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WEAKLY SINGULAR, SINGULAR AND HYPERSINGULAR INTEGRALS IN 3-D ELASTICITY AND FRACTURE MECHANICS

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ABSTRACT

This article considers weakly singular, singular and hypersingular integrals which arise when the boundary integral equations (BIE) are used to solve problems in the theory of elasticity and fracture mechanics. For their regularization, an approach based on the application of the Gauss-Ostrogradskii and the Green theorems has been used. The expressions, which allow an easy calculation of the weakly singular, singular and hypersingular integrals for any convex polygon, have been constructed. Such an approach may be generalized easily and applied for the calculation of multidimensional integrals with various singularities.

I. INTRODUCTION

The method of potentials is one of the most powerful and effective methods for the solution of different problems in science and engineering (Gunter, 1953; Muskhelishvili, 1968; Kupradze *et al.*, 1976; Michlin, 1962). The essence of this method is to transform a boundary value problem into the BIE. One of the most important advantages of such a transformation, when a problem is being solved numerically, using the boundary element method (BEM), is the dimension reduction of the problem by one. One of the difficulties found with such an approach, is the presence of the divergent integrals and the integral operators with kernels that contain different kinds of singularities.

In mathematics, singular integrals and integral operators with singular kernels have a well-established theoretical basis (Muskhelishvili, 1968; Kupradze *et al.*, 1976; Michlin, 1962). For example, the weakly singular (*WS*) integrals are considered as improper integrals, the singular integrals are considered in the sense of Cauchy as principal values (*PV*) and the hypersingular integrals are considered in the sense of Hadamard as finite parts (*FP*) (Gunter, 1953; Muskhelishvili, 1968; Kupradze *et al.*, 1976; Michlin, 1962). The theory of distributions (generalized functions) lets us consider divergent integrals and integral operators with kernels containing different kind of singularities using the same approach (Gel'fand and Shilov, 1962).

The divergent integrals must be calculated when the BIEs are solved numerically using the BEM. There are several methods for the calculation of the weakly singular and singular integrals (Muskhelishvili, 1968; Kupradie *et al.*, 1976; Michlin, 1962). Hypersingular integrals are more complex and there are some problems with their numerical calculation. Therefore, the BIE with singular integrals (in the sense of Cauchy *PV*) have been used

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until recently. However, there are some kinds of problems where the BIE with hypersingular integrals are preferable and closer to the physical sense of the problem. Such a situation takes place in the theory of elasticity and fracture mechanics when the BIE method is used to solve problems for bodies with cuts and cracks.

Several approaches to solve the BIE with hypersingular integral operators have been developed. For example, the BIE with hypersingular integrals may be transformed into the BIE with weakly singular or at the most with singular integrals (Tanaka, *et al.*, 1994). Then, the theoretical and applied results developed for those last two integral operators may be used. The essence of another approach is to calculate the finite part of hypersingular integrals, which consists of their regularization. There are different regularization techniques (Krishnasamy *et al.*, 1992). The standard one consists of subtracting the divergent part of the hypersingular integral, followed by its calculation and then adding the result obtained to the regular part (Chen and Hong, 1999). Such an approach has some disadvantages, which will be discussed briefly later. A detailed discussion and comprehensive review of these problems and their solution methods can be found in (Krishnasamy *et al.*, 1992; Tanaka *et al.*, 1994; Chen and Hong, 1999).

Based on the theory of distribution an approach has been developed for the regularization and numerical calculation of the hypersingular integrals that arise in the BIE of elasticity and fracture mechanics(Zozulya, 1991; Guz and Zozulya, 1993). The mathematical methodology of this approach is well known and widely discussed in the mathematical literature (Gel'fand and Shilov, 1962) but until recently, it had not been used for the numerical solution of the BIE with hypersingular integrals. The advantage of this method is that it can not only be applied for the numerical calculation of hypersingular integrals, but also for integrals with different kinds of singularities, for example weakly singular and singular ones. One-dimensional (1-D) and multi-dimensional divergent integrals can also be calculated using this method, for example, two-dimensional (2-D) hypersingular integrals from the BIE solution of the 3-D static and dynamic problems of fracture mechanics (Zozulya, 1991; Guz and Zozulya, 1993).

In the present paper, an approach based on the theory of distribution is developed for the solution of 3-D problems of the theory of elasticity and fracture mechanics. The equations that permit easy calculation of the weakly singular, the singular and the hypersingular integrals over any convex polygonal area, are presented here.

II. INTEGRAL EQUATIONS FOR BODIES WITH CRACKS

Consider a homogeneous, isotropic and linearly elastic body, which in the 3-D Euclidean space $\Re^3$ occupies the volume V with a smooth boundary. The boundary of the body contains two parts ∂V_u and ∂V_p, such that $\partial V_u \cap \partial V_p = \varnothing$ and $\partial V_u \cup \partial V_p = \partial V$. On the part ∂V_u, the displacements $u_i(\boldsymbol{x})$ of the body points and on the part ∂V_p, the tractions $p_i(\boldsymbol{x})$ are prescribed respectively. There are N arbitrarily oriented cracks in the body which are described by their surfaces $\Omega_n^+ \cup \Omega_n^-$, where Ω_n^+ and Ω_n^- are the opposite crack edges. The body may be affected by volume forces $b_i(\boldsymbol{x})$. We assume that the displacements of the body points and their gradients are small, so its stress-strain state is described by the small strain $\varepsilon_{ij}(\boldsymbol{x})$ and stress $\sigma_{ij}(\boldsymbol{x})$ tensors, which are connected by Hook's law.

