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A COMPLEX VARIABLE BOUNDARY ELEMENT METHOD FOR ANTIPLANE STRESS ANALYSIS AROUND A CRACK IN SOME NONHOMOGENEOUS BODIES

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ABSTRACT

A boundary element method based on the Cauchy integral formulae, i.e. a complex variable boundary element method (CVBEM), is proposed for the numerical solution of an antiplane crack problem involving an elastic body with shear modulus that varies continuously in space. The shear modulus assumes a certain form which is quite general to allow for multiparameter fitting of its variation. The method reduces the problem to a system of linear algebraic equations and can be readily implemented on the computer. For clarity, the CVBEM formulation is first carried out for a straight crack and then its extension to include an arbitrary curved crack is indicated.

I. INTRODUCTION

The boundary element method (BEM) is a useful and efficient numerical technique for stress analysis in solids. Its application to problems involving cracks is, however, not a straightforward task, as it is not easy to model the opposite crack faces that are distinct yet lie on one and the same surface. Furthermore, the displacement field changes rapidly near the edge of a crack. For further details on the difficulties involved in the numerical solution of crack problems, refer to Aliabadi and Rooke (1991).

During the last two decades or so, significant progress has been made in the use of the BEM as a numerical tool for crack problems. There are now several BEM strategies for solving crack problems accurately. The approach which avoids integration over the crack faces through the derivation of suitable Green's functions was pioneered by Snyder and Cruse (1975) in the 1970s and extended to more complicated problems by other investigators, e.g. Clements and Haselgrove (1983), Ang and Clements (1987), Ang (1987) and Telles, *et al.* (1997). Another approach by Ang (1994) and Chen and Chen (1995) which employs the usual boundary integral equations to deal with the conditions on the exterior boundary of the solid but uses a differentiated form of the integral equations to express the conditions on the crack faces leads to hypersingular-boundary integral equations that are numerically tractable. For two-dimensional elastostatic crack problems, a complex variable approach to the BEM is possible. Denda and Dong (1997) introduced one such approach for solving problems involving straight cracks in homogeneous isotropic bodies. More recently, Ang, *et al.* (2000) proposed a different version of the complex variable BEM (CVBEM) for the numerical solution of a curved crack in a homogeneous anisotropic body.

The CVBEM is based on the Cauchy integral formulae. Apparently, it was first introduced in the 1980s by Hromadka II and Lai (1987) for solving boundary value problems governed by the two-dimensional Laplace's equation. More recently, introducing the theory of complex hypersingular integrals,

Linkov and Mogilevskaya (1994, 1997, 1998) described a CVBEM formulation for certain boundary value problems in plane isotropic elastostatics. Ang and Park (1998) extended the approach to a generalized system of second-order elliptic partial differential equations. Application of the CVBEM for the numerical solution of an anisotropic thermoelastic problem was carried out by Ang, *et al.* (1999).

In the present paper, we propose a CVBEM (which follows quite closely that of Ang, *et al.* (2000)) to solve the problem of a straight crack in a nonhomogeneous isotropic elastic body under antiplane deformation. The shear modulus μ of the body varies with the Cartesian spatial coordinates x_1 and x_2 and takes the form given by (6) (refer to Section 2). Examples of multiparameter forms which μ can assume include

$$\mu=(a_0+a_1x_1+a_2x_2)^2$$

$$\mu=(\mathrm{Re}\{a_0+a_1(x_1+ix_2)+a_2(x_1+ix_2)^2+\ldots+a_N(x_1+ix_2)^N\})^2 \tag{1}$$

where $i=\sqrt{-1}$ and a_k are constants which may be chosen to fit the variation of the shear modulus. The CVBEM reduces the crack problem to a system of linear algebraic equations. For some specific problems, the system of linear algebraic equations is set up and solved using a computer. Once the system is solved, the relevant stress intensity factors at the crack tips are computed. An extension of the proposed CVBEM to include arbitrary curved cracks is also discussed.

II. STATEMENT OF THE PROBLEM

With reference to a Cartesian coordinate frame $0x_1x_2x_3$, consider an isotropic elastic body whose geometry does not vary in the x_3-direction. The interior of the body contains a crack. For clarity in presentation, let us first consider the case where the crack is straight and lying in the region $-a<x_1<a$, $x_2=0$, $-\infty<x_3<\infty$, where a is a given positive number. (An extension of the problem to include an arbitrary curved crack is discussed in Section 6.) On the $x_3=0$ plane, the exterior boundary of the body is the simple closed curve C, the crack is a straight cut of finite length $2a$ with endpoints $(-a,0)$ and $(a,0)$ and the region enclosed by C with the cut is R. We assume that the crack does not intersect the exterior boundary C.

At each and every point on the exterior boundary of the body, either the Cartesian displacement u_k or traction T_k is prescribed in such a way that the crack becomes traction-free. The specified displacement or traction on the exterior boundary is assumed to be independent of time and the coordinate x_3 and such that $u_1=u_2=0$ or $T_1=T_2=0$, i.e. the body is assumed to undergo an antiplane deformation. The problem is then to determine the displacement $u_3(x_1, x_2)$ or the stress $\sigma_{k3}(x_1, x_2)$ throughout the body. Of particular interest is the calculation of the stress intensity factors at the tips of the crack.

