Journal of the Chinese Institute of Engineers, Vol. 22, No. 5, pp. 721-727 (1999)

HOMOGENIZATION OF NONLINEAR COMPOSITES USING TRANSFORMATION FIELD ANALYSIS

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Key Words: nonlinear material behavior of composites, homogenization, transformation field analysis, boundary element method.

ABSTRACT

In the early nineties, Dvorak (1991), and Dvorak and Procházka (1996), put forward Transformation Field Analysis (TFA), which consists of expressing a linear relationship between either eigenstrains or eigenstresses and on the other hand strains and stresses in a structure. By means of these eigenparameters, mechanical and thermal phenomena in materials can be simulated. In this paper, the TFA is applied to homogenization of nonlinear composites. A very useful numerical tool, the BEM, has been applied to the solution of integral equations. A sample example shows the ability of the procedure proposed. The problem is solved on a unit cell at microscopic level, but more complicated structures of fibers on, say, reference volume elements with stochastically distributed fibers may be studied.

I. INTRODUCTION

During the past decades, the boundary element method as a solving tool for integral equations governing both linear and nonlinear problems for two and three-dimensional bodies, isotropic or anisotropic, has rapidly become very efficient. Application to homogenization problems, especially in connection with transformation field analysis, seems to be very promising.

In recent years some papers (Lene, 1978; Suquet, 1985) have been devoted to homogenization of composite or laminated materials, mostly making use of Eshelby's forces. He used integral equations as a basic tool. In the present paper we concentrate our attention on homogenization of locally physically nonlinear, bodies by means of a special treatment proposed by Dvorak (1991) and extended by Dvorak and Procházka (1996). The main idea consists of separating the mutual effect of eigenstrains (eigenstresses) from one inclusion (internal cell) to the other. One of the most suitable numerical techniques dealing with homogenization is the boundary element method (BEM). This method provides many advantageous features in comparison with the finite element method. The influence functions (concentration factors) may be computed at each point of the internal cells with high accuracy, nonlinearity in a matrix can easily be introduced by the procedure proposed in this paper and the computation can effectively be carried out.

The influence functions enable one, according to Dvorak (1991), to solve the elastic-plastic and viscoplastic composite systems based on a change of eigenstrains or eigenstresses, while the other quantities remain unchanged during the incremental process.

Basic formulations for boundary element techniques including a brief description of the numerical approach are presented. For more details the reader is referred to Bittnar and Šejnoha (1997), Procházka and Šejnoha (1996) and Telles *et al.* (1981), where more theoretical arguments are discussed.

It is worth noting the paper by Castenada (1991),

where explicit overall mechanical properties are presented for nonlinear isotropic com-posites.

II. BASIC RELATIONS

In this section we will deal with the either the eigenstrain or the eigenstress field and involve it in the formulation. Assume that no body forces are present.

Let us consider a coordinate system $0y_1y_2$ in 2D (for the sake of simplicity we restrict our discussion to 2D, the case of 3D is similar), and a bounded domain (unit cell) $\Omega \equiv \Omega^f \cup \Omega^m$ with a boundary $\partial\Omega$, see Fig. 1. In what follows, the subscripts run the set $\{1, 2\}$, considering only the case of 2D. The interfacial boundary fiber–matrix is denoted by Γ.

Define average quantities $\langle \cdot \rangle$ by

$$\langle a \rangle_\Omega = \frac{1}{\text{meas}\,\Omega} \int_\Omega a(y) d\Omega , \tag{1}$$

where meas Ω means the volume of Ω.

