv \frac{1}{2\pi i}\int_L [-\Delta\sigma\varsigma(\tau-t)d\tau - \Delta\sigma\frac{\partial}{\partial t}k_{d1}(\tau,t)d\tau] \qquad (29)$$

$$\frac{\partial}{\partial t}k_{d1}(\tau,t) \equiv -\varsigma(\tau-t)+\frac{\partial\overline{t}}{\partial t}\overline{\varsigma(\tau-t)} \qquad (30)$$

$$\frac{\partial}{\partial t}k_{d2}(\tau,t) \equiv \frac{\partial}{\partial t}[(\tau-t)\overline{\varsigma(\tau-t)}] - \frac{\partial\overline{t}}{\partial t}\overline{Q(\tau-t)} \qquad (31)$$

A solution of (26) should satisfy conditions

$$\int_{L_j}\Delta u'd\tau = 0 \qquad (32)$$

which express that displacements are single-valued functions of coordinates.

Differentiation accompanied with integration by parts is also applied to (14), (15) resulting in K-M functions $\Phi(z)=\varphi'(z)$ and $\Psi(z)=\psi'(z)$. We shall not write down its quite obvious result, as well as the result of integration by parts in integrals entering (20), (21).

4. Hypersingular CV-BIE

A complex variable, hypersingular boundary, integral Eq. (CVH-BIE) with respect to Δu is obtained by either integration by parts terms containing $\Delta u'$ in (26) or differentiation of (24) accompanied with integration by parts only terms containing Δf. (These transformations involving hypersingular integrals are substantiated by regularization formulae (see e.g. Linkov and Mogilevskaya (1994; 1998)). In both ways we arrive at the CVH-BIE

$$2\mu hH_t\Delta u + hS_t\Delta\sigma + R_t\Delta\sigma - (2\mathrm{Re}A_\alpha + \frac{\partial\overline{t}}{\partial t}\overline{A}_\beta)2\mu h + \sigma^\infty(t)$$
$$=\frac{1}{2}(\sigma^+ + \sigma^-) \qquad (33)$$

where

$$H_t\Delta u \equiv \frac{1}{2\pi i}\int_L \{2\Delta u[-\varsigma'(\tau-t)]d\tau - \Delta u\frac{\partial}{\partial t}dk_{d1}(\tau,t)$$
$$-\overline{\Delta u}\frac{\partial}{\partial t}dk_{d2}(\tau,t)\} \qquad (34)$$

operators S_t and R_t are given by (28)-(31); formulae for $dk_{d1}(\tau,t)$, $dk_{d2}(\tau,t)$ are given by (7), (8).

The CVH-BIE (33) is the most convenient in applications because it contains displacement discontinuities and traction, the very values which characterize contact interaction at crack surfaces. Besides, in contrast with singular Eqs. (2), (24) and (26), it neither contains unknown constants C_j as (2), (24), nor needs to satisfy additional conditions (32) as (26).

5. Average Strains and Effective Compliances

From (22) we have a relation between cyclic constants $2\alpha_k$ of the function $\varphi(z)$, cyclic constants $2\gamma_k$ of $f(z)$ and cyclic constants $2\rho_k$ of displacements:

$$2\alpha_k=[2\rho_k+2\gamma_k/(2\mu)]2\mu h \qquad (35)$$

Consider uniform strains ε_{xx}, ε_{yy}, ε_{xy} in a plane without cracks and holes. A corresponding displacement field with zero rotation is

$$u_0(z)=\frac{1}{2}[(\varepsilon_{xx}+\varepsilon_{yy})z+(\varepsilon_{xx}-\varepsilon_{yy}+2i\varepsilon_{xy})\overline{z}\,]$$

This field, being linear, is quasiperiodic for arbitrary periods. For periods $2\omega_1$, $2\omega_2$ its cyclic constants are

$$2\rho_{0k}=u_0(z+2\omega_k)-u_0(z)=\frac{1}{2}[(\varepsilon_{xx}-\varepsilon_{yy})\omega_k$$
$$+(\varepsilon_{xx}-\varepsilon_{yy}+2i\varepsilon_{xy})\overline{\omega}_k\,]$$

They will be the same as those in a plane with cracks and holes if $2\rho_{0k}=2\rho_k$. Hence, the cyclic constants $2\rho_k$ are expressed through average strains ε_{xx}, ε_{yy}, ε_{xy} as

$$2\rho_k=\frac{1}{2}[(\varepsilon_{xx}-\varepsilon_{yy})\omega_k+(\varepsilon_{xx}-\varepsilon_{yy}+2i\varepsilon_{xy})\overline{\omega}_k\,] \qquad (36)$$