Mathematically, the problem is to solve

$$\frac{\partial}{\partial x_1}(\mu\frac{\partial u_3}{\partial x_1})+\frac{\partial}{\partial x_2}(\mu\frac{\partial u_3}{\partial x_2})=0 \text{ in } R, \tag{2}$$

subject to

$$u_3=w(x_1, x_2) \text{ on } C_1, \tag{3}$$

$$\mu[n_1\frac{\partial u_3}{\partial x_1}+n_2\frac{\partial u_3}{\partial x_2}]=p(x_1, x_2) \text{ on } C_2, \tag{4}$$

$$\lim_{x_2\to 0}\mu\frac{\partial u_3}{\partial x_2}=0 \quad \text{for } -a<x_1<a, \tag{5}$$

where $\mu>0$ is the shear modulus of the material occupying the body, w and p are suitably prescribed functions of x_1 and x_2, $[n_1, n_2]$ is the unit normal vector to C pointing away from R, and C_1 and C_2 are non-intersecting curves such that $C=C_1\cup C_2$.

For homogeneous materials, the shear modulus μ is a constant and the equilibrium Eq. (2) reduces to the two-dimensional Laplace's equation. In the present work, we take the shear modulus to be a spatial function of the form

$$\mu(x_1, x_2)=(\mathrm{Re}(g(x_1+ix_2)\})^2, \tag{6}$$

where $i=\sqrt{-1}$ and g is an arbitrary holomorphic function of the complex variable $z=x_1+ix_2$ in $R\cup C$ such that $g\neq 0$ for any $(x_1, x_2)\in R\cup C$. Notice that (6) implies that $\mu^{1/2}$ satisfies the two-dimensional Laplace's equation in $R\cup C$. Although this places some restriction on the choice of μ, it does allow for rather general multiparameter forms like the one in (1). In some other work on cracks in nonhomogeneous bodies, investigators assume even more restrictive form on the shear modulus, e.g. linear or exponential variations. As we shall see, the choice of (6) is to allow (2) to be transformed to the two-dimensional Laplace's equation.

With (6), if we make the substitution

$$u_3(x_1, x_2)=\mu^{-1/2}\phi(x_1, x_2), \tag{7}$$

we find that (2) transforms to become

$$\frac{\partial^2\phi}{\partial x_1^2}+\frac{\partial^2\phi}{\partial x_2^2}=0 \quad \text{in } R, \tag{8}$$

and (3), (4) and (5) become

$$\phi=\mu^{1/2}w(x_1, x_2) \text{ on } C_1, \tag{9}$$

$$\mu[n_1\frac{\partial\phi}{\partial x_1}+n_2\frac{\partial\phi}{\partial x_2}]-\frac{1}{2}\phi[n_1\frac{\partial\mu}{\partial x_1}+n_2\frac{\partial\mu}{\partial x_2}]$$

$$=\mu^{1/2}p(x_1,x_2) \text{ on } C_2, \tag{10}$$

$$\lim_{x_2\to 0}[\mu\frac{\partial\phi}{\partial x_2}-\frac{1}{2}\frac{\partial\mu}{\partial x_2}\phi]$$

$$=0 \text{ for } -a<x_1<a. \tag{11}$$

The Laplace's Eq. (8) admits solution of the general form

$$\phi(x_1, x_2)=\mathrm{Re}\{f(x_1+ix_2)\}, \tag{12}$$

where f is a holomorphic function of $z=x_1+ix_2$ in $R\cup C$.

In view of (12), the crack problem under consideration may be formulated as a mathematical problem which requires the construction of a complex function f which is holomorphic in $R\cup C$ and which satisfies the conditions

$$\mathrm{Re}\{f(x_1+ix_2)\}=\mu^{1/2}w(x_1, x_2) \text{ on } C_1, \tag{13}$$

$$\mathrm{Re}\{(n_1+in_2)\mu f'(x_1+ix_2)-\frac{1}{2}[n_1\frac{\partial\mu}{\partial x_1}+n_2\frac{\partial\mu}{\partial x_2}]f(x_1+ix_2)\}$$

$$=\mu^{1/2}p(x_1, x_2) \text{ on } C_2, \tag{14}$$

$$\lim_{x_2\to 0}\mathrm{Re}\{i\mu f'(x_1+ix_2)-\frac{1}{2}\frac{\partial\mu}{\partial x_2}f(x_1+ix_2)\}$$

$$=0 \text{ for } -a<x_1<a. \tag{15}$$

IV. CVBEM

Visualizing the straight crack as an elliptical hole $x_1^2/a^2+x_2^2/\epsilon^2<1$ with $\epsilon\to 0$ and applying the Cauchy integral formulae for the holomorphic function f in $R\cup C$, for $(\xi_1, \xi_2)\in R$, we obtain

$$2\pi i f(\xi_1+i\xi_2)=\oint_C\frac{f(z)dz}{z-(\xi_1+i\xi_2)}+\int_{-a}^{a}\frac{F(x)dx}{x-(\xi_1+i\xi_2)}, \tag{16}$$

$$2\pi i f'(\xi_1+i\xi_2)=\oint_C\frac{f(z)dz}{[z-(\xi_1+i\xi_2)]^2}+\int_{-a}^{a}\frac{F(x)dx}{[x-(\xi_1+i\xi_2)]^2}, \tag{17}$$

where C is assigned a counterclockwise direction and

$$F(x)=\lim_{y\to 0}[f(x+i|y|)-f(x-i|y|)] \text{ for } -a<x<a. \tag{18}$$

From either (17) or (18), it can be shown that

$$F'(x)=\lim_{y\to 0}[f'(x+i|y|)-f'(x-i|y|)] \text{ for } -a<x<a, \tag{19}$$

where $F'(x)=dF/dx$.