The homogenization can start by introducing an overall (average) strain tensor $\boldsymbol{E}$, or a stress tensor Σ, defined as

$$E_{ij} = \left\langle \varepsilon_{ij}(\boldsymbol{y}) \right\rangle_\Omega , \quad \Sigma_{ij} = \left\langle \sigma_{ij}(\boldsymbol{y}) \right\rangle_\Omega \tag{2}$$

where σ_{ij} are components of the stress tensor and ε_{ij} are components of the strain tensor. There are a couple of possibilities for introducing the boundary conditions in connection with the solution of elastic or elasto-plastic problems. Without loss of generality, we focus only on the given $\boldsymbol{E}$. Then we have to solve the problem:

$$\text{div}\ \sigma(\boldsymbol{y}) = 0 \text{ in } \Omega,\ u_i(\boldsymbol{y}) = E_{ij} y_j \text{ on } \partial\Omega. \tag{3}$$

The real displacements $u_i(\boldsymbol{y})$ and the real strains $\varepsilon_{ij}(\boldsymbol{y})$ may be written in the form of the sum of E_{ij} and the fluctuating terms u_i' and $\varepsilon_{ij}'(\boldsymbol{y})$ as,

$$u_i(\boldsymbol{y}) = E_{ij} y_j + u_i'(\boldsymbol{y}),\ \varepsilon_{ij}(\boldsymbol{y}) = E_{ij} + \varepsilon_{ij}'(\boldsymbol{y}). \tag{3a}$$

In the case of elastic behavior of both fiber and matrix it holds

$$\left\langle \boldsymbol{\varepsilon}' \right\rangle_\Omega = 0 , \tag{3b}$$

cf. Suquest (1985).

The solution procedure $\boldsymbol{\varepsilon}$ is split into two steps, as done in Procházka and Šejnoha (1996). Here, special boundary conditions have to be considered.

First, let u_i^0, $\varepsilon_{ij}^0 \equiv E_{ij}$ and σ_{ij}^0 be known displacement, strain and stress fields, respectively, on a comparative homogeneous medium $\boldsymbol{L}^0$. The

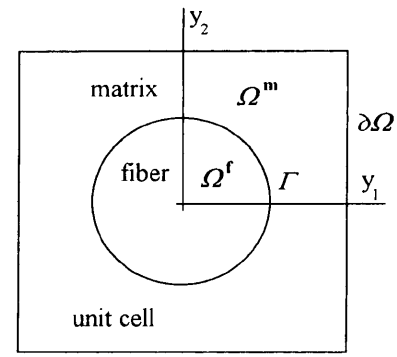

Fig. 1. The domain (unit cell) - geometry and denotation.

stresses σ_{ij}^0 and small strains $\varepsilon_{ij}^0 \equiv E_{ij}$ are related by the linear homogeneous Hooke's law:

$$\sigma_{ij}^0 = L_{ijkl}^0 E_{ij} \text{ in } \Omega, \tag{4a}$$

$$u_i^0 = E_{ij} y_j \text{ on } \partial\Omega. \tag{4b}$$

The matrix L_{ijkl}^0 is so far not fixed.

In the second step a geometrically identical body is considered, which is anisotropic, non-homogeneous, and may exhibit nonlinear behavior. Displacements u_i, strains ε_{ij} and stresses σ_{ij} are unknown and the generalized Hooke's law, including the eigenstresses λ_{ij}, or eigenstrains μ_{ij}, can be written as:

$$\sigma_{ij} = L_{ijkl}\varepsilon_{kl} + \lambda_{ij} \text{ in } \Omega, \tag{5a}$$

$$u_i = E_{ij} y_j \text{ on } \partial\Omega, \tag{5b}$$

$$\lambda_{ij} = -L_{ijkl}\mu_{kl}. \tag{5c}$$

Note that the eigenparameters may stand for prestressing, change of temperature and/or plastic stresses, or plastic strains.

Similar to the classical Hashin-Shtrikman theorem, (Hashin and Shtrikman, 1962), define the symmetric stress polarization tensor τ_{ij} by:

$$\sigma_{ij} = L_{ijkl}^0 \varepsilon_{kl} + \tau_{ij}. \tag{6}$$

Also define

$$u_i' = u_i - u_i^0 \text{ in } \Omega,\ u_i' = 0 \text{ on } \partial\Omega, \tag{7}$$

and consequently

$$\varepsilon_{ij}' = \varepsilon_{ij} - E_{ij},\ \sigma_{ij}' = \sigma_{ij} - \sigma_{ij}^0 \text{ in } \Omega. \tag{8}$$