Substitution of (12) and (36) in (35) and the latter in (3), (4), after some algebra yields

$$\varepsilon_{xx}=\frac{1}{2S}\mathrm{Im}\int_L\{\Delta u(d\overline{\tau}-d\tau)-\frac{\Delta f}{2\mu}(d\tau+\frac{\chi-1}{2}d\overline{\tau})\}+\varepsilon_{xx}^{\infty}$$

$$\varepsilon_{yy}=\frac{1}{2S}\mathrm{Im}\int_L\{\Delta u(d\overline{\tau}+d\tau)+\frac{\Delta f}{2\mu}(d\tau-\frac{\chi-1}{2}d\overline{\tau})\}+\varepsilon_{yy}^{\infty}$$

$$\varepsilon_{xy}=\frac{1}{2S}\mathrm{Re}\int_L(\Delta u+\frac{\Delta f}{2\mu})d\tau+\varepsilon_{xy}^{\infty} \qquad (37)$$

where $\varepsilon_{xx}^{\infty}$, $\varepsilon_{yy}^{\infty}$, $\varepsilon_{xy}^{\infty}$ are strains in a plane without cracks and holes under uniform stresses s_{xx}, s_{yy}, s_{xy}; they are given by Hooke's law:

$$\varepsilon_{xx}^{\infty}=\frac{1}{8\mu}[(\chi+1)s_{xx}-(3-\chi)s_{yy}],$$

$$\varepsilon_{yy}^{\infty}=\frac{1}{8\mu}[(\chi+1)s_{yy}-(3-\chi)s_{xx}],$$

$$\varepsilon_{xy}^{\infty}=\frac{1}{2\mu}s_{xy}$$

The first terms in the r.h.s. of (37) represent additional average strains due to presence of cracks and holes. They provide additional compliances to those defined by Hooke's law. The formulae (37) serve to find effective compliances after solving any of Eqs. (24), (26), (33). To find all the compliances we need to have solutions under three sets of average stresses

1) $s_{xx}=1,\ s_{yy}=0,\ s_{xy}=0;$

2) $s_{xx}=0,\ s_{yy}=1,\ s_{xy}=0;$

3) $s_{xx}=0,\ s_{yy}=0,\ s_{xy}=1.$

Denote with the superscript "1" average strains for the first set, with the superscript "2" for the second, and "3" for the third. Then the matrix $\boldsymbol{B}$ with the components

$$b_{11}=\varepsilon_{xx}^{1} \quad b_{12}=\varepsilon_{xx}^{2} \quad b_{13}=\varepsilon_{xx}^{3}$$

$$b_{21}=\varepsilon_{yy}^{1} \quad b_{22}=\varepsilon_{yy}^{2} \quad b_{23}=\varepsilon_{yy}^{3}$$

$$b_{31}=2\varepsilon_{xy}^{1} \quad b_{32}=2\varepsilon_{xy}^{2} \quad b_{33}=2\varepsilon_{xy}^{3}$$

is the *matrix of effective compliances*. From (37) it follows that $\boldsymbol{B}$ is the sum of the matrix $\boldsymbol{B}_0$, corresponding to Hooke's law, and the matrix $\boldsymbol{B}_a$, of additional compliances generated by additional average strains. In fact, it is sufficient to find the latter.

III. CV-BIE FOR DOUBLY PERIODIC SYSTEMS OF BLOCKS AND/OR INCLUSIONS

1. CV-BIE for Blocks and Inclusions in a Matrix

Consider a doubly periodic system of n blocks (grains) in a matrix. In particular cases, the problem refers to inclusions (individual blocks surrounded by the matrix). We will show below that the results are also true for blocks, inclusions and matrices containing internal cracks, cracks terminating at boundaries, holes and inclusions of a smaller rank. For this problem, omitting lengthy derivation, we arrive at the singular Eq.

$$K_t\Delta u+T_tf-(t\bullet 2\mathrm{Re}A_\alpha+\overline{t}\,\overline{A}_\beta)+\frac{\chi_0+1}{2\mu_0}f^{\infty}(t)=\frac{1}{2}a_2f(t)$$
$$t\in L \qquad (38)$$

where the operator K_t is given by (5) with L now being the total boundary of the system of blocks in the main cell (the contact between surfaces of adjacent blocks or blocks and matrix is accounted as a single line at which mechanical values may experience discontinuities); it also includes contours of cracks and holes in the matrix;

$$T_t f = \frac{1}{2\pi i}\int_L \{(2a_1 - a_3)f \bullet [\varsigma(\tau - t) - \varsigma(\tau)]d\tau$$

$$+ (a_3 - a_1)f \bullet dk_{d1}(\tau, t) - a_1\overline{f} \bullet dk_{d2}(\tau, t)\}$$

the differentials $dk_{d1}(\tau, t)$, $dk_{d2}(\tau, t)$ are defined by (7), (8);

$$a_1 = \frac{1}{2\mu^+} - \frac{1}{2\mu^-},\quad a_2 = \frac{\chi^+ + 1}{2\mu^+} + \frac{\chi^- + 1}{2\mu^-},$$

$$a_3 = \frac{\chi^+ + 1}{2\mu^+} - \frac{\chi^- + 1}{2\mu^-} \tag{39}$$

the traveling direction is taken arbitrarily for contacts and cracks in the matrix, but for holes we assume it to be clockwise; the normal, as usual, is directed to the right of the traveling path; remember also that the sign "plus" ("minus") refers to values of a block (matrix), to left (right) of the traveling path; for holes we take $u^-=0$, $f^-=0$, $1/\mu^-=0$; the constants A_α and A_β are now defined as

$$A_\alpha = -\frac{1}{2\pi i}\frac{\eta_1}{\omega_1}\int_L (\Delta u + a_1 f)d\tau$$

$$-\frac{1}{2S}\{\mathrm{Im}\int_L [\Delta u + \frac{1}{2}(2a_1 - a_3)f]d\overline{\tau}$$

$$+ i\frac{\overline{\omega}_1}{\omega_1}\int_L (\Delta u + a_1 f)d\tau\}$$

$$A_\beta = -\frac{1}{\pi}\frac{\eta_1}{\omega_1}[\mathrm{Im}\int_L (\Delta u + a_1 f)d\overline{\tau} - \frac{i}{2}\int_L a_3\overline{f}\,d\tau]$$