The crack surfaces Ω_n^+ and Ω_n^- are locally parallel and their curvature is relatively small. Therefore we assume that $\Omega_n^+ = \Omega_n^- = \Omega_n$. They will be distinct by the direction of their external normal unit vectors, $\boldsymbol{n}^+(\boldsymbol{x}) = -\boldsymbol{n}^-(\boldsymbol{x}) = \boldsymbol{n}(\boldsymbol{x})$. The vector of discontinuity displacements ,

$$\Delta \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}^+(\boldsymbol{x}) - \boldsymbol{u}^-(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in \Omega^+ \cup \Omega^-,$$

$$\Omega^+ = \bigcup_{n=1}^{N} \Omega_n^+, \quad \Omega^- = \bigcup_{n=1}^{N} \Omega_n^-$$

characterizes the mutual displacements of the crack edges.

During the deformation process, the overlapping of the opposite crack edges is not allowed. This means that between the crack edges may arise unilateral contact with friction, which has been investigated in (Chen and Hong, 1999; Zozulya and Lukm, 1998). Here, the contact crack edges will not be taken into account.

The problem formulated above may be transformed into the BIE of the following form (1)

$$\pm \frac{1}{2} u_i(\boldsymbol{y}) = \int_{\partial V} (p_i(\boldsymbol{x}) U_{ji}(\boldsymbol{x} - \boldsymbol{y}) - u_j(\boldsymbol{x}) W_{ji}(\boldsymbol{x}, \boldsymbol{y})) dS + \int_{\Omega} \Delta u_j(\boldsymbol{x}) W_{ji}(\boldsymbol{x}, \boldsymbol{y}) dS + \int_{V} p_i(\boldsymbol{x}) U_{ji}(\boldsymbol{x} - \boldsymbol{y}) dV$$

$$\mp \frac{1}{2} p_i(\boldsymbol{y}) = \int_{\partial V} (p_i(\boldsymbol{x}) K_{ji}(\boldsymbol{x}, \boldsymbol{y}) - u_j(\boldsymbol{x}) F_{ji}(\boldsymbol{x}, \boldsymbol{y})) dS + \int_{\Omega} \Delta u_j(\boldsymbol{x}) F_{ji}(\boldsymbol{x}, \boldsymbol{y}) dS + \int_{V} p_i(\boldsymbol{x}) K_{ji}(\boldsymbol{x}, \boldsymbol{y}) dV$$

The kernels in these integral representations are the fundamental solutions of the elasticity theory. Any book regarding the BEM contains the expressions of these kernels (see for example (Kupradze *et al.*, 1976; Guz' and Zozulya, 1993)).

As is well known (Guz and Zozulya, 1993) when $y \to x$ in the 3-D case

$$U_{ji}(\boldsymbol{x}, \boldsymbol{y}) \to r^{-1}, \ W_{ji}(\boldsymbol{x}, \boldsymbol{y}) \to r^{-2},$$

$$K_{ji}(\boldsymbol{x}, \boldsymbol{y}) \to r^{-2}, \ F_{ji}(\boldsymbol{x}, \boldsymbol{y}) \to r^{-3}$$

where $r = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$ is a distance between the points $\boldsymbol{x}$ and $\boldsymbol{y}$. In the 2-D case

$$U_{ji}(\boldsymbol{x}, \boldsymbol{y}) \to \ln(r), \ W_{ji}(\boldsymbol{x}, \boldsymbol{y}) \to r^{-1},$$

$$K_{ji}(\boldsymbol{x}, \boldsymbol{y}) \to r^{-1}, \ F_{ji}(\boldsymbol{x}, \boldsymbol{y}) \to r^{-2}$$

here $r = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$.

These kernels are *W.S.*, singular and hypersingular. The integrals with such kernels are divergent, they can not be considered in the usual sense (Reimann or Lebegue). These integrals need a special consideration in order to make some sense,

III. DIVERGENT INTEGRALS AND DISTRIBUTIONS

Divergent integrals and integral operators with divergent kernels are used in mathematics, applied science and engineering (Gunter, 1953; Muskhelishvili, 1968; Kupradze *et al.*, 1976; Michlin, 1962; Hadamard, 1932; Gel'fand and Shilov, 1962; Krishnasamy *et al.*, 1992; Tanaka *et al.* 1994; Chen and Hang, 1999; Zozulya, 1991; Guz' and Zozulya, 1993; Zozulya and Lukn, 1998; Zozulya and Menshicov, 1999; Guz' and Zozulya, 1995). Nevertheless, their correct mathematical interpretation has recently been shwon by the theory of distributions (generalized functions) (Gel'fand and Shilov, 1962). This aspect of the problem is not discussed often by specialists in mechanics. For this reason, we will briefly consider divergent integrals from the point of view of the theory of distributions and compare it with traditional approaches.

Definition. Consider two points with coordinates $\boldsymbol{x}, \boldsymbol{y} \in \Re^m$ (where m=3 or m=2) and a region V with smooth boundary ∂V. The boundary integrals are of the type:

$$\int_{\partial V} \frac{f(\boldsymbol{x})}{r^{\alpha}} dS(\boldsymbol{x}), \ \alpha > 0, \ \forall \boldsymbol{y} \in \partial V$$

where $f(\boldsymbol{x})$ is bounded in ∂V and their kernels are *weakly singular* if $0<\alpha<m-1$, *singular* if $m-1 \le \alpha < m$ and *strongly singular* or *hypersingular* if $\alpha > m$.

1. 1-D Divergent Integrals

For clear and easy consideration, 1-D divergent integrals will be studied first. We apply the definition of integrals with different singularities from (Gunter, 1953; Muskhelishvili *et al.*, 1968; Kupradze *et al.*, 1976; Michlin, 1962; Krishnasamy *et al.*, 1992; Tanaka *et al.*, 1994) to the boundary integrals which are used in the BIE methods.