Our aim is to obtain relations between strains or stresses and eigenstrains or eigenstresses. Since

both σ_{ij} and σ^0_{ij} are statically admissible, the following equations have to be satisfied in the sense of distributions:

$$\frac{\partial \sigma'_{ij}}{\partial y_j} = 0 \text{ in } \Omega, \tag{9}$$

$$\tau_{ij} - [L_{ijkl}]\varepsilon_{kl} - \lambda_{ij} = 0 \text{ in } \Omega, \tag{10}$$

$$u'_i = 0 \text{ on } \partial\Omega, \tag{11a}$$

where

$$[L_{ijkl}(y)] = L_{ijkl}(y) - L^0_{ijkl}. \tag{11b}$$

Subtracting (4a) from (6) we get

$$\sigma'_{ij} = L^0_{ijkl}\varepsilon'_{kl} + \tau_{ij}. \tag{12}$$

III. INTEGRAL FORMULATION

Solution of problems involving linear or nonlinear behavior of composite bodies is mostly formulated in terms of integral equations. Consequently, the natural way to solve these problems is to describe the behavior of such bodies subject to external or internal loading (including eigenstrains and eigenstresses) by the boundary integral equation method.

Integral equation equivalent to (6) and (10) can be expressed as:

$$u'_i(\xi) = \int_{\partial\Omega_u} u^*_{ik}(y, \xi) p'_k(y) d\Gamma(y) - \int_{\partial\Omega_p} p^*_{ik}(y, \xi) u'_k(y) d\Gamma(y) - \int_{\Omega} \varepsilon^*_{ikl}(y, \xi) \tau_{kl}(y) d\Omega(y), \quad \xi \in \Omega, \tag{13}$$

where the starred quantities are known kernels, see Telles and Brebbia (1981). The domains $\partial\Omega_u$ and $\partial\Omega_p$ are the disjoint parts of $\partial\Omega$ where displacements and tractions, respectively, are given. Because of the validity of the boundary condition (2.4_2) and $\partial\Omega \equiv \partial\Omega_u$, the second integral on the right hand side of (13) disappears and we get:

$$u'_i(\xi) = \int_{\partial\Omega} u^*_{ik}(y, \xi) p'_k(y) d\Gamma(y) - \int_{\Omega} \varepsilon^*_{ikl}(y, \xi) \tau_{kl}(y) d\Omega(y) \quad \xi \in \Omega. \tag{13a}$$

Differentiating the latter equation with respect to ξ_j we arrive at the expression

$$\varepsilon'_{ij}(\xi) = \int_{\partial\Omega} h^*_{ijk}(y, \xi) p'_k(y) d\Gamma(y) - \int_{\Omega} \psi^*_{ijkl}(y, \xi) \tau_{kl}(y) d\Omega(y) - C[\tau_{ij}(\xi)] \quad \xi \in \Omega. \tag{14}$$

The appropriate convected term C can be found in Bittnar and Šejnoha (1997), and arise at the internal point $\xi \in \Omega$ by the exchange of integration and differentiation when deriving (14) from (13a).

Note that an important property of the kernel ψ^*_{ijkl} is its symmetry with respect to indices i, j and k, l. The same result holds for h^*_{ijk} with respect to i and j.

It is worth noting that several useful conclusions follow from (14). Since h^*_{ijk} are decaying as r^{-1} in 2D and as r^{-2} in 3D (see e. g. [1] for exact expressions for kernels needed), and if p'_k is bounded, the first integral on the right hand side in (14) tends obviously to zero for supp τ_{ij} (closure of a set of nonvanishing values of τ_{ij}) far from the boundary $\partial\Omega$. This occurs, for example, when L^0_{ijkl} represents the stiffness of the matrix, with the fibers being distributed far enough from the boundary (assumption of Eshelby, Mori & Tanaka, etc.). Then (14) is simplified as

$$\varepsilon'_{ij}(\xi) = -\int_{\Omega \cap \operatorname{supp}\tau_{ij}} \psi^*_{ijkl}(y, \xi) \tau_{kl}(y) d\Omega(y) - C[\tau_{ij}(\xi)], \tag{15}$$

which can be transformed to the form:

$$\varepsilon_{ij}(\xi) = E_{ij}(\xi) - \int_{\Omega \cap \operatorname{supp}\tau_{ij}} [\psi^*_{ijkl}(y, \xi)(L_{klmn}(y) - L^0_{klmn})\varepsilon_{mn}(y) + \lambda_{kl}(y)] d\Omega(y) - C[L_{ijkl}(\xi) - L^0_{ijkl})\varepsilon_{kl}(\xi) + \lambda_{ij}(y)]. \tag{16}$$