Thus, from (18) and (19), to ensure the existence of the limit on the left hand side of (15), we impose the condition

$$-\mu(x,0)B'(x)-\frac{1}{2}\frac{\partial\mu}{\partial x_2}\bigg|_{(x_1,x_2)=(x,0)}A(x)=0$$

$$\text{for } -a<x<a, \tag{20}$$

if we write $F(x)$ in the form $F(x)=A(x)+iB(x)$ where A and B are *real* functions of the *real* variable x.

We shall apply (16) and (17) together with (13)-(15) and (20) to construct the required holomorphic function f. We proceed as follows.

Put M well spaced out points $(x_1^{(1)}, x_2^{(1)})$, $(x_1^{(2)}, x_2^{(2)})$, ..., $(x_1^{(M-1)}, x_2^{(M-1)})$ and $(x_1^{(M)}, x_2^{(M)})$ on C in a counterclockwise order. For $k=1, 2, ..., M$, define $C^{(k)}$ to be the straight line segment from $(x_1^{(k)}, x_2^{(k)})$ to $(x_1^{(k+1)}, x_2^{(k+1)})$ where $(x_1^{(M+1)}, x_2^{(M+1)})=(x_1^{(1)}, x_2^{(1)})$. We make the approximation

$$C\approx C^{(1)}\cup C^{(2)}\cup\ldots\cup C^{(M-1)}\cup C^{(M)}. \tag{21}$$

For $(\xi_1, \xi_2)\in R$, we rewrite (16) (approximately) as:

$$2\pi i f(\xi_1+i\xi_2)=\sum_{m=1}^{M}\int_{C^{(m)}}\frac{f(z)dz}{z-(\xi_1+i\xi_2)}+\int_{-a}^{a}\frac{F(x)dx}{x-(\xi_1+i\xi_2)}, \tag{22}$$

To evaluate the integral over $C^{(m)}$, we expand $f(z)$ as a Taylor-Maclaurin series about $z=\hat{z}^{(m)}$ where $\hat{z}^{(m)}=(z^{(m)}+z^{(m+1)})/2$ and $z^{(m)}=x_1^{(m)}+ix_2^{(m)}$, i.e.

$$f(z)=f(\hat{z}^{(m)})+(z-\hat{z}^{(m)})f'(\hat{z}^{(m)})+\frac{1}{2}(z-\hat{z}^{(m)})^2f''(\hat{z}^{(m)})+\ldots. \tag{23}$$

It follows that

$$\int_{C^{(m)}}\frac{f(z)dz}{z-(\xi_1+i\xi_2)}=f(\hat{z}^{(m)})\int_{C^{(m)}}\frac{f(z)dz}{z-(\xi_1+i\xi_2)}$$

$$+f'(\hat{z}^{(m)})\int_{C^{(m)}}\frac{(z-\hat{z}^{(m)})dz}{z-(\xi_1+i\xi_2)}$$

$$+\frac{1}{2}f''(\hat{z}^{(m)})\int_{C^{(m)}}\frac{(z-\hat{z}^{(m)})^2dz}{z-(\xi_1+i\xi_2)}+\ldots. \tag{24}$$

If we ignore all terms whose magnitudes are $O(|z^{(m+1)}-z^{(m)}|^2)$ in (24), we obtain the approximation

$$\int_{C^{(m)}}\frac{f(z)dz}{z-(\xi_1+i\xi_2)}\approx f(\hat{z}^{(m)})\int_{C^{(m)}}\frac{dz}{z-(\xi_1+i\xi_2)}$$

$$\text{for } (\xi_1, \xi_2)\in R. \tag{25}$$

If we write

$$f(\hat{z}^{(m)})=\phi^{(m)}+i\psi^{(m)}, \tag{26}$$

where $\phi^{(m)}$ and $\psi^{(m)}$ are constants (yet to determined), with (25), we find that (22) can be approximately replaced by

$$\begin{aligned}&2\pi i f(\xi_1+i\xi_2)\\&=\sum_{m=1}^{M}(\phi^{(m)}+i\psi^{(m)})[\gamma(z^{(m)},z^{(m+1)},\xi_1+i\xi_2)\\&\quad+i\theta(z^{(m)},z^{(m+1)},\xi_1+i\xi_2)]\\&\quad+\int_{-a}^{a}\frac{F(x)dx}{x-(\xi_1+i\xi_2)}\quad\text{for }(\xi_1,\xi_2)\text{ in }R,\end{aligned} \tag{27}$$

where

$$\theta(z,w,c)=\begin{cases}\Omega(z,w,c) & \text{if}\quad -\pi\le\Omega(z,w,c)\le\pi\\ \Omega(z,w,c)+2\pi & \text{if}\quad -2\pi\le\Omega(z,w,c)<-\pi\\ \Omega(z,w,c)-2\pi & \text{if}\quad \pi<\Omega(z,w,c)\le 2\pi\end{cases}$$

$$\Omega(z,w,c)=\mathrm{Arg}(w-c)-\mathrm{Arg}(z-c),$$

$$\gamma(z,w,c)=\ln|w-c|-\ln|z-c|, \tag{28}$$

where $\mathrm{Arg}(z)$ denotes the principal value of the argument of the complex number z.