The *WS* integrals must be considered as improper. They are defined as

$$W.S.\int_a^b \frac{f(x)}{(x-y)^{\alpha}} dx = \lim_{\varepsilon_1, \varepsilon_2 \to 0} \Big[\int_a^{x-\varepsilon_1} \frac{f(x)}{(x-y)^{\alpha}} dx + \int_{x+\varepsilon_2}^b \frac{f(x)}{(x-y)^{\alpha}} dx\Big],$$

$$(0 \le \alpha < 1), \ (a<y<b)$$

where $f(x)$ is bounded in $[a, b]$ function. In the same way as the *WS* integrals are defined with the function $\ln|x-y|$ instead of $1/(x-y)^{\alpha}$.

The singular integrals must be considered in the sense of the Cauchy PV. They are defined as

$$P.V.\int_a^b \frac{f(x)}{(x-y)} dx = \lim_{\varepsilon \to 0} \Big[\int_a^{x-\varepsilon} \frac{f(x)}{x-y} dx + \int_{x+\varepsilon}^b \frac{f(x)}{x-y} dx\Big],$$

$$(a<y<b)$$

where $f(x)$ is bounded in $[a, b]$ and Holder continuous function at y.

The hypersingular integrals must be considered in the sense of the Hadamard *FP*. They are defined as

$$F.P.\int_a^b \frac{f(x)}{(x-y)^2} dx$$

$$= \lim_{\varepsilon \to 0} \Big[\int_a^{x-\varepsilon} \frac{f(x)}{(x-y)^2} dx + \int_{x+\varepsilon}^b \frac{f(x)}{(x-y)^2} dx - \frac{2f(y)}{\varepsilon}\Big] \ (a < x < b)$$

The smoothness requirement at the function $f(x)$ has been discussed in (Krishnasamy *et al.*, 1992; Tanaka *et al.*, 1994). Here this problem will not be considered. We assume that all the functions considered here are sufficiently smooth.

Now, we will show how to calculate some of the divergent integrals using these definitions. For simplicity, the divergent integrals with $f(x)=1$ will be considered. The following integrals are easy to calculate

$$W.S.\int_a^b \ln|x-y|dx$$

$$=\lim_{\varepsilon\to 0}\left[\int_a^{x-\varepsilon}\ln|x-y|dx+\int_{x+\varepsilon}^{b}\ln|x-y|dx\right]$$

$$=(b-y)\ln|b-y|-(a-y)\ln|a-x|,\quad (a<y<b)$$

$$P.V.\int_a^b \frac{dx}{x-y}=\lim_{\varepsilon\to 0}\left[\int_a^{x-\varepsilon}\frac{dx}{x-y}+\int_{x+\varepsilon}^{b}\frac{dx}{x-y}\right]=\ln\left|\frac{b-y}{a-y}\right|,$$

$$(a<y<b)$$

$$F.P.\int_a^b \frac{dx}{(x-y)^2}=\lim_{\varepsilon\to 0}\left[\int_a^{x-\varepsilon}\frac{dx}{(x-y)^2}+\int_{x+\varepsilon}^{b}\frac{dx}{(x-y)^2}-\frac{2}{\varepsilon}\right]$$

$$=-\frac{1}{(b-y)}+\frac{1}{(a-y)},\ (a<y<b)$$

At the last equation can be found an interesting example of a function that is positive everywhere in the integration region, but, its integral is a negative one

$$F.P.\int_{-a}^{a}\frac{dy}{y^2}=-\frac{2}{a},\quad a>0$$

In the same way the divergent integrals for some another functions $f(x)$ may be calculated.

2. 2-D Divergent Integrals

The WS integrals are defined as:

$$W.S.\int_{\partial V}\frac{f(\boldsymbol{x})}{(\boldsymbol{x}-\boldsymbol{y})^{\alpha}}dS(\boldsymbol{x})=\lim_{\varepsilon\to 0}\int_{\partial V\backslash\partial V_{\varepsilon}}\frac{f(\boldsymbol{x})}{(\boldsymbol{x}-\boldsymbol{y})^{\alpha}}dS(\boldsymbol{x}),$$

$$0<\alpha<2,\ \forall \boldsymbol{y}\in\partial V$$

Here ∂V_{ε} is a part of the boundary, its projection on a tangential plane is the neighborhood of the point $\boldsymbol{x}$. In the same way, the WS integrals are defined with the function $\ln|\boldsymbol{x}-\boldsymbol{y}|$ instead of $1/(\boldsymbol{x}-\boldsymbol{y})^{\alpha}$.

The Cauchy PV of the singular integrals are defined as:

$$P.V.\int_{\partial V}\frac{f(\boldsymbol{x})}{(\boldsymbol{x}-\boldsymbol{y})^{2}}dS(\boldsymbol{x})=\lim_{\varepsilon\to 0}\int_{\partial V\backslash\partial v(r<\varepsilon)}\frac{f(\boldsymbol{x})}{(\boldsymbol{x}-\boldsymbol{y})^{2}}dS(\boldsymbol{x})$$

Here $\partial V(r<\varepsilon)$ is a part of the boundary, its projection on a tangential plane is the circle $C_{\varepsilon}(\boldsymbol{x})$ of radivs ε with center at the point $\boldsymbol{x}$.

The Hadamard FP of the hypersingular integrals are defined as:

$$F.P.\int_{\partial V}\frac{f(\boldsymbol{x})}{(\boldsymbol{x}-\boldsymbol{y})^{3}}dS(\boldsymbol{x})=\lim_{\varepsilon\to 0}\left(\int_{\partial V\backslash\partial v(r<\varepsilon)}\frac{f(\boldsymbol{x})}{(\boldsymbol{x}-\boldsymbol{y})^{3}}dS(\boldsymbol{x})+\frac{f(\boldsymbol{y})}{\partial V(r<\varepsilon)}\right)$$

In the above definitions, assume sufficient smoothness for function $f(\boldsymbol{x})$ and boundary ∂V.