The nature of the kernels requires only 3D problems, while the 2D problems are "too singular". It will not be necessary to impose this requirement in the following text.

Willis (1977) proved that the relation (16) is a positive definite and symmetric integral operator in the standard norm. He considers the Green's tensor of homogeneous media vanishing on the boundary. This is virtually possible only for some particular unbounded domains in 3D, not in 2D, which case is also used in this study.

In the same manner it can be proved that the relation (16) is not bounded with respect to the norm induced by the scalar product of L_2–functions and their L_2–derivatives.

Levin (1976) uses other assumptions leading to the vanishing boundary terms in (14). Levin derived(15) without considering the effect of eigenstresses.

IV. TRANSFORMATION FIELD ANALYSIS

Our goal now is to derive the relation between the strains and the eigenstresses of the form

$$\boldsymbol{\varepsilon} = \boldsymbol{A}\boldsymbol{E} + \boldsymbol{G}\boldsymbol{\lambda}, \tag{17}$$

where $\boldsymbol{A}$ and $\boldsymbol{G}$ are the influence function tensors ($\boldsymbol{A}$ is mostly referred to as the mechanical concentration function tensor). Note that once computed, these matrices do not change their values during incremental processes for nonlinear solution of plasticity, optimization, etc. They completely depend only on the shape of the body under study and its material properties in the initial stage of the iteration.

Let us concentrate our attention on a special case of material properties. The starting elastic homogeneous body possesses a stiffness $\boldsymbol{L}^0$ while the body under study is divided into 2 subdomains Ω^f and Ω^m representing the phases with the stiffnesses $\boldsymbol{L}^f$ and $\boldsymbol{L}^m$, respectively. The tensor $[\boldsymbol{L}]$ admits the representation

$$[\boldsymbol{L}(\boldsymbol{y})] = \sum_{\alpha=f,m} [\boldsymbol{L}]_\alpha \kappa_\alpha(\boldsymbol{y}) \tag{18a}$$

where

$$[\boldsymbol{L}]_\alpha = \boldsymbol{L}^\alpha - \boldsymbol{L}^0, \tag{18b}$$

and κ_α is a characteristic function of the phase $\alpha=m$, f, being equal to one inside and zero outside the phase Ω^α. Substituting (18) to (16) and using the definition of polarization tensor τ in (9), we arrive at the following expression

$$\varepsilon_{ij}(\xi) = E_{ij} - \int_\Omega \psi^*_{ijkl}(y,\xi)\{\sum_{\alpha=m,f} [L_{klmn}]_\alpha \kappa_\alpha(y)\varepsilon_{mn}(y) + \lambda_{kl}(y)\}d\Omega(y) - C[\sum_{\alpha=f,m} [L_{ijkl}]_\alpha \kappa_\alpha(\xi)\varepsilon_{kl}(\xi) + \lambda_{ij}(\xi)], \tag{19}$$

and, after removing the characteristic functions, we have

$$\varepsilon_{ij}(\xi) = E_{ij} - \{\sum_{\alpha=m,f} \int_{\Omega_\alpha} \psi^*_{ijkl}(y,\xi)\{[L_{klmn}]_\alpha \varepsilon_{mn}(y) + \lambda_{kl}(y)\}d\Omega(y) - C[\sum_{\alpha=m,f} [L_{ijkl}]_\alpha \kappa_\alpha(\xi)\varepsilon_{kl}(\xi) + \lambda_{ij}(\xi)], \quad \xi \in \Omega. \tag{20}$$