If the simple closed curve C is such that the region it encloses is convex, then for $c\in R$ and z and w lying on C, $\theta(z,w,c)$ can be computed directly from

$$\theta(z,w,c)=\cos^{-1}\left(\frac{|w-c|^2+|z-c|^2-|w-z|^2}{2|w-c||z-c|}\right). \tag{29}$$

If we push the point (ξ_1,ξ_2) in (27) to approach (from within R) the midpoint of $C^{(p)}$ then the real part of (27) gives (for $p=1, 2, \ldots, M$)

$$\begin{aligned}-2\pi\psi^{(p)}&=\sum_{m=1}^{M}\{\phi^{(m)}\gamma(z^{(m)},z^{(m+1)},\hat{z}^{(p)})\\&\quad-\psi^{(m)}\theta(z^{(m)},z^{(m+1)},\hat{z}^{(p)})\}\\&\quad+\mathrm{Re}\left\{\int_{-a}^{a}\frac{[A(x)+iB(x)]dx}{x-\hat{z}^{(p)}}\right\}.\end{aligned} \tag{30}$$

Notice that in (30) $\theta(z^{(p)},z^{(p+1)},\hat{z}^{(p)})=\pi$ and $\gamma(z^{(p)},z^{(p+1)},\hat{z}^{(p)})=0$.

The system (30) consists of M equations but there are $2M$ unknown constants $\phi^{(m)}$ and $\psi^{(m)}$ ($m=1, 2, \ldots, M$) and two unknown real functions $A(x)$ and $B(x)$ ($-a<x<a$). More equations are obviously needed to complete the system. They come from (13)-(15) and (20).

Condition (13) gives

$$\phi^{(p)}=\mu^{1/2}(\hat{x}_1^{(p)},\hat{x}_2^{(p)})w(\hat{x}_1^{(p)},\hat{x}_2^{(p)})$$

$$\text{if } u_3 \text{ is specified on } C^{(p)}, \tag{31}$$

where $(\hat{x}_1^{(p)},\hat{x}_2^{(p)})$ is the midpoint of $C^{(p)}$.

To deal with (14), for $(\xi_1,\xi_2)\in R$, we rewrite (17)

$$\begin{aligned}2\pi i f'(\xi_1+i\xi_2)&=\sum_{m=1}^{M}\int_{C^{(m)}}\frac{f(z)dz}{[z-(\xi_1+i\xi_2)]^2}\\&\quad+\int_{-a}^{a}\frac{F(x)dx}{[x-(\xi_1+i\xi_2)]^2}.\end{aligned} \tag{32}$$

Proceeding, as before, in calculating the integral over $C^{(m)}$ and omitting terms having magnitude $O(|z^{(m+1)}-z^{(m)}|^2)$, we obtain

$$\int_{C^{(m)}}\frac{f(z)dz}{[z-(\xi_1+i\xi_2)]^2}\approx f(\hat{z}^{(m)})\int_{C^{(m)}}\frac{dz}{[z-(\xi_1+i\xi_2)]^2}$$

$$\text{for }(\xi_1,\xi_2)\in R. \tag{33}$$

Furthermore, if we repeat the task of calculating the integral over $C^{(m)}$ but with (ξ_1,ξ_2) approaching $(\hat{x}_1^{(m)},\hat{x}_2^{(m)})$ (from within R), we find that

$$\int_{C^{(m)}}\frac{f(z)dz}{[z-\hat{z}^{(m)})]^2}\approx f(\hat{z}^{(m)})\int_{C^{(m)}}\frac{dz}{[z-\hat{z}^{(m)}]^2}+\pi i f'(\hat{z}^{(m)}), \tag{34}$$

after neglecting terms having magnitude $O(|z^{(m+1)}-z^{(m)}|)$.

Together with (32) and (34), condition (14) gives

$$\begin{aligned}&\frac{1}{\pi}\mu(\hat{x}_1^{(p)},\hat{x}_2^{(p)})\sum_{m=1}^{M}\{[n_1^{(p)}r(z^{(m)},z^{(m+1)},\hat{z}^{(p)})\\&+n_2^{(p)}q(z^{(m)},z^{(m+1)},\hat{z}^{(p)})]\phi^{(m)}\\&+[n_1^{(p)}q(z^{(m)},z^{(m+1)},\hat{z}^{(p)})-n_2^{(p)}r(z^{(m)},z^{(m+1)},\hat{z}^{(p)})]\psi^{(m)}\}\\&+\frac{1}{\pi}\mu(\hat{x}_1^{(p)},\hat{x}_2^{(p)})\mathrm{Re}\{(n_2^{(p)}-in_1^{(p)})\int_{-a}^{a}\frac{[A(t)+iB(t)]dt}{(t-\hat{z}^{(p)})^2}\}\\&-\frac{1}{2}[n_1^{(p)}\frac{\partial\mu}{\partial x_1}+n_2^{(p)}\frac{\partial\mu}{\partial x_2}]\Big|_{(x_1,x_2)=(\hat{x}_1^{(p)},\hat{x}_2^{(p)})}\phi^{(p)}\\&=\mu^{1/2}(\hat{x}_1^{(p)},\hat{x}_2^{(p)})p(\hat{x}_1^{(p)},\hat{x}_2^{(p)})\end{aligned}$$