The calculation of the divergent integrals for the 2-D case using these definitions is not as simple as for the 1-D one. Now we will consider the divergent integrals which can be easily calculated using polar coordinates:

$$F.P.\int_{C(\boldsymbol{y})}\frac{dS(\boldsymbol{x})}{(\boldsymbol{x}-\boldsymbol{y})^{3}}=\int_0^{2\pi}d\varphi F.P.\int_0^{r}\frac{d\rho}{\rho^2}=-\frac{2\pi}{r}$$

Here $C(\boldsymbol{y})$ is the circle of radius r with center at the point $\boldsymbol{y}$.

Using the above definition of the Hadamard FP for a hypersingular integral, even for this simple kernel, only small changes increase dramatically the problem of this integral calculation. For example if the point y is not located in the center of the circle $C(\boldsymbol{y})$ or if $C(\boldsymbol{y})$ is a rectangle or triangle with the point $\boldsymbol{y}$ located inside it, the calculation becomes not so simple.

We will demonstrate here that using the approach developed in (Zozulya, 1991; Guz' and Zozulya, 1993, 1995; Zozulya and Menshicov, 1999) one can easily calculate the divergent integrals which arise in the application of the BIE method to elasticity and fracture mechanics for any polygonal region, analytically or numerically.

3. Distributions and Divergent Integrals

Most of the distributions, which arise in applied science and engineering may be presented in the form

$$f(\boldsymbol{x})=\partial^{r}g(\boldsymbol{x})$$

where $g(\boldsymbol{x})$ is a continuous function. In general $\partial^{r}=\partial_1^{r_1}\ldots\partial_m^{r_m}$ is a partial derivative of order r with respect to x_1, ..., x_n and $\partial_1^{r_1}=\partial^{r_1}/\partial x_1^{r_1}$ is a partial derivative of order r_1 with respect to x_1.

As usual, $f(\boldsymbol{x})$ is a regular function everywhere in a region V except in the smaller subregion V^s. In the region $V^r=V/V^s$ the generated function $g(\boldsymbol{x})$ has continuous derivatives $\partial^r g(\boldsymbol{x})$, but in the region V^s the functions $\partial^r g(\boldsymbol{x})$ can have singularities concentrated in separated points, curves and surfaces. These singularities are taken into account automatically when we operate with them according to the theory of distributions (Gel'fand and Shilov, 1962).

Now, we will consider the concept of a definite integral of the distribution. First consider a function of one variable $f(\boldsymbol{x})$ with strong singularities which concentrate on $x\in V^s=[a, b]$. What does this symbol means for such distribution?:

$$I_0 = \int_a^b f(x)dx$$

To define the definite integral for functions with strong singularities in the sense of the distributions, let us use the test function $\varphi(x)\in C^\infty(V)$, such that $\varphi(x)=1$, $\forall x\in[a, b]$ and $\varphi(x)=0$, $\forall x\notin V$. We arbitrarily prolong the function $\varphi(x)$ in the region V^r. In this case, its derivatives are equal to zero in $\partial V^s=\{a, b\}$. Also we assume that the function $\partial^r g(x)$ is continuous near the points $x=a$ and $x=b$. Now consider the scalar product:

$$(f, \varphi) = \int_V f(x)\varphi(x)dx = \int_V \partial^r g(x)\varphi(x)dx$$

In the 1-D case, $\partial^r=d^r/dx^r$ is an ordinary derivative of r order.

Because the derivatives of the test function $\varphi(x)$ equal zero in ∂V^s, the integration by parts gives

$$\int_V \partial^r g(x)\varphi(x)dx = (-1)^r\int_V g(x)\partial^r\varphi(x)dx$$

$$= (-1)^r\int_{V^r} g(x)\partial^r\varphi(x)dx$$

At the last right integral the integration by parts in reverse order leads to the result:

$$\int_V g(x)\partial^r\varphi(x)dx = (-1)^r\int_{V^r}\partial^r g(x)\varphi(x)dx + \partial^{r-1}g(x)\Big|_{x=a}^{x=b}$$

Taking into account that:

$$\int_{V^s} f(x)\varphi(x)dx = \int_V f(x)\varphi(x)dx - \int_{V^r} f(x)\varphi(x)dx$$

we will find the finite part of the divergent integral according to Hadamard in the form:

$$F.P.\int_{V^s} f(x)dx = F.P.\int_a^b f(x)dx = \partial^{r-1}g(x)\Big|_{x=a}^{x=b} \qquad (2)$$

We can use this equation to calculate weakly singular, singular and hypersingular integrals.

For regular functions this is a usual formula from integral calculus which connects infinite and finite integrals. Obviously for $r=1$ we have:

$$F.P.\int_a^b f(x)dx = g(x)\Big|_{x=a}^{x=b} \qquad (3)$$

Using this formula one can easily calculate the 1-D divergent integrals which were calculated before, using a standard technique. Evidently, the function $f(x)=1/(y-x)^2$ can be represented in the form $f(x)=-\partial^2\ln|y-x|$ and $\partial g(x)=-1(y-x)$, then from (2) it follows

$$F.P.\int_{V^s}\frac{dx}{(x-y)^2}dx = -F.P.\int_a^b \partial^2\ln|y-x|dx$$

$$= -\frac{1}{x-y}\Bigg|_{x=a}^{x=b} = -\frac{1}{(b-y)} + \frac{1}{(a-y)}, \quad (a<y<b)$$

The same result may be obtained for such representation of $f(x)=-\partial(y-x)^{-1}$ and $g(x)=-1(y-x)$.