$$+ \frac{1}{2\pi i}\frac{\overline{\omega}_1}{\omega_1}(\frac{\eta_1}{\omega_1} - \frac{\lambda_1}{\overline{\omega}_1})\int_L (\Delta u + a_1 f)d\tau$$

$$+ \frac{\overline{\omega}_1}{\omega_1}\frac{1}{S}\mathrm{Im}\{\int_L [\Delta u + \frac{1}{2}(2a_1 - a_3)f]d\overline{\tau}$$

$$-(\frac{\overline{\omega}_1}{|\omega_1|})^2\int_L (\Delta u + a_1 f)d\tau\} \tag{40}$$

f^∞(t) is given by (25).

Equation (38) and those below are naturally extended to cases when blocks themselves contain cracks, inclusions and/or holes. It is sufficient to include cracks and boundaries of inclusions into L. It is clear that inclusions themselves may have cracks and internal inclusions; the latter, in their turn, also may have cracks and inclusions and so on. For a hole we have an alternative, either, as mentioned, assume $u^-=0$, $f^-=0$, $1/\mu^-=0$ or consider it filled but disconnected with surrounding media: $f^+=f^-=f$, $\mu^+=\mu^-$, $\chi^+=\chi^-$; the latter choice reproduces the line of the displacement discontinuity method (see Crouch and Starfield, 1983).

The operations, previously used to derive (24) and (26) from (2), being applied to (38) yield a singular CV-BIE

$$S_t\Delta u' + P_t\sigma - (2\mathrm{Re}A_\alpha + \frac{\partial\overline{t}}{\partial t}\overline{A}_\beta) + \frac{\chi_0 + 1}{2\mu_0}\sigma^\infty(t) = \frac{1}{2}a_2\sigma(t) \tag{41}$$

and a hypersingular CV-BIE

$$H_t\Delta u + P_t\sigma - (2\mathrm{Re}A_\alpha + \frac{\partial\overline{t}}{\partial t}\overline{A}_\beta) + \frac{\chi_0 + 1}{2\mu_0}\sigma^\infty(t) = \frac{1}{2}a_2\sigma(t) \tag{42}$$

where operators S_t and H_t are defined by (28) and (34) respectively with L taken as explained above;

$$P_t\sigma = \frac{1}{2\pi i}\int_L \{(2a_1 - a_3)\sigma\varsigma(\tau - t)d\tau - (a_3 - a_1)\sigma\frac{\partial}{\partial t}k_{d1}(\tau, t)d\tau$$

$$+ a_1\overline{\sigma}\frac{\partial}{\partial t}k_{d2}(\tau, t)d\overline{\tau}\} \tag{43}$$

the kernels $(\partial/\partial t)k_{d1}(\tau, t)$, $(\partial/\partial t)k_{d2}(\tau, t)$ are given by (30), (31); $\sigma^\infty(t)$ is defined by (27). The constants A_α and A_β, given by (40), are also transformed by appropriate integration by parts; we shall not write down these obvious formulae.

Equations (38), (41), (42) turn into Eqs. (24), (26), (33) respectively for a homogeneous media.

2. Stresses Inside The Blocks and The Matrix

A solution of the derived equations makes known both displacement discontinuities and tractions at L. Having these values, one may find fields inside the blocks and the matrix. To this end, find, first, K-M functions $\varphi_l(z)$ and $\psi_l(z)$ inside the blocks (l=1, ..., n) and the matrix (l=0). The result is

$$\frac{\chi_l + 1}{2\mu_l}\varphi_l(z) = \frac{1}{2\pi i}\int_L (\Delta u + a_1 f)[\varsigma(\tau - z) - \varsigma(\tau)]d\tau$$

$$-A_\alpha z + \frac{1}{4}(s_{xx} + s_{yy})z\frac{\chi_0 + 1}{2\mu_0} \quad z \in D_l,\ l=0,\ ...,\ n \tag{44}$$

$$\frac{\chi_l+1}{2\mu_l}\psi_l(z)=\frac{1}{2\pi i}\int_L \{[-\Delta\overline{u}+(a_3-a_1)\overline{f}][\varsigma(\tau-z)-\varsigma(\tau)]d\tau$$

$$+(\Delta u+a_1 f)d[\overline{\tau}(\varsigma(\tau-z)-\varsigma(\tau))]+(\Delta u+a_1 f)Q(\tau-z)d\tau\}-A_\beta z$$

$$+\frac{1}{2}(s_{yy}-s_{xx}+2is_{xy})z\frac{\chi_0+1}{2\mu_0}\quad z\in D_l,\ l=0,\ ...,\ n \qquad (45)$$

where L is the total boundary as explained above; a_1, a_2, a_3 are given by (39); A_α and A_β are defined by (40).

These are integral representations of the K-M functions. They could serve as a start point to derive (38). Using (44), (45) in K-M formulae (16) for displacements and resultant force, we obtain these values. The functions $\Phi_l(z)=\varphi_l'(z)$, $\Psi_l(z)=\psi_l'(z)$, obtained from (44), (45) by differentiation, serve to find stresses. The latter also may be found directly from (41)-(43) by applying $dt=dz$ and $a_2=2(\chi_l+1)/(2\mu_l)$ for $t=z\in D_l$ ($l=0$, ..., n).