$$\text{if the traction } T_3 \text{ is specified on } C^{(p)}, \tag{35}$$

where

$$q(z, w, c) + ir(z, w, c) = \frac{-1}{w-c} + \frac{1}{z-c}. \tag{36}$$

Condition (15) can be rewritten as

$$\mu(x, 0)[\sum_{m=1}^{M} \{q(z^{(m)}, z^{(m+1)}, x)\phi^{(m)} - r(z^{(m)}, z^{(m+1)}, x)\psi^{(m)}\}$$

$$+ \mathcal{H}\int_{-a}^{a} \frac{A(t)dt}{(t-x)^2}] - \frac{1}{2}\frac{\partial\mu}{\partial x_2}\bigg|_{x_2=0} [\sum_{m=1}^{M} \{\theta(z^{(m)}, z^{(m+1)}, x)\phi^{(m)}$$

$$+ \lambda(z^{(m)}, z^{(m+1)}, x)\psi^{(m)}\} + \mathcal{P}\int_{-a}^{a} \frac{B(t)dt}{(t-x)}] = 0$$

for $-a<x<a$, (37)

where $\mathcal{P}$ and $\mathcal{H}$ denote that the real integrals over $(-a, a)$ are to be interpreted in the Cauchy principal and Hadamard finite-part sense respectively, i.e. more specifically (for $-a<x<a$)

$$\mathcal{P}\int_{-a}^{a} \frac{B(t)dt}{[t-x]} \overset{\text{def}}{=} \lim_{\varepsilon\to 0^+} \int_{-a}^{a} \frac{(t-x)B(t)dt}{[(t-x)^2+\varepsilon^2]}$$

$$\mathcal{H}\int_{-a}^{a} \frac{A(t)dt}{[t-x]^2} \overset{\text{def}}{=} \lim_{\varepsilon\to 0^+} \{\int_{-a}^{a} \frac{(t-x)^2 A(t)dt}{[(t-x)^2+\varepsilon^2]^2} - \frac{\pi}{2\varepsilon}A(x)\}. \tag{38}$$

The unknown function $A(t)$ $(-a<t<a)$ is directly related to the 'crack-opening displacement.' To solve (20), (31), (35) and (37) for the unknown constants $\phi^{(m)}$ and $\psi^{(m)}$ and the unknown functions $A(t)$ and $B(t)$, there are several approaches which can be used for the approximation of $A(t)$. One such approach is to approximate $A(t)$ in the style of Kaya and Erdogan (1987), i.e.

$$A(t) \approx \sqrt{a^2-t^2}\sum_{j=1}^{J} c_j U_{j-1}(t/a) \quad \text{for } -a<t<a, \tag{39}$$

where c_k are real coefficients yet to be determined and $U_k(x)$ denotes the k-th order Chebyshev polynomial of the second kind.

From (20), if we write

$$B(x) = -\frac{1}{2}\int_{-a}^{x} \frac{1}{\mu(t,0)}\frac{\partial\mu}{\partial x_2}\bigg|_{(x_1,x_2)=(t,0)} A(t)dt$$

for $-a<x<a$, (40)

then (30) together with (39) gives (for p=1, 2, ..., M)

$$-2\pi\psi^{(p)} = \sum_{m=1}^{M} \{\phi^{(m)}\chi(z^{(m)}, z^{(m+1)}, \hat{z}^{(p)})$$

$$- \psi^{(m)}\theta(z^{(m)}, z^{(m+1)}, \hat{z}^{(p)})\}$$

$$+ \sum_{j=1}^{J} c_j[\int_{-a}^{a} \sqrt{a^2-t^2}\, U_{j-1}(t/a)(\mathrm{Re}\{\frac{1}{t-\hat{z}^{(p)}}\}$$

$$+ \frac{1}{2}\frac{\theta(t, a, \hat{z}^{(p)})}{\mu(t,0)}\frac{\partial\mu}{\partial x_2}\bigg|_{(x_1,x_2)=(t,0)})dt], \tag{41}$$

and (35) becomes

$$\frac{1}{\pi}\mu(\hat{x}_1^{(p)}, \hat{x}_2^{(p)})\sum_{m=1}^{M} \{[n_1^{(p)}r(z^{(m)}, z^{(m+1)}, \hat{z}^{(p)})$$

$$+ n_2^{(p)}q(z^{(m)}, z^{(m+1)}, \hat{z}^{(p)})]\phi^{(m)}$$

$$+ [n_1^{(p)}q(z^{(m)}, z^{(m+1)}, \hat{z}^{(p)}) - n_2^{(p)}r(z^{(m)}, z^{(m+1)}, \hat{z}^{(p)})]\psi^{(m)}\}$$

$$+ \frac{1}{\pi}\mu(\hat{x}_1^{(p)}, \hat{x}_2^{(p)})\sum_{j=1}^{J} c_j \mathrm{Re}\{(n_2^{(p)} - in_1^{(p)})\int_{-a}^{a} \sqrt{a^2-t^2}\, U_{j-1}(t/a)$$

$$[\frac{1}{\mu(t,0)}\frac{\partial\mu}{\partial x_2}\bigg|_{(x_1,x_2)=(t,0)} \phi^{(p)}]dt\}$$