If we consider the functions $f(x)=(y-x)^{-1}$ and $g(x)=\ln|y-x|$, the singular integral, which is considered in the sense of Cauchy PV, comes from (3)

$$P.V.\int_a^b\frac{dx}{(y-z)} = P.V.\int_a^b \partial\ln|y-x|dx = \ln\left|\frac{b-x}{a-x}\right|,$$

$$(a<x<b)$$

If we consider the functions $f(x)=\ln|y-x|$ and $g(x)=-|y-x|\ln|y-x|-(y-x)$, the WS integral, which is considered as an improper integral comes from (3)

$$W.S.\int_a^b \ln|y-x|dy = (b-x)\ln|b-x| - (a-x)\ln|a-x|,$$

$$(a<x<b)$$

Now we will find multidimensional analogies for Eqs. (2) and (3). The symbol ∂_i^{r-1} in the multidimensional case may be represented in the form $\partial_i^{r-1} = \partial_i^{-1}\partial_i^{r_i}$, where symbol ∂_i^{-1} is defined as an inverse operator for the operator of partial derivative ∂_i and also as an indefinite integral operator with respect to x_i. If the regions V and V^s and their boundaries satisfy some special conditions, which are discussed in any standard course of analysis, we can find the equation:

$$F.P.\int_{V^s} f(\boldsymbol{x})dV = \int_{S^*}\partial_i^{r-1}g(\boldsymbol{x})n_i^*(\boldsymbol{x})dS^*$$

or, for r=1, the equation:

$$F.P.\int_{V^s} f(\boldsymbol{x})dV=\int_{S^*}\partial_i g(\boldsymbol{x})n_i^*(\boldsymbol{x})dS^*$$

Here S^* is the boundary of V^s with such parameterization that the right integrals do not have integration with respect to x_i. $\boldsymbol{n}^*(\boldsymbol{x})$ is a unit vector normal to the surface S^*. For the multidimensional case, the regularization of the divergent integrals use, instead of these equations, the Ostrogradskii-Gauss theorem in the form:

$$\int_V f(\boldsymbol{x})dV=\int_V \Delta g(\boldsymbol{x})dV=\int_{\partial V}\partial_n g(\boldsymbol{x})dS \tag{4}$$

and the second Green theorem in the form:

$$\int_V [\varphi(\boldsymbol{x})\Delta g(\boldsymbol{x})-g(\boldsymbol{x})\Delta\varphi(\boldsymbol{x})]dV$$

$$=\int_{\partial V}[\varphi(\boldsymbol{x})\partial_n g(\boldsymbol{x})-g(\boldsymbol{x})\partial_n\varphi(\boldsymbol{x})]dS \tag{5}$$

as it has been shown in (Zozulya and Lukin, 1998; Zozulya and Menshicov, 1999).

In the next section we will apply this approach for the calculation of integrals with singularities that arise during the application of boundary integral equations to three dimensional elasticity and fracture mechanics.

IV. TRANSFORMATION OF DIVERGENT INTEGRALS INTO REGULAR INTEGRALS

Ostrogradskii-Gauss theorem will be applied for the regularization of divergent integrals, which arise at numerical applications of the BIE using the BEM in elasticity and fracture mechanics. For that purpose the boundary of body ∂V is divided into N boundary elements S_n such that

$$\partial V=S=\bigcup_{n=1}^{N}S_n\,,\ S_n\cap S_k=\varnothing \text{ if } n\neq k$$

For simplicity, only the boundary elements that are plane convex polygons and the interpolation polynomials that are constants with nodes of interpolation located inside of elements will be considered. The 1-D singular and hypersingular integrals for 2-D problems in the theory of elasticity and fracture mechanics are obviously very simple (Chen and Hong, 1999). For this reason, 3-D problems and the corresponding 2-D weakly singular, singular and hypersingular integrals will be considered here.

The rectangular coordinate system will be considered with the x_1 and x_2 axes located in the plane of the boundary element and the x_3 axis perpendicular to this plane. For this case, all the integrals with singularities in the integral Eqs. (2) and (3) may be presented in the form

$$J_k^{l,m}=\int_{S_n}\frac{(x_1-y_1)^l(x_2-y_2)^m}{r^k}dS \tag{6}$$

where $r=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$, x_α, y_α are the coordinates connected with the boundary element n and α=1, 2.

The main idea used for the regularization of the divergent integrals may be illustrated by the example

$$F.P.\int_{S_n}\frac{dS}{r^3}=F.P.\int_{S_n}\Delta_2\frac{1}{r}dV=\int_{\partial S_n}\nabla_2\frac{1}{r}\bullet\boldsymbol{n}(\boldsymbol{x})dS$$

$$=\int_{\partial S_n}\partial_n\frac{1}{r}dS \tag{7}$$

Here $\Delta_2=\partial_1^2+\partial_2^2$ is a two dimensional Laplace operator, $\nabla_2=\partial_1+\partial_2$ is a two dimensional Hamilton operator, ∂S_n is the boundary of the element S_n or its perimeter. In the 2-D case, the Ostrogradskii-Gauss theorem with $g(\boldsymbol{x})=\frac{1}{r}$ and $f(\boldsymbol{x})=\frac{1}{r^3}=\Delta_2\frac{1}{r}$ has been applied. Eq. (7) shows that for the regularization of the divergent integrals using the Ostrogradskii-Gauss theorem it is necessary to find the function $g(\boldsymbol{x})$ such that $f(\boldsymbol{x})=\Delta_2 g(\boldsymbol{x})$.

Here the Ostrogradskii-Gauss theorem (4) and the Green theorem (5) will be used for the transformation of the 2-D divergent integrals into 1-D regular contour integrals. The divergent integrals, which will be considered here, may be divided according to the type of kernel in the integral (6). Each type of divergent integral will be considered separately.