3. Average Strains and Effective Compliances

The formulae (37) for average strains are also transformed for the case of a doubly periodic system of blocks. The result is

$$\varepsilon_{xx}=\frac{1}{2S}\mathrm{Im}\int_L \{(\Delta u(d\overline{\tau}-d\tau)+\sigma[a_1\tau$$

$$+\frac{1}{2}(a_3-2a_1)\overline{\tau}]d\tau\}+\varepsilon_{xx}^{\infty}$$

$$\varepsilon_{yy}=\frac{1}{2S}\mathrm{Im}\int_L \{(\Delta u(d\overline{\tau}+d\tau)-\sigma[a_1\tau$$

$$-\frac{1}{2}(a_3-2a_1)\overline{\tau}]d\tau\}+\varepsilon_{yy}^{\infty}$$

$$\varepsilon_{xy}=\frac{1}{2S}\mathrm{Re}\int_L (\Delta u-a_1\tau\sigma)d\tau+\varepsilon_{xy}^{\infty} \qquad (46)$$

Equation (46) serves to find effective elastic compliances as explained in point 2.5.

In conclusion of this Section, note that its results open new facilities to model media with internal structure. One may consider a representative parallelogram with a finite number of blocks (grains), inclusions and cracks as a main cell of a doubly periodic system. As equations involve a contour only in the main cell, we obtain a facility to study an infinite medium by solving a problem for its finite representative region. In the next Section we will make a further step to simplify the problem by transforming it in a way which allows us to use conventional codes of CV-BEM for finite systems avoiding developing special codes for doubly periodic systems. It should also be noted that the equations derived are hardly may be obtained following the path of real variables; the theory of Kolosov-Muskhelishvili provided us with a mighty means to tackle the problem.

IV. FORMS OF CV-BIE CONVENIENT IN COMPUTATIONS

The equations derived contain dzeta-function $\zeta(z)$ by Weierstrasse and the function $Q(z)$ by Natanzon. They are defined by series (9). In practical calculations, it is sufficient to keep a finite number of terms because the series absolutely converge (their terms decrease as $1/w^3$). Suppose we took a reasonable number of terms to provide the needed accuracy (in the next Section we will make conclusions on this issue). For certainty, assume that integers j_1 and j_2 in $w=j_1\cdot 2\omega_1+j_2\cdot 2\omega_2$ run the sequences $-N_1\le j_1\le N_1$, $-N_2\le j_2\le N_2$. In other words, account for $(2N_1+1)(2N_2+1)$ cells including the main cell. Then we have

$$\varsigma(z)\approx\sum_{\substack{j_1=-N_1\\ j_2=-N_2}}^{N_1,N_2}\frac{1}{z-w}+\sum_{\substack{j_1=-N_1\\ j_2=-N_2}}^{N_1,N_2}{}'\frac{1}{w}+z\sum_{\substack{j_1=-N_1\\ j_2=-N_2}}^{N_1,N_2}{}'\frac{1}{w^2}$$

$$Q(z)\approx\sum_{\substack{j_1=-N_1\\ j_2=-N_2}}^{N_1,N_2}{}'\frac{\overline{w}}{(z-w)^2}-\sum_{\substack{j_1=-N_1\\ j_2=-N_2}}^{N_1,N_2}{}'\frac{\overline{w}}{w^2}-2z\sum_{\substack{j_1=-N_1\\ j_2=-N_2}}^{N_1,N_2}{}'\frac{\overline{w}}{w^3} \qquad (47)$$

Note, that each sum in the r.h.s. of (47), taken itself, does not converge when N_1, $N_2\to\infty$. But for finite sums taken to represent $\zeta(z)$ and $Q(z)$ we may use the forms of (47). They provide forms of CV-BIE, convenient in numerical implementation. Indeed, using, for example, (47) in the CVH-BIE (33) or (42), we obtain after transformations:

$$-\varsigma'(\tau-t)d\tau\approx\sum_{\substack{j_1=-N_1\\ j_2=-N_2}}^{N_1,N_2}\frac{d\tau}{(\tau-w-t)^2}-bd\tau$$

$$\frac{\partial}{\partial t}dk_{d1}(\tau,t)\approx\sum_{\substack{j_1=-N_1\\ j_2=-N_2}}^{N_1,N_2}\frac{\partial}{\partial t}dk_1(\tau-w,t)-bd\tau+\frac{\partial\overline{t}}{\partial t}\overline{b}\,d\overline{\tau}$$

$$\frac{\partial}{\partial t}dk_{d2}(\tau,t)\approx\sum_{\substack{j_1=-N_1\\ j_2=-N_2}}^{N_1,N_2}\frac{\partial}{\partial t}dk_2(\tau-w,t)-\overline{b}\,d\overline{\tau}$$

$$-\frac{\partial\overline{t}}{\partial t}(\overline{b}\,d\tau-2\overline{b}_Q d\,\overline{\tau}) \qquad (48)$$

where $k_1(\tau,t)=\ln[(\tau-t)/(\overline{\tau}-\overline{t})]$, $k_2(\tau,t)=[(\tau-t)/$

$(\overline{\tau}-\overline{t})$, $b=\sum_{\substack{j_1=-N_1\\ j_2=-N_2}}^{N_1,N_2}{}'\frac{1}{w^2}$, $b_Q=\sum_{\substack{j_1=-N_1\\ j_2=-N_2}}^{N_1,N_2}{}'\frac{\overline{w}}{w^3}$. Note that $k_1(\tau, t)$, $k_2(\tau, t)$ are kernels common in equations for non-periodic problems (e.g. Linkov, 1974; Linkov and Mogilevskaya, 1994; Muskhelishvili, 1975; Panasiuk et al., 1976). The sums explicitly written in (48) include w=0; consequently, they are not marked with a prime.