Now we describe the procedure leading to the influence functions. Let us assume that both the original problem involving Hooke's law according to (4) and the problem involving the polarization tensor $\boldsymbol{\tau}$ have the same geometry as well as the same geometrical boundary conditions. Placing the point ξ on the boundary, (13) yields

$$0 = \int_{\partial\Omega_u} u^*_{ik}(y,\xi)p'_k(y)d\Gamma(y) - \sum_{\alpha=m,f} \int_\Omega \varepsilon^*_{ikl}(y,\xi)\{[L_{klmn}]_\alpha \varepsilon_{mn}(y) + \lambda_{kl}(y)\}d\Omega(x), \quad \xi \in \partial\Omega. \tag{21}$$

After discretizing the boundary and after discretizing the domain Ω into M internal cells, both (21) and (14) take the form

$$\boldsymbol{U}\boldsymbol{p}' - \boldsymbol{S}'\boldsymbol{\varepsilon} - \boldsymbol{S}\boldsymbol{\lambda} = \boldsymbol{0}, \tag{22}$$

$$\boldsymbol{\varepsilon}' = \boldsymbol{H}\boldsymbol{p}' - \boldsymbol{\Psi}'\boldsymbol{\varepsilon} - \boldsymbol{\Psi}\boldsymbol{\lambda}, \tag{23}$$

where $\boldsymbol{U}$ is a square matrix ($2N\times 2N$) and $2N$ is a number of degrees of freedom on the boundary in 2D (using, say, linear approximation of tractions), $\boldsymbol{p}'$ is the vector ($2N$) of discretized unknowns p' at nodal points on $\partial\Omega$, $\boldsymbol{S}$, $\boldsymbol{S}'$ are the matrices ($2N\times 3M$) of influences of the strains and eigenstrains in the discretized domain (in 2D three components of strain tensor are independent), $\boldsymbol{H}$ is a ($3M\times 2N$) matrix and, finally, $\boldsymbol{\Psi}$ and $\boldsymbol{\Psi}'$ are square matrices ($3M\times 3M$).

Since the system is well-posed, the regular matrix $\boldsymbol{U}$ may be inverted. Elimination of $\boldsymbol{p}'$ from (22) and (23) provides

$$\boldsymbol{W}\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_0 + \boldsymbol{T}\boldsymbol{\lambda}, \text{ or } \boldsymbol{W}\boldsymbol{\varepsilon} = \boldsymbol{E} + \boldsymbol{T}\boldsymbol{\lambda}, \tag{24}$$

where

$$\boldsymbol{W} = \boldsymbol{I} + \boldsymbol{\Psi}' - \boldsymbol{H}\boldsymbol{U}^{-1}\boldsymbol{S}', \quad \boldsymbol{T} = -\boldsymbol{\Psi} + \boldsymbol{H}\boldsymbol{U}^{-1}\boldsymbol{S}. \tag{24a}$$

Obviously, $\boldsymbol{W}$ is a regular ($6M\times 6M$) matrix, as for a given λ_{ij}, (i, j being fixed) a unique response $\boldsymbol{\varepsilon}$ may be expected. The sought influence function tensor $\boldsymbol{G}$ is equal to $\boldsymbol{W}^{-1}\boldsymbol{T}$ while $\boldsymbol{W}^{-1}$ is the mechanical concentration function tensor $\boldsymbol{A}$.

V. APPLICATION IN COMPOSITE MATERIALS

We start with concentration factors for the phases. There is a certain freedom in selecting the matrix $\boldsymbol{L}^0$, as this does not affect the relation (17), setting $\boldsymbol{\lambda}=\boldsymbol{0}$. Setting it successively at $\boldsymbol{L}^0=\boldsymbol{L}^m$ and $\boldsymbol{L}^0=\boldsymbol{L}^f$ in (20), we obtain the integral over Ω_m in the first case and the integral over Ω_f in the second case disappears:

$$\varepsilon_{ij}(\xi) = E_{ij} - \int_{\Omega^f} \psi^*_{ijkl}(y, \xi) L^f_{klmn} \varepsilon_{mn}(y) d\Omega^f(y)$$