1. Integrals with Kernels of the Type r^{-k}, k>0

Integrals of this type may be regularized using the Ostrogradskii-Gauss theorem (4), from which it is easy to calculate the following representation of the kernel for this type of integral. In this case and and

$$f(\boldsymbol{x})=\frac{1}{r^k}\text{ and } g(\boldsymbol{x})=\frac{1}{(k-2)^2r^{k-2}}\text{ and}$$

$$\frac{1}{r^k}=\frac{1}{(k-2)^2}\Delta_2\frac{1}{r^{k-2}}\,,\ k>0,\ k\neq 2$$

Replacing the kernel at integral (6) by this equation and taking into account the Ostrogradskii-Gauss theorem (4) it is easy to find that

$$J_k^{0,0} = F.P.\int_{S_n} \frac{dS}{r^k} = \frac{1}{(k-2)^2} F.P.\int_{S_n} \Delta_2 \frac{1}{r^{k-2}} dS$$

$$= \frac{1}{(k-2)^2} \int_{\partial S_n} \partial_n \frac{1}{r^{k-2}} dl \qquad (8)$$

After the calculation of the normal derivative, the regular contour integral has the form

$$J_k^{0,0} = F.P.\int_{S_n} \frac{dS}{r^k} = -\frac{1}{(k-2)} \int_{\partial S_n} \frac{r_n}{r^k} dl \qquad (8a)$$

Here $r_n=(x_\alpha-y_\alpha)n_\alpha$ and $\alpha=1, 2$.

These equations may be used for the regularization of integrals with the kernels $1/r^k$ for every integer $k>0$ and $k\neq 2$.

Let us consider, for example, the hypersingular integral with kernel $1/r^3$. In this case $f(\boldsymbol{x})=1/r^3$ and $g(\boldsymbol{x})=1/r$ and Eqs. (8) and (8a) with $k=3$ are transformed into the following

$$J_3^{0,0} = F.P.\int_{S_n} \frac{dS}{r^3} = \int_{\partial S_n} \partial_n \frac{1}{r} dl = -\int_{\partial S_n} \frac{r_n}{r^3} dl \qquad (9)$$

This result completely coincides with (7).

For $k=1$ we have the weakly singular integral with kernel $1/r$, which may be transformed using the Eqs. (8) and (8a). In this case $f(\boldsymbol{x})=1/r$, $g(\boldsymbol{x})=r$ and $\frac{1}{r}=\Delta_2 r$. Now from the Ostrogradskii-Gauss theorem (4) comes the regular contour integral in the form

$$J_1^{0,0} = W.S.\int_{S_n} \frac{dS}{r} = W.S.\int_{S_n} \Delta_2 r dS = \int_{\partial S_n} \partial_n r dl = \frac{1}{2}\int_{\partial S_n} \frac{r_n}{r} dl \qquad (10)$$

The Eqs. (8) and (8a) are used to calculate this type of integral for the general case, but they can not be used to calculate the integral with kernel $1/r^2$. The singular integral with kernel $1/r^2$ must be considered separately. After the regularization of this integral the final result is the following regular contour integral

$$J_2^{0,0} = P.V.\int_{S_n} \frac{dS}{r^2} = \frac{1}{2}\int_{\partial S_n} [(\ln r)^3 \partial_n \frac{1}{\ln r} - \frac{1}{\ln r}\partial_n (\ln r)^3] dl$$

$$= \int_{\partial S_n} \frac{r_n \ln r}{r^2} dl \qquad (11)$$

Now any divergent integral with a kernel of the type $1/r^k$ for any positive integer k can be calculated.

2. Integrals with Kernels of the Type $\frac{x_\alpha - y_\alpha}{r^k}$, $k>0$

Like the previous case, for the regularization of this type divergent integral, the Ostrogradskii-Gauss theorem (4) may be used. The kernels of this type of integral may be presented in the following form

$$\frac{x_\alpha - y_\alpha}{r^k} = \frac{1}{(k-2)(k-4)} \Delta_2 \frac{x_\alpha - y_\alpha}{r^{k-2}},$$

$$k>0,\ k\neq 2,\ k\neq 4.$$

For k=2 and k=4 this representation is not valid.

Using this representation with the Ostrogradskii-Gauss theorem (4) divergent integrals with kernels of the type $(x_\alpha-y_\alpha)/r^k$ may be transformed into the following regular contour integrals

$$J_k^{1,0} = F.P.\int_{S_n} \frac{x_\alpha - y_\alpha}{r^k} dS = \frac{1}{(k-2)(k-4)} F.P.\int_{S_n} \Delta_2 \frac{x_\alpha - y_\alpha}{r^{k-2}} dS$$

$$= \frac{1}{(k-2)(k-4)} \int_{\partial S_n} \partial_n \frac{x_\alpha - y_\alpha}{r^{k-2}} dl$$

$$= \frac{1}{(k-2)(k-4)} \int_{\partial S_n} [\frac{n_\alpha}{r^{k-2}} - \frac{(k-2)(x_\alpha - y_\alpha) r_n}{r^k} dl = ,$$

$$k>0,\ k\neq 2,\ k\neq 4 \qquad (12)$$

Now let us consider several examples. The integral with kernel $\frac{x_\alpha - y_\alpha}{r}$ is in fact a regular one, but it may also be transformed into the regular contour integral, which is easy to calculate. Taking into account that $\frac{x_\alpha - y_\alpha}{r} = \frac{1}{3}\Delta_2 r(x_\alpha - y_\alpha)$ and using Eq. (12) it can be shown that

$$J_1^{1,0} = \int_{S_n} \frac{x_\alpha - y_\alpha}{r} dS = \frac{1}{3}\int_{S_n} \Delta_2 r(x_\alpha - y_\alpha) dS$$

$$= \frac{1}{3}\int_{\partial S_n} \partial_n r(x_\alpha - y_\alpha) dl = \frac{1}{3}\int_{\partial S_n} [n_n r + \frac{(x_\alpha - y_\alpha) r_n}{r}] dl \qquad (13)$$