Equations (48) are of direct use for terms in (33) and (42) containing displacement discontinuities Δu. For integrals in (29), (43) containing tractions, their application is not so immediate. One may first use integration by parts coming from tractions to the resultant force; this transforms kernels to forms similar to those for displacement discontinuities. Then (48) are applied. At last, we again use integration by parts returning to tractions. As a result, Eq. (33), for example, reads

$$\frac{1}{2\pi i}\sum_{\substack{j_1=-N_1\\ j_2=-N_2}}^{N_1,N_2}\int_L\{2\Delta u\frac{d\tau}{(\tau-w-t)^2}-\Delta u\frac{\partial}{\partial t}dk_1(\tau-w,t)$$

$$-\Delta\overline{u}\frac{\partial}{\partial t}dk_2(\tau-w,t)+(2a_1-a_3)\sigma\frac{d\tau}{\tau-w-t}$$

$$-(a_3-a_1)\sigma\frac{\partial k_1}{\partial t}d\tau+a_1\overline{\sigma}\frac{\partial k_2}{\partial t}d\overline{\tau}\}$$

$$-[2\mathrm{Re}(A_\alpha+A)+\frac{\partial\overline{t}}{\partial t}(\overline{A_\beta}+\overline{B})]+\frac{\chi_0+1}{2\mu_0}\sigma^\infty(t)=\frac{1}{2}a_2\sigma(t) \tag{49}$$

where A and B are new constants:

$$A=b(I_{u1}-I_{\sigma1})/(2\pi i),$$

$$B=2b\mathrm{Re}[(I_{u2}-I_{\sigma2})/(2\pi i)]-b\overline{I}_{\sigma3}/(2\pi i)-b_Q(I_{u1}-I_{\sigma1})/(\pi i),$$

$$I_{u1}=\int_L\Delta u d\tau,\ I_{u2}=\int_L\Delta u d\overline{\tau},\ I_{\sigma1}=\int_L a_1\sigma\tau d\tau,$$

$$I_{\sigma2}=\int_L a_1\sigma\overline{\tau}d\tau,\ I_{\sigma3}=\int_L a_3\sigma\overline{\tau}d\tau$$

The integrals $I_{u1},\ldots,I_{\sigma3}$ are integrals which are also present in A_α and A_β. We used equality sign in (49) assuming that the numbers N_1, N_2 are great enough to provide needed accuracy.

The CVH-BIE (49) resembles the common CVH-BIE for a *non-periodic* problem (Linkov and Mogilevskaya, 1998) involving $(2N_1+1)(2N_2+1)$ cells with two reservations: (i) we consider Δu to be repeated in congruent points; (ii) (49) contains additional term $-[2\mathrm{Re}(A_\alpha+A)+(\partial\overline{t}/\partial t)(\overline{A_\beta}+\overline{B})]$, which being multiplied by $2\mu_0/(\chi_0+1)$, may be interpreted as generated by some additional stresses at infinity. The first feature allows us to solve the equation only for the contour L in the main cell. The second serves to use successive steps starting from zero values of constants A_α, A_β, A, B. Hence, one can easily adjust a CV-BEM program for a *finite* system of cracks, blocks, holes and/or inclusions in the main cell to solve problems for *doubly periodic* systems.

V. NUMERICAL TESTS AND EXAMPLES

1. Data on Numerical Experiments

A program of the CVH-BEM, worked out by the authors for finite systems of cracks served as a basis for solution of doubly periodic problems following the line of the previous Section. The program employs three-point ordinary elements for internal parts of L. For tips of cracks it employs three-point tip elements accounting for square root asymptotic. All integrals for straight and circular-arc elements are evaluated exactly by using basis functions and quadrature formulae given in (Linkov and Mogilevskaya, 1998). This provides accurate results (four-five correct digits) for displacement discontinuities and SIFs even for a moderate number of boundary elements which normally does not exceed twenty.

These features retained when applying the code to the CVH-BIE (49). Meanwhile, to keep accuracy when using (49) we need to choose appropriate parameters: (i) the numbers N_1 and N_2 in truncated series, and (ii) the number of iterations if we do not change the algebraic matrix by accounting for integrals in the constants A_α, A_β, A, B but find the latter with successive steps starting from their zero values. The necessary numbers were found in preliminary numerical experiments.

We took the same number $N_1=N_2=N$ along each of the periods $2\omega_1$, $2\omega_2$. In this case, the main cell is embraced by "chains" of cells enclosing previous ones with growing number N. The total number of cells and consequently the number of terms in truncated series is $(2N+1)^2$. From our numerical tests we found out, that, even for strongly interacting cracks, it was sufficient to take N=10 to reproduce results to five digits.

We stated also that ten iterations are always enough to have five digits reproduced even for cracks with the length $2a$ close to the length 2ω of the smallest period (a/ω=0.9).