$$- C[L^f_{ijkl} \kappa_\alpha(\xi) \varepsilon_{kl}(\xi)], \quad \xi \in \Omega^f, \tag{25}$$

and

$$\varepsilon_{ij}(\xi) = E_{ij} - \int_{\Omega^m} \psi^*_{ijkl}(y, \xi) L^m_{klmn} \varepsilon_{mn}(y) d\Omega^m(y)$$

$$- C[L^m_{ijkl} \kappa_\alpha(\xi) \varepsilon_{kl}(\xi)], \quad \xi \in \Omega^m \tag{26}$$

where $\boldsymbol{L}^f$ and $\boldsymbol{L}^m$ are stiffness matrices of fiber and matrix, respectively.

Discretizing the integral equations in the sense described in Section 4, we get the discretized strains in each subdomain:

$$\boldsymbol{\varepsilon}^f = \boldsymbol{A}^f \boldsymbol{E}, \ \boldsymbol{\varepsilon}^m = \boldsymbol{A}^m \boldsymbol{E}. \tag{27a}$$

From the latter equations it immediately follows that

$$\boldsymbol{A}^f + \boldsymbol{A}^m = \boldsymbol{I}. \tag{27b}$$

where $\boldsymbol{I}$ is the unit matrix (unit tensor). The last relation makes clear why we concentrated our attention on the splitting of integrals into integration over Ω^f and over Ω^m. Because of (27b) we can simply calculate integrals either over matrix or fiber, exceptionally. Then, using (27b), we get the second needed concentration factor.

Hence, in the case of elastic behavior, the homogenization is straightforward:

$$\Sigma = \langle \boldsymbol{\sigma} \rangle_\Omega = \left\langle \overline{\boldsymbol{L}}^f \boldsymbol{\varepsilon}^f \right\rangle_{\Omega^f} + \left\langle \overline{\boldsymbol{L}}^m \boldsymbol{\varepsilon}^m \right\rangle_{\Omega^m}$$

$$= \left(\left\langle \overline{\boldsymbol{L}}^f \boldsymbol{A}^f \right\rangle_{\Omega^f} + \left\langle \overline{\boldsymbol{L}}^m \boldsymbol{A}^m \right\rangle_{\Omega^m} \right) \boldsymbol{E}. \tag{28}$$

Since we deal with discretized fields, $\overline{\boldsymbol{L}}^\alpha$, $\alpha = f, m$, have the form:

$$\overline{\boldsymbol{L}}^\alpha = \begin{bmatrix} \boldsymbol{L}^\alpha & 0 & 0 & \cdots & 0 \\ 0 & \boldsymbol{L}^\alpha & 0 & \cdots & 0 \\ \vdots & & \ddots & & \\ 0 & 0 & 0 & \cdots & \boldsymbol{L}^\alpha \end{bmatrix}, \tag{29}$$

and $\boldsymbol{L}^\alpha$ is a $(3M \times 3M)$ matrix under the assumption that we have ordered tensors $\boldsymbol{\varepsilon}$, and so on, in the vector form, as is usual in numerical applications. The overall stiffness matrix $\boldsymbol{L}^*$ becomes,

$$\boldsymbol{L}^* = \left\langle \overline{\boldsymbol{L}}^f \boldsymbol{A}^f \right\rangle_{\Omega^f} + \left\langle \overline{\boldsymbol{L}}^m \boldsymbol{A}^m \right\rangle_{\Omega^m}. \tag{30}$$

It is worth noting that multiplying (20) by the appropriate stiffness matrix and discretizing the integral equation into boundary elements and internal cells, we get stress concentration function $\boldsymbol{B}^\alpha$ as

$$\boldsymbol{\sigma}^\alpha = \boldsymbol{B}^\alpha \boldsymbol{\Sigma}. \tag{31}$$

Note that in a general case the relations (31) is calculated from given $\boldsymbol{\Sigma}$. This leads to similar integral equations, but the solution is not unique. Since the external forces are equilibrated, the rigid motion of the unit cell is disregarded, and the solution is then unique. Here we use the relation (28) and the possibility to invert $\boldsymbol{L}^*$.