The singular integral with kernel $(x_\alpha-y_\alpha)/r^3$ is calculated using Eq. (12). In this case, $f(\boldsymbol{x})=(x_\alpha-y_\alpha)/r^3$ and $g(\boldsymbol{x})=-(x_\alpha-y_\alpha)/r$. Taking into account that $\frac{x_\alpha - y_\alpha}{r^3} = -\Delta_2 \frac{x_\alpha - y_\alpha}{r}$ Eq. (12) with $k=3$ is transformed into the following one

$$J_3^{1,0} = P.V. \int_{S_n} \frac{x_\alpha - y_\alpha}{r^3} dS = -P.V. \int_{S_n} \Delta_2 \frac{x_\alpha - y_\alpha}{r} dS$$

$$= -\int_{\partial S_n} \partial_n \frac{x_\alpha - y_\alpha}{r} dl = \int_{\partial S_n} [\frac{(x_\alpha - y_\alpha) r_n}{r^3} - \frac{n_\alpha}{r} + dl \quad (14)$$

For k=2 and k=4 the general Eq. (12) is not valid. But these integrals may be evaluated using the Green theorem (5). For the regularization of the weakly singular integral with kernel $(x_\alpha - y_\alpha)/r^2$, the functions $f(\boldsymbol{x})=1/r$ and $\varphi(\boldsymbol{x})=r(x_\alpha - y_\alpha)$ must be placed into the Green theorem (5). Taking into account that $\Delta_2 r(x_\alpha - y_\alpha) = \frac{3(x_\alpha - y_\alpha)}{r}$ it is easy to show that

$$J_2^{1,0} = W.S. \int_{S_n} \frac{x_\alpha - y_\alpha}{r^2} dS$$

$$= -\frac{1}{2} \int_{\partial S_n} [r(x_\alpha - y_\alpha) \partial_n \frac{1}{r} - \frac{1}{r} \partial_n r(x_\alpha - y_\alpha)] dl$$

$$= \int_{\partial S_n} [\frac{(x_\alpha - y_\alpha) r_n}{r^2} + \frac{n_\alpha}{r}] dl \quad (15)$$

The hypersingular integral with kernel $(x_\alpha - y_\alpha)/r^4$ is calculated with $f(\boldsymbol{x})=1/4r^2$ and $\varphi(\boldsymbol{x})=x_\alpha - y_\alpha$. Taking into account that $\frac{1}{r^4} = \frac{1}{4} \Delta_2 \frac{1}{r^2}$ it is easy to show that

$$J_4^{1,0} = F.P. \int_{S_n} \frac{x_\alpha - y_\alpha}{r^4} dS$$

$$= \frac{1}{4} \int_{\partial S_n} [(x_\alpha - y_\alpha) \partial_n \frac{1}{r^2} - \frac{1}{r^2} \partial_n (x_\alpha - y_\alpha)] dl$$

$$= -\frac{1}{4} \int_{\partial S_n} [\frac{2(x_\alpha - y_\alpha) r_n}{r^4} + \frac{n_\alpha}{r^2}] dl \quad (16)$$

Also, divergent integrals with kernels of the type $(x_\alpha - y_\alpha)/r^k$ can be calculated using Eq. (12) for any positive integer k.

3. Integrals with Kernels of the Type $\frac{(x_\alpha - y_\alpha)^2}{r^k}$, $k>0$

This type of divergent integrals will be regularized using the Ostrogradskii-Gauss theorem (4). The kernels of these integrals may be presented in the following form

$$\frac{(x_\alpha - y_\alpha)^2}{r^k} = \frac{1}{(k-2)(k-6)} (\Delta_2 \frac{(x_\alpha - y_\alpha)^2}{r^{k-2}} - \frac{2}{r^{k-2}}),$$

$$k>0,\ k \neq 2,\ k \neq 6.$$

This representation is not valid for k=2 and k=6.

Using this representation, divergent integrals of this type may be presented in the form

$$J_k^{2,0} = F.P. \int_{S_n} \frac{(x_\alpha - y_\alpha)^2}{r^k} dS$$

$$= \frac{1}{(k-2)(k-6)} (F.P. \int_{S_n} \Delta_2 \frac{(x_\alpha - y_\alpha)^2}{r^{k-2}} dS - 2F.P. \int_{S_n} \frac{dS}{r^{k-2}})$$

At the right part of the previous expression the last divergent integral may be calculated using Eq. (8), and the first one may be regularized using the Ostrogradskii-Gauss theorem (4). In this case $f(\boldsymbol{x}) = \frac{(x_\alpha - y_\alpha)^2}{r^k}$ and $g(\boldsymbol{x}) = \frac{(x_\alpha - y_\alpha)^2}{(k-2)(k-6) r^{k-2}}$. Replacing these functions in the integral (6) and taking into account the Ostrogradskii-Gauss theorem (4) it is easy to find that

$$J_k^{2,0} = F.P. \int_{S_n} \frac{(x_\alpha - y_\alpha)^2}{r^k} dS$$

$$= \frac{1}{(k-2)(k-6)} (\int_{\partial S_n} \partial_n \frac{(x_\alpha - y_\alpha)^2}{r^{k-2}} dl - 2F.P. \int_{S_n} \frac{dS}{r^{k-2}}),$$

$$k>0,\ k \neq 2,\ k \neq 6$$

After the calculation of the normal derivative and the divergent integral, the regular contour integral has the form

$$J_k^{2,0} = F.P. \int_{S_n} \frac{(x_\alpha - y_\alpha)^2}{r^k} dS$$

$$= \frac{1}{(k-2)(k-6)} \int_{\partial S_n} [\frac{2(x_\alpha - y_\alpha) n_\alpha}{r^{k-2}} + \frac{2 r_n}{(k-4) r^{k-2}}$$

$$- \frac{(k-2)(x_\alpha - y_\alpha)^2 r_n}{r^{k-2}}] dl \quad (17)$$

These equations may be used for the regularization of integrals with the kernels $\frac{(x_\alpha - y_\alpha)^2}{r^k}$ for every integer such that $k>0$, $k\neq 2$ and $k\neq 6$.