Numerical data below are obtained with the mentioned number (ten) of "chains" embracing the main cell and with ten iterations. To compare the

Table 1. Normalized values of k_I for a rectangular lattice ($\omega_2/\omega_1=0.4i$)

a/ω_1	0.3	0.4	0.5	0.6	0.7	0.75
CVH-BEM	0.8532	0.8381	0.8864	0.9771	1.141	1.263
[21]	0.85	0.84	0.89	0.98	1.14	1.27

Table 2. Normalized values of k_I and k_{II} for a square lattice ($\omega_2/\omega_1=i$)

a/ω_1	0.3	0.4	0.5	0.6	0.7	0.8	0.9
k_I CVH-BEM	1.0304	1.0624	1.1136	1.1941	1.325	1.558	2.112
k_I [20]	1.03	1.06	1.12	1.19	1.32		
k_{II} CVH-BEM	1.0468	1.0853	1.1399	1.2197	1.345	1.571	2.116
k_{II} [20]	1.0461	1.0827	1.131	1.193	1.27	1.37	

Table 3. Normalized effective moduli E_{eff} and μ_{eff} for a square lattice ($\omega_2/\omega_1=i$)

a/ω_1	0.3	0.4	0.5	0.6	0.7	0.8	0.9
E_{eff}, CVH-BEM	0.873	0.790	0.698	0.602	0.504	0.405	0.300
E_{eff} Filshtinski (1974)	0.86	0.80	0.71	0.61	0.50	0.41	0.29
μ_{eff}, CVH-BEM	0.946	0.905	0.853	0.791	0.718	0.632	0.521
μ_{eff}, Filshtinski (1974)	0.93	0.90	0.85	0.80	0.72	0.63	0.50

results with those of other authors, we considered *straight* cracks of the length $2a$ in rectilinear, square and triangle lattices. The length of tip elements was taken as $0.15a$; three neighboring ordinary elements have the length $0.05a$; the remaining part of a crack was represented with eleven equal ordinary elements with the length $1.4a/11$. So, the total number of boundary elements was nineteen. From calculation for an isolated crack, we could see that such a choice of elements always provided five correct digits.

To ensure accuracy in cases of high crack density, we sometimes doubled precision. The results always had five reproduced digits. The case of periodic collinear cracks for which analytical solution is available, was studied in detail to examine accuracy for closely located cracks. We conclude that our numerical data have at least four correct significant digits.

2. Comparison with Results of Other Authors

Tables 1-3 contain our data obtained with CVH-BEM compared with available results published by other authors (Filshtinski, 1974; Panasiuk *et al.*, 1976). SIFs are normalized by $p\sqrt{\pi a}$, where p is a uniform traction at cracks. The effective modulus E_{eff} is normalized by Young's modulus of a plate; the effective shear modulus μ_{eff} is normalized by the shear modulus of a plate.

Rectangular lattice. Consider a rectangular lattice with straight cracks parallel to a real period $2\omega_1$. Graphs of k_I for this problem are given in the Handbook (1987). They provide SIFs with error of about 1.5 percent.

Square lattice. Results for a square lattice are given in Table 2. It presents normalized values of k_I for tension normal to cracks and k_{II} for shear along cracks. For comparison, data obtained in Panasiuk et al., (1976) are also given.

We see that the results of the CVH-BEM agree with those obtained in Panasiuk *et al.*, (1976). Note, that the CVH-BEM provides reliable accuracy data for crack concentrations up to $a/\omega_1=0.9$.

Data on effective moduli are presented in Table 3. The Poisson's ratio is taken $\nu=0.3$ to compare the results with those obtained in Filshtinski (1974) for this value.

Again we see satisfactory agreement within the accuracy obtained in Filshtinski (1974).

Quite similar results and the same conclusions are given in our paper (Koshelev and Linkov, 1999) for straight cracks in a triangular lattice.

3. New Examples

Results for angular (Fig. 2a) and semi-circular (Fig. 2b) cracks in a square lattice with periods $2\omega_1=2\omega$, $2\omega_2=i\cdot2\omega$ are given in Tables 4, 5. They include data on normalized SIFs and additional compliances. The additional compliances $b_{11a}=b_{22a}$ and b_{33a} are normalized by the values respectively $b_{110}=b_{220}=(\chi+1)/(8\mu)=1/E$ and $b_{330}=1/\mu$,

Table 4. Normalized values of SIFs k_I, k_{II} and additional compliances b_{11a}, b_{33a} for a square lattice with angle cracks

$a/(2\omega)$	0.3	0.4	0.5	0.6	0.7	0.8
k_I^A	0.8197	0.8572	0.9242	1.0391	1.2366	1.594
k_{II}^A	-0.0094	-0.0054	0.0180	0.0342	0.0448	0.056
k_I^B	-0.0309	0.0083	0.0113	-0.0093	-0.0567	-0.140
k_{II}^A	-0.3251	-0.3381	-0.3416	-0.3297	-0.3091	-0.298
$b_{11a}=b_{22a}$	0.2110	0.3895	0.6457	1.0157	1.581	2.552
b_{33a}	0.2416	0.4782	0.8451	1.3645	2.0845	3.136

Table 5. Additional compliances b_{11a}, b_{33a} for a square lattice with semi-circular cracks

R/ω	0.3	0.4	0.5	0.6	0.7	0.8
b_{22a}	0.1627	0.2965	0.4828	0.7432	1.126	1.756
b_{33a}	0.1987	0.3887	0.6921	1.190	2.073	3.914

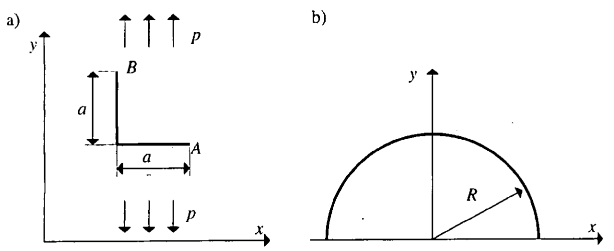

Fig. 2. Angle (a) and semi-circular (b) crack in the main cell of a square lattice.

corresponding to a plane without cracks.