$$-\frac{(t-\hat{z}^{(p)})}{2}[n_1^{(p)}\frac{\partial\mu}{\partial x_1} + n_2^{(p)}\frac{\partial\mu}{\partial x_2}]\bigg|_{(x_1,x_2)=(\hat{x}_1^{(p)},\hat{x}_2^{(p)})} - \frac{i}{2}\frac{(q(t, a, \hat{z}^{(p)}) + ir(t, a, \hat{z}^{(p)}))}{\mu(t,0)}\frac{\partial\mu}{\partial x_2}\bigg|_{(x_1,x_2)=(t,0)}$$

$$= \mu^{1/2}(\hat{x}_1^{(p)}, \hat{x}_2^{(p)})p(\hat{x}_1^{(p)}, \hat{x}_2^{(p)})$$

if the traction T_3 is specified on $C^{(p)}$. (42)

If we collocate (37) by choosing x to be given (in turn) by

$$x = y^{(n)} = a\cos(\frac{[2n-1]\pi}{2J}) \quad \text{for } n=1, 2, ..., J, \tag{43}$$

then using (39) we obtain

$$\mu(y^{(n)},0)\sum_{m=1}^{M} \{q(z^{(m)},z^{(m+1)},y^{(n)})\phi^{(m)} - r(z^{(m)},z^{(m+1)},y^{(n)})\psi^{(m)}\}$$

$$- \pi\mu(y^{(n)}, 0)\sum_{j=1}^{J} jc_j U_{j-1}(a^{-1}y^{(n)}) - \frac{1}{2}\frac{\partial\mu}{\partial x_2}\bigg|_{(x_1,x_2)=(y^{(n)},0)}$$

$$\times \{\sum_{m=1}^{M} \{\theta(z^{(m)}, z^{(m+1)}, y^{(n)})\phi^{(m)} + \lambda(z^{(m)}, z^{(m+1)}, y^{(n)})\psi^{(m)}\}$$

$$- \frac{1}{2}\sum_{j=1}^{J} c_j[\int_{-a}^{a} \frac{\{D_j(t) - D_j(y^{(n)})\}dt}{(t-y^{(n)})} + D_j(y^{(n)})\ln\left|\frac{a-y^{(n)}}{a+y^{(n)}}\right|]\}$$

$=0$ for n=1, 2, ..., J, (44)

where

$$D_j(t)=\int_{-a}^{t}\frac{1}{\mu(\xi,0)}\frac{\partial\mu}{\partial x_2}\bigg|_{(x_1,x_2)=(\xi,0)}U_{j-1}(\xi/a)\sqrt{a^2-\xi^2}\,d\xi\,. \tag{45}$$

Now (31), (41), (42) and (44) constitute a system of linear algebraic equations in the unknowns $\phi^{(m)}$ and $\psi^{(m)}$ (m=1, 2, ..., M) and c_j (j=1, 2, ..., J). Once these unknowns are determined, the function f (and hence u_3) can be computed at any point (ξ_1, ξ_2) in R.

A much easier-to-implement method of solving (20), (31), (35) and (37) is to discretize the crack into smaller elements and approximate $A(t)$ as either a constant or a linear function over a crack element. In general, such a simple approximation of $A(t)$, that ignores the asymptotic behavior of the 'crack-opening displacement' near the crack tips, cannot be expected to yield highly accurate results if fewer crack elements are used in the computation. However, as demonstrated in Ang, *et al.* (2000) and in Linkov and Mogilevskaya (1994) for a circular arc crack, reasonable results can be achieved even with a simple constant approximation of $A(t)$ over a crack element, if a sufficiently large number of crack elements is employed. With highly advanced modern computers and the ability to carry out parallel processing with multiple processors, the need to use a large number of elements is not necessarily a disadvantage. In fact, it appears that a simpler numerical procedure often opens up a better way for designing a more efficient parallel algorithm. Thus, simple approximation of the crack-opening displacement should not be ruled out, but, is still an option which is worth considering.

IV. STRESS INTENSITY FACTORS

The mode III stress intensity factors at the crack tips ($-a$, 0) and (a, 0) are respectively defined by

$$K^-=\lim_{\varepsilon\to 0^+}\sqrt{2\varepsilon}\,\sigma_{32}(-a-\varepsilon,0)\text{ and}$$

$$K^+=\lim_{\varepsilon\to 0^+}\sqrt{2\varepsilon}\,\sigma_{32}(a+\varepsilon,0). \tag{46}$$

From the analysis in Section 3, the stress intensity factors are approximately given by

$$K^{\pm}\approx\frac{\mu^{1/2}(\pm a,0)}{2\sqrt{a}}\sum_{j=1}^{J}ac_jU_{j-1}(\pm 1)\,. \tag{47}$$

which can be easily computed once the constants c_j are determined.

V. SPECIFIC EXAMPLES

We take the shear modulus to be given by $\mu=\mu_0[\varepsilon x_1/(2a)+1]^2$ and the boundary C to be a rectangle with vertices $A(\ell_1, \ell_2)$, $B(-\ell_1, \ell_2)$, $C(-\ell_1, -\ell_2)$ and $D(\ell_1, -\ell_2)$ [where μ_0 is a positive constant, ℓ_1 and ℓ_2 positive constants such that $\ell_1>a$ and ε is a non-dimensionalized constant such that $|\varepsilon|<2a/\ell_1$].