Including (5c) in the calculation, one can derive from (20) and discretization of integral equations the relations

$$\boldsymbol{\varepsilon}^\alpha = \boldsymbol{A}^\alpha \boldsymbol{E} + \boldsymbol{D}_{\alpha k} \boldsymbol{\mu}^k, \ \boldsymbol{\sigma}^\alpha = \boldsymbol{B}^\alpha \boldsymbol{\Sigma} + \boldsymbol{F}_{\alpha k} \boldsymbol{\lambda}^k,$$

$$\alpha = f, m, \ k = 1, \ldots, M. \tag{32}$$

The relations (32) are starting points for theories established by Dvorak (1991). He assumes that the concentration functions are estimated by approximate formulas following Mori-Tanaka, or the Self-consistent method. The calculations presented in this paper may be very accurate and fast.

VI. EXAMPLE

In the examples we did not used the useful relation (27a), in order to enhance the accuracy of the computation. Since the procedure leads to a *linear relation* at each stage of $\boldsymbol{E}$, for given (increasing step by step) values of components of either the stress tensor (elastic and relaxation part) or strain tensor (elastic and plastic part) of "unit impulses" we need not compute influence tensors. Now the procedure fully described in Dvorak (1992) can be used. Note that in Dvorak (1992) the values of concentration tensors, which are the most important quantities for numerical computation, are computed by very approximate methods.

The quarter of a unit cell is considered with fiber volume ratio equal to 0.6, according to Fig. 2. We used the following elastic material properties of phases: Young's modulus of fiber E^f=210 MPa, Poisson's ratio ν^f=0.16; on the matrix E^m=17 MPa, and ν^m=0.3. Moreover, the ideally elasto-plastic constitutive law is imposed on the matrix, whereas the fiber remains elastic in every stage of loading and unloading.

For fiber volume ratio 0.6, the radius of the fiber is r=0.714. The homogenized elastic matrix $\boldsymbol{L}^*$ in this case possessed the following values:

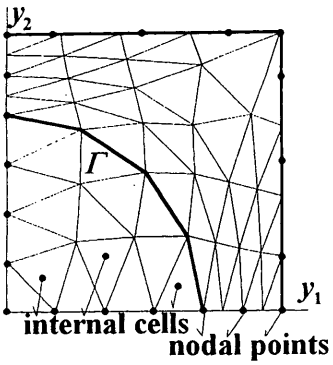

Fig. 2. Discretization.

$$(\boldsymbol{L}^*)^T = \begin{bmatrix} 182 & 62 & -0.05 \\ 62 & 182 & 0.034 \\ -0.75 & -1.1 & 98 \end{bmatrix}$$

From the above matrix one can conclude that the responses on normal unit strains are computed with high accuracy (comparing the symmetry), while the results from shearing strains are less accurate, but still very precise.

In Fig. 2, the mesh of internal cells and boundary elements is depicted. In Fig. 3 a draft of ideally elasto-plastic material behaviour of the matrix is shown.

Using von Mises's hypothesis without hardening, and choosing σ_0=10 MPa, E=0.65 is the limit value for elastic behaviour of the matrix. Increasing $\boldsymbol{E}$, and using von Mises's hypothesis, (Dvorak, 1991), we get an elasto-plastic overall stiffness matrix $(\boldsymbol{L}^*)^{ep}$. In this step we use (32), where we apply iterative computations on linear relations (it is therefore possible only to multiply matrices, not solve equations) to get the appropriate values of $\boldsymbol{\lambda}^k$ in the internal cells k. From the relation (5c) we obtain eigenstresses, and, consequently, the appropriate strains.

The number of iterations depends on the difference between the currently applied $\boldsymbol{E}$ and the previous one. We applied seven steps to reach the value 0.1. At each step ten iterations were supposed to get the one percent error.