There could be presented numerous other examples. The developed code of the CVH-BEM allows us to consider arbitrary lattices with arbitrary contours represented or approximated with a set of straight and circular-arc elements. Having a friendly graphic interface, the program may serve as a specific handbook to find SIFs and effective compliances for numerous configurations which can not be listed in a conventional handbook.

VI. CONCLUSIONS

The conclusions of the paper are as follows.

(1) New singular (26) and hypersingular (33) CV-BIE for doubly periodic systems of cracks and holes do not involve additional conditions for cyclic constants. Meanwhile, the latter conditions written as (37) may serve to find effective compliances after a CV-BIE is solved.

(2) New singular (38), (41) and hypersingular (42) CV-BIE serve to solve doubly periodic problems for systems of blocks and/or inclusions. Blocks, inclusions and the embedding matrix may have cracks and/or holes. Effective compliances may be found by using (46) after a CV-BIE is solved. The derived CV-BIE open new facilities to model media with internal structure by repeating infinitely a typical region in the form of a parallelogram.

(3) Conventional programs based on CV-BIE for non-periodic problems may be promptly adjusted to solve doubly periodic problems by using the approach of Sec. 4. Applied to the hypersingular Eq. (42) this approach results in the CVH-BIE (49) which is easily implemented in the CVH-BEM.

(4) Numerical tests confirm easy implementation, high efficiency and accuracy of the CVH-BEM code obtained from a code for non-periodic problems. SIFs and effective compliances are calculated with at least four correct digits even for strongly interacting cracks with the number of boundary elements normally not exceeding twenty. The data obtained provide benchmarks to validate approximate approaches used to calculate effective compliances.

NOMENCLATURE

$2a$	the length of a straight crack
b_{11}, ..., b_{33}	effective compliances
C_α, C_β	the main constants in equations for effective compliances
f	the resultant force
G_t	singular operator in integral equation
H_t	hypersingular operator in integral equation
i	the imaginary unit
k_1, k_2	kernels of integral operators for a non-periodic problem
k_{d1}, k_{d2}	kernels corresponding to k_1, k_2 for a double-periodic problem

k_I, k_{II}	normalized stress intensity factors
K_t	singular integral operator for a double-periodic problem
K-M	Kolosov-Muskhelishvili
L	the total contour of cracks and holes in the main cell
L_j	the j-th particular contour in the main cell
$\mathbf{n}$	the unit normal
Q	the function by Natanzon
R_t	integral operator acting on the derivative of displacement discontinuities
s_{xx}, s_{yy}, s_{xy}	components of the average stress in the plane
S	the area of a cell
S_t	integral operator acting on tractions
$\mathbf{t}$	the unit tangent to the traveling path
T_t	integral operator acting on the resultant force
u	the complex displacement at a point
w	translation vector between congruent points in a lattice
x, y	the real coordinates of a point in a global system
$z=x+iy$	the complex coordinate of a point
$2\alpha_1$, $2\alpha_2$	the cyclic constants of K-M function φ
χ	Muskhelishvili's parameter
Δ	the symbol of the discontinuity of a function (e.g. $\Delta u=u^+-u^-$)
ε_{xx}, ε_{yy}, ε_{xy}	the components of average strains in a plane
$2\gamma_1$, $2\gamma_2$	the cyclic constants of the resultant force
$2\eta_1$, $2\eta_2$	the cyclic constants of the dzeta-function by Weierstrasse
φ	the first K-M function
$2\lambda_1$, $2\lambda_2$	the constants of the function by Natanzon
μ	shear modulus
ν	Poisson's ratio
$2\rho_1$, $2\rho_2$	the cyclic constants of displacements
σ	complex traction
σ_n, σ_t	the components of traction in local coordinates
τ	the complex coordinate of a point at a contour
$2\omega_1$, $2\omega_2$	the complex periods of a lattice
ψ	the second K-M function
ζ	the dzeta-function by Weierstrasse

The bar over a symbol marks complex conjugation.