Table 1. Numerical stress intensity factors.

ε	CVBEM $K^-/(s_0\sqrt{2a})$	Ref. [5] $K^-/(s_0\sqrt{2a})$	CVBEM $K^+/(s_0\sqrt{2a})$	Ref. [5] $K^+/(s_0\sqrt{2a})$
0.00	0.702	0.710	0.702	0.710
0.05	0.686	0.692	0.722	0.728
0.10	0.671	0.675	0.742	0.746
0.15	0.656	0.656	0.765	0.764
0.20	0.641	–	0.791	–

For a test problem, we first consider the case where the sides AB and CD are acted on by the stress $\sigma_{k3}=s_0$ (s_0 is a given constant) and the remaining sides BC and AD are traction-free. For $\ell_1/a=\ell_2/a=8.0$, we divide the square boundary into elements of equal length, 0.25 units, and put 8 collocation points on the crack to execute the CVBEM. The numerical values of the non-dimensionalized stress intensity factors $K^{\pm}/(s_0\sqrt{2a})$ thus obtained for selected values of ε are compared with those given by Ang, Clements and Cooke (1999) in Table 1. The two sets of results computed by different methods are in good agreement with each other.

We now study the case where the sides AB and CD are acted on by the stress $\sigma_{k3}=s_0$ (s_0 is a given constant) and BC and AD are fixed (with u_3=0) for ε=0.20. To the best of the author's knowledge, there is no prior calculation of the stress intensity factors in the literature for this specific case. For a fixed ℓ_1/a=2.0, we compute the stress intensity factors $K^{\pm}/(s_0\sqrt{2a})$ against various values of ℓ_2/a in Table 2. Similarly, for a fixed ℓ_2/a=2.0, the stress intensity factors $K^{\pm}/(s_0\sqrt{2a})$ are tabulated against various values of ℓ_1/a in Table 3. It appears that for a fixed ℓ_1/a the magnitudes of the stress intensity factors decrease as ℓ_2/a increases, while for a fixed ℓ_2/a they increase with increasing ℓ_1/a. This observation is qualitatively acceptable.

The numerical results in Tables 2 and 3 are obtained by dividing the rectangular boundary up into 160 elements and putting 10 collocation points on the crack. When the number of elements is doubled, convergence of the results to at least 2 significant figures is observed.

VI. EXTENSION TO A CURVED CRACK

The CVBEM analysis in Section 3 can be extended to a crack whose shape (on the $0x_1x_2$ plane) is

Table 2. Numerical stress intensity factors.

ℓ_2/a	1.00	1.50	2.00	2.50	3.00	3.50	4.00
$K^-/(s_0\sqrt{2a})$	0.538	0.399	0.281	0.194	0.132	0.0895	0.0603
$K^+/(s_0\sqrt{2a})$	0.615	0.467	0.334	0.232	0.159	0.108	0.0737

Table 3. Numerical stress intensity factors.

ℓ_2/a	1.25	1.50	2.00	3.00	4.00	6.00	8.00
$K^-/(s_0\sqrt{2a})$	0.0849	0.149	0.281	0.481	0.590	0.675	0.701
$K^+/(s_0\sqrt{2a})$	0.103	0.179	0.334	0.557	0.670	0.750	0.770

given by the curve Γ by following closely the work in Ang, *et al.* (2000) or by using using recent results on the complex hypersingular integrals in Linkov and Mogilevskaya (1994, 1997, 1998). A practical indication of how the extension can possibly be carried out is given below.

Let us discretize the crack by putting N closely-packed consecutive points $(y_1^{(1)}, y_2^{(1)})$, $(y_1^{(2)}, y_2^{(2)})$, ..., $(y_1^{(N-1)}, y_2^{(N-1)})$ and $(y_1^{(N)}, y_2^{(N)})$ on it, with $(y_1^{(1)}, y_2^{(1)})$ and $(y_1^{(N)}, y_2^{(N)})$ as crack tips. Let us denote the crack element (straight line segment) from $(y_1^{(k)}, y_2^{(k)})$ to $(y_1^{(k+1)}, y_2^{(k+1)})$ by $\Gamma^{(k)}$. We make the approximation:

$$\Gamma \approx \Gamma^{(1)} \cup \Gamma^{(2)} \cup \ldots \cup \Gamma^{(N-1)}. \tag{48}$$

The exterior boundary C is discretized as before.

For any $(\xi_1, \xi_2) \in R$, the Cauchy integral formulae can now be (approximately) written as:

$$2\pi i f(\xi_1 + i\xi_2)$$

$$= \sum_{m=1}^{M} f(\hat{z}^{(m)})[\gamma(z^{(m)}, z^{(m+1)}, \xi_1 + i\xi_2) + i\theta(z^{(m)}, z^{(m+1)}, \xi_1 + i\xi_2)]$$

$$+ \sum_{k=1}^{N-1} \frac{1}{2}[(y_1^{(k+1)} - y_1^{(k)}) + i(y_2^{(k+1)} - y_2^{(k)})] \int_{-1}^{1} \frac{F^{(k)}(t)dt}{w^{(k)}(t) - (\xi_1 + i\xi_2)}, \tag{49}$$

$$2\pi i f'(\xi_1 + i\xi_2)$$

$$= \sum_{m=1}^{M} f(\hat{z}^{(m)})[q(z^{(m)}, z^{(m+1)}, \xi_1 + i\xi_2) + ir(z^{(m)}, z^{(m+1)}, \xi_1 + i\xi_2)]$$