Recall that a very important property of this procedure is the fixed, unchanged values of the matrices $\boldsymbol{F}_{\alpha k}$ and $\boldsymbol{D}_{\alpha k}$. Although the mesh was not very fine, the results were very reasonable, see Fig. 2 and the overall matrix.

The computation was run on a Pentium II PC, 366 MHz in FORTRAN. The program for generation of meshes of internal cells as well as the boundary nodes had been prepared, as obviously seen from Fig. 2. According to the wish of the user, the meshing can be improved. The consumption of time

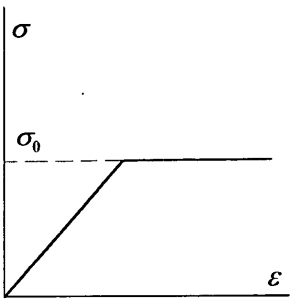

Fig. 3. Ideally elasto-plastic σ–ε diagram.

for computation of even large systems of equations (150×150), which can be stored in memory without use of hard disc or extended/expanded memory was negligible in each step of $\boldsymbol{E}$. Our illustration does not reach such dimensions of computation. It is also not necessary for such problems to increase the precision of the meshing, it loses efficiency. The iterations at each step of loading were also very fast.

It is worth noting that similar computations were carried out by the FEM, but finer meshing had to be imposed to get a result comparable with the BEM in the procedure presented. The comparison has been tested in such a way that the sum of the concentration factors should be the unit tensor.

VII. CONCLUSIONS

In this contribution we have presented the fundamental idea of a numerical procedure leading to an overall elasto-plastic stiffness matrix on a unit cell. Based on Transformation Field Analysis it is possible to obtain the *strain* and *stress influence matrices*, relating the strains and stresses and on the other hand the eigenstrains and eigenstresses. Since the problem leads to integral equations, the most suitable numerical tool appears to be the BEM. The obvious advantage of this procedure can be found in *a priori* computed influence matrices (concentration tensors) $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{D}$, and $\boldsymbol{F}$ in (32). They may be stored into a computer and hence, the iteration process for solving nonlinear material behavior of structures is very efficient. Moreover, a concentration may be focused on integrals over either the matrix or the fiber.

A very important property of the above procedure is the linearity of the problem at each stage of $\boldsymbol{E}$. The accuracy of the overall elasto-plastic stiffness (compliance) matrix is not dependant of the size of the step of $\boldsymbol{E}$, providing there is not "unwanted" unloading in any internal cell at the current stage of $\boldsymbol{E}$. When it is so, the time prolongs slightly, as the standard iterative process has to be carried out. It was not the case in our computations.

Although there is no intent to discuss cases of eigenstrain fields, it is appropriate to mention the

connection between the present formulas and those obtained in studies of the thermoelastic response in composite materials subjected to a uniform change of temperature θ. The corresponding eigenstrains assume the form $\mu_{ij}=m\theta\delta_{ij}$, where δ_{ij} is Kronecker's delta, and m is the thermal expansion coefficient. There are some methods which are based on homogenization starting with the change of temperature. Such methods may also use the above integral formulations and solution by the BEM.

ACKNOWLEDGEMENT

This paper was supported by GAČR, grant # 103/99/0756.

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Discussions of this paper may appear in the discussion section of a future issue. All discussions should be submitted to the Editor-in-Chief.

Manuscript Received: Aug. 04, 1998
Revision Received: May 10, 1999
and Accepted: May 15, 1999

非線性的複合材料使用轉換場分析的均質化行爲

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摘 要

在90年代的早期，Dvorak (1991)，及Dvorak和Prochazla (1996)，提出轉換場分析(TFA)。它包含表達特徵應變或特徵應力與應變和應力之間的線性關係。借助這些特徵參數，材料中的力學和熱的現象能被模擬出來。在這篇論文中，TFA被應用於非線性的複合材料的均質化。邊界元素法是非常有用的數值方法，已應用於解積分方程式。一個範例顯示出本文方法的可行性。雖然此例題在微觀尺度的單位晶格下討論，但也可推廣到隨機分佈纖維的整體單元問題。

關鍵字：非線性複合材料行爲，均質化行爲，轉換場分析，邊界元素法。