REFERENCES

1. Crouch, S.L., and Starfield, A.M., 1983, *Boundary Element Method in Solid Mechanics*. London, George Allen & Unwin.
2. Filshtinski, L.A., 1974, "Interaction of Doubly Periodic System of Straight Cracks in an Isotropic Medium," *Journal of Applied Mathematics and Mechanics (Translation of Prikladnaya Matematika i Mekhanika, PMM (U.S.S.R.)*, Vol. 38, No. 5, pp. 906-914).
3. Grigoliuk, E.I., and Filshtinski, L.A., 1970, *Perforated Plates and Shells*. Moscow, Nauka. (In Russian.).
4. Ioakimidis, N.I., and Theocaris, P.S., 1978, "Doubly-periodic Array of Cracks in an Infinite Isotropic Medium," *Journal of Elasticity*, **8**, pp. 157-169.
5. Koiter, W.T., 1959, "Some General Theorems on Doubly-periodic and Quasi-periodic Functions," *Proceedings of the Koninklijke Nederlandse Akademie van Wetensehappen, Ser.* A, Vol. 62, pp. 120-128.
6. Koiter, W.T., 1960, "Stress Distribution in an Elastic Sheet with a Doubly-periodic Set of Equal Holes," *Boundary Problems of Differential Equations*, Medison, Univ. Wisconsin Press, pp. 191-213.
7. Koshelev, V.F., and Linkov, A.M., 1998, "Complex Boundary Element Method: Application to Periodic and Doubly Periodic Problems," Proc. of XVI Int. Conference "*Mathematical Modeling in Solid Mechanics. Methods of Boundary and Finite Elements*". A.N.Snitko (ed.), Vol. 2, St-Petersburg, pp. 11-13 (In Russian.).
8. Koshelev, V.F., and Linkov, A.M., 1999, "Complex BEM in Periodic and Doubly Periodic Elasticity Problems," *Proc. 13th ASCE Engineering Mechanics Division Conference*. Baltimore, (CD-ROM).
9. Linkov, A.M., 1974, "Integral Equations of Elasticity Theory for a Plane with Cuts Loaded with Balanced System of Forces," *Doklady Akademii Nauk S.S.S.R.*, Vol. 218, No.6, pp.1294-1297. (In Russian; translated into English in *Soviet Physics. Doklady.*).
10. Linkov, A.M., 1976, "Integral Equations for the Plane Problem of Elasticity Theory of a Doubly Periodic System of Slots Acted by Self-balancing Loads," *Mechanics of Solids (Translation of Akademia Nauk SSSR, Izvestia, Mekhanika Tverdogo Tela (U.S.S.R.)*, No. 11, pp. 60-63 .
11. Linkov, A.M., and Mogilevskaya, S.G., 1991, "Complex Hypersingular Integrals and Integral Equations in Plane Problems of Elasticity Theory," *Researches on Structure Mechanics and Materials*, Leningrad Institute for Building Engineering, Leningrad, pp.17-34. (In Russian.) .
12. Linkov, A.M., Zoubkov, V.V., and Mogilevskaya, S.G., 1994, *Complex Integral Equations: an*

Effective Means to Solve Plane Problems, Preprint No. 118 of the Institute for Problems of Mech. Eng. (Russian Academy of Sciences). Izdatelstvo Instituta Problem Engenernoy Mekhaniki, St-Petersburg. (In Russian.).
13. Linkov, A.M., and Mogilevskaya, S.G., 1994, "Complex Hypersingular Integrals and Integral Equations in Plane Elasticity," *Acta Mechanica*, Vol. 105, No. 1-4, pp. 189-205.
14. Linkov, A.M., and Mogilevskaya, S.G., 1998, "Complex Hypersingular BEM in Plane Elasticity Problems," *Singular Integrals in Boundary Element Methods*, Ed. V.Sladek & J.Sladek, Computational Mechanics Publications, Southampton, pp. 299-364.
15. Mogilevskaya, S.G., 1996, "The Universal Algorithm Based on Complex Hypersingular Integral Equation to Solve Plane Elasticity Problems," *Computational Mechanics*, Vol. 18, pp. 127-138.
16. Muskhelishvili, N.I., 1934, "A New General Method of Solution of the Fundamental Boundary Problems of Plane Theory of Elasticity," *Doklady Akademii Nauk S.S.S.R.*, Vol. III, No.1, pp. 7-11. (In Russian.).
17. Muskhelishvili, N.I., 1975, *Some Basic Problems of the Mathematical Theory of Elasticity*. The 3-rd English edition, Groningen, Nordhoff.
18. Muskhelishvili, N.I., 1953, *Singular Integral Equations*. Groningen, Nordhoff.
19. Natanzon, V.Ya., 1935, "Stresses in an Extended Plate Weakened by Identical Holes in Checkerboard Pattern," *Matematicheskiy Sbornik*, Vol. 42, No. 5, pp. 617-633 (In Russian).
20. Panasiuk, V.V., Savruk, M.P., and Datsishin, A.P., 1976, *Stress Distribution in Plates and Shells*. Kiev, Naukova Dumka. (In Russian.).
21. *Stress Intensity Factors Handbook*, 1987, Ed. Y. Murakami, Oxford-New York, Pergamon Press.

Discussions of this paper may appear in the discussion section of a future issue. All discussions should be submitted to the Editor-in-Chief.

Manuscript Received: May 15, 1999
Revision Received: June 28, 1999
and Accepted: July 20, 1999

雙週期裂縫系統複變邊界積分方程與邊界元素法

Alexander M. Linkov and Vadim F. Koshelev
俄國科學研究院機械工程研究所

摘　要

本文導出雙週期平面彈性問題的新複變數奇異和超奇異邊界積分方程。考慮的問題包含顆粒，置入物，有孔洞和含裂縫,這些問題可使用合適的非週期性系統的程式來修改，本文提出計算複數型式柔度的公式。複變數邊界元素法非週期性問題先前已發展出數值工具導出複變數超奇異邊界積分方程，並且被有效推廣到含裂縫雙週期系統。正方晶格和三角晶格直線裂縫的應力強度因子和柔度係數計算與實驗資料吻合，並且計算的精確度都在可接受的範圍內，裂縫尖端和含曲線多重裂縫雙週期系統的應力強度因子和柔度係數均在本文中說明。

關鍵詞：複變數積分方程，雙週期裂縫，彈性力學。