$$+ \sum_{k=1}^{N-1} \frac{1}{2}[(y_1^{(k+1)} - y_1^{(k)}) + i(y_2^{(k+1)} - y_2^{(k)})] \int_{-1}^{1} \frac{F^{(k)}(t)dt}{[w^{(k)}(t) - (\xi_1 + i\xi_2)]^2}, \tag{50}$$

where

$$w^{(k)}(t) = \hat{y}_1^{(k)} + i\hat{y}_2^{(k)} + \frac{1}{2}t[(y_1^{(k+1)} - y_1^{(k)}) + i(y_2^{(k+1)} - y_2^{(k)})]$$

$$F^{(k)}(t) = \lim_{\varepsilon \to 0^+} [f(w^{(k)}(t) - \varepsilon[m_1^{(k)} + im_2^{(k)}])$$

$$- f(w^{(k)}(t) + \varepsilon[m_1^{(k)} + im_2^{(k)}])], \tag{51}$$

where $[m_1^{(k)}, m_2^{(k)}] = [(y_2^{(k+1)} - y_2^{(k)})/L^{(k)}, (y_1^{(k)} - y_1^{(k+1)})/L^{(k)}]$ is a unit normal vector to $\Gamma^{(k)}$, $L^{(k)}$ is the length of $\Gamma^{(k)}$ and $(\hat{y}_1^{(k)}, \hat{y}_2^{(k)})$ is the midpoint of $\Gamma^{(k)}$. As shown in Section 3, if (ξ_1, ξ_2) lies on the exterior boundary C, (49) still holds but (50) is valid only if the factor $2\pi i$ is replaced by πi.

Now, for the curved crack, condition (15) should be modified (over each crack element) to become:

$$\lim_{(x_1, x_2) \to (X_1^{(k)}(t), X_2^{(k)}(t))} \mathrm{Re}\{(m_1^{(k)} + im_2^{(k)})\mu f'(x_1 + ix_2)$$

$$- \frac{1}{2}[m_1^{(k)} \frac{\partial \mu}{\partial x_1} + m_2^{(k)} \frac{\partial \mu}{\partial x_2}] f(x_1 + ix_2)\}$$

$$= 0 \text{ for } -1 < t < 1 \ (k = 1, 2, \ldots, N-1), \tag{52}$$

where $X_1^{(k)}(t) = \mathrm{Re}\{w^{(k)}(t)\}$ and $X_2^{(k)}(t) = \mathrm{Im}\{w^{(k)}(t)\}$.

To ensure that the limit in (52) exists, it is required that

$$\mathrm{Re}\{\frac{2(m_1^{(k)} + im_2^{(k)})\mu(X_1^{(k)}(t), X_2^{(k)}(t))}{[(y_1^{(k+1)} - y_1^{(k)}) + i(y_2^{(k+1)} - y_2^{(k)})]} \cdot \frac{d}{dt}[A^{(k)}(t) + iB^{(k)}(t)]\}$$

$$- \frac{1}{2}[m_1^{(k)} \frac{\partial \mu}{\partial x_1} + m_2^{(k)} \frac{\partial \mu}{\partial x_2}]\Big|_{(x_1, x_2) = (X_1^{(k)}(t), X_2^{(k)}(t))} A^{(k)}(t)$$

$$= 0 \text{ for } -1 < t < 1 \ (k = 1, 2, \ldots, N-1), \tag{53}$$

if we write $F^{(k)}(t) = A^{(k)}(t) + iB^{(k)}(t)$ where $A^{(k)}(t)$ and $B^{(k)}(t)$ are real unknown functions to be determined.

If we proceed as in Section 3, we can use (13),

(14), (49)-(50) (including the modified form of (50) for $(\xi_1, \xi_2) \in C$), (52) and (53) to set up a system of equations from which the unknown constants $f(\hat{z}^{(m)})$ $(m=1, 2, ..., M)$ and functions $F^{(k)}(t)$ $(k=1, 2, ..., M-1)$ can be determined.

VII. SUMMARY

A CVBEM is described for the antiplane problem of a straight crack in a nonhomogeneous elastic body with an arbitrary exterior boundary. The shear modulus of the material assumes a form which allows for multiparameter fittings of the shear variation. For a specific shear modulus, the method is applied to compute the crack tip stress intensity factors of a straight crack in a rectangular slab. For a particular constant shear loading on the boundary of the slab, the results obtained are in reasonable agreement with those given in Ang, *et al.* (1999). A new set of numerical results for a different loading condition is also obtained.

A discussion on how the method can be extended to include a curved crack is given. Generalization to multiple cracks is a trivial matter.

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複變數邊界元素法在含裂縫非均質物體的反平面應力分析

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摘 要

本文根據柯西積分公式所得的邊界元素法（複變數邊界元素法，CVBEM）來求彈性體含連續變化的剪力模數之反平面裂縫問題的數值解，剪力模數假設成一種可適合多重變化參數之普遍模式。本文所採用的方法可將問題縮減成線性代數方程系統，並且可由電腦迅速地算出其解。明顯地，CVBEM 公式可先解直線裂縫的問題，而後可延伸至含隨機曲線裂縫的問題上。

關鍵詞：複變數邊界元素法，裂縫，非均質彈性體。