The weakly singular integral with kernel $\frac{(x_\alpha - y_\alpha)^2}{r^3}$ is calculated using Eq. (17). Taking into account that $\frac{(x_\alpha - y_\alpha)^2}{r^3} = \frac{1}{3}(\frac{2}{r} - \Delta_2 \frac{(x_\alpha - y_\alpha)^2}{r})$, it is easy to show that

$$J_3^{2,0} = W.S. \int_{S_n} \frac{(x_\alpha - y_\alpha)^2}{r^3} dS$$

$$= \frac{2}{3} W.S. \int_{S_n} \frac{dS}{r} - \frac{1}{3} W.S. \int_{S_n} \Delta_2 \frac{(x_\alpha - y_\alpha)^2}{r} dS$$

$$= \frac{2}{3} \int_{\partial S_n} \partial_n r dl - \frac{1}{3} \int_{\partial S_n} \partial_n \frac{(x_\alpha - y_\alpha)^2}{r} dl$$

$$= \frac{1}{3} \int_{\partial S_n} [\frac{(x_\alpha - y_\alpha)^2 r_n}{r^3} - \frac{2(x_\alpha - y_\alpha) n_\alpha}{r} + \frac{2r_n}{r}] dl \quad (18)$$

The singular integral with kernel $\frac{(x_\alpha - y_\alpha)^2}{r^4}$ is calculated using the Green theorem (5). Taking into account that $\Delta_2 \frac{(x_\alpha - y_\alpha)^2}{r^4} = \frac{1}{4}(\frac{2}{r^2} - \Delta_2 \frac{(x_\alpha - y_\alpha)^2}{r^2})$ it is easy to show that

$$J_4^{2,0} = P.V. \int_{S_n} \frac{(x_\alpha - y_\alpha)^2}{r^4} dS$$

$$= \frac{1}{2} P.V. \int_{S_n} \frac{dS}{r^2} - \frac{1}{4} P.V. \int_{S_n} \Delta_2 \frac{(x_\alpha - y_\alpha)^2}{r^2} dS$$

$$= \frac{1}{2} \int_{S_n} \frac{dS}{r^2} - \frac{1}{4} \int_{\partial S_n} \partial_n \frac{(x_\alpha - y_\alpha)^2}{r^2} dl$$

$$= \frac{1}{2} \int_{\partial S_n} [\frac{(x_\alpha - y_\alpha)^2 r_n}{r^4} - \frac{(x_\alpha - y_\alpha) n_\alpha}{r^2} + \frac{r_n \ln r}{2r^2}] dl \quad (19)$$

The hypersingular integral with kernel $\frac{(x_\alpha - y_\alpha)^2}{r^5}$ is calculated using Eq. (17). Taking into account that $\frac{(x_\alpha - y_\alpha)^2}{r^5} = \frac{1}{3}(\frac{2}{r^3} - \Delta_2 \frac{(x_\alpha - y_\alpha)^2}{r^3})$, it is easy to show that

$$J_5^{2,0} = F.P. \int_{S_n} \frac{(x_\alpha - y_\alpha)^2}{r^5} dS$$

$$= \frac{1}{3} F.P. (2\int_{S_n} \frac{dS}{r^3} - \int_{S_n} \Delta_2 \frac{(x_\alpha - y_\alpha)^2}{r^3} dS)$$

$$= \frac{2}{3} \int_{\partial S_n} \partial_n \frac{1}{r} dS - \frac{1}{3} \int_{\partial S_n} \partial_n \frac{(x_\alpha - y_\alpha)^2}{r^3} dl$$

$$= \int_{\partial S_n} [\frac{(x_\alpha - y_\alpha)^2 r_n}{r^5} - \frac{2(x_{\alpha'} - y_{\alpha'}) n_{\alpha'}}{3r^3} + \frac{2r_n}{3r^3}] dl \quad (20)$$

For $k=2$ and $k=6$ the general Eq. (17) is not valid. The corresponding integrals can be calculated using the Green theorem (5). For example, the integral with kernel $\frac{(x_\alpha - y_\alpha)^2}{r^2}$ is in reality a regular one, but it may also be transformed into a regular contour integral which is easier to calculate. In this case $f(\boldsymbol{x}) = \frac{1}{r}$ and $\varphi(\boldsymbol{x}) = r(x_\alpha - y_\alpha)^2$. Taking into account that $\Delta_2 r(x_\alpha - y_\alpha)^2 = 2r + \frac{5(x_\alpha - y_\alpha)^2}{r}$ it is easy to show that

$$\int_{S_n} \frac{(x_\alpha - y_\alpha)^2}{r^2} dS = -\frac{1}{4} \int_{\partial S_n} [r(x_\alpha - y_\alpha)^2 \partial_n \frac{1}{r} - \frac{1}{r} \partial_n r (x_\alpha - y_\alpha)^2] dl$$

$$-\frac{1}{2} \int_{S_n} dS$$

and

$$J_2^{2,0} = \int_{S_n} \frac{(x_\alpha - y_\alpha)^2}{r^2} dS = \frac{1}{2} \int_{\partial S_n} [\frac{(x_\alpha - y_\alpha)^2 r_n}{r^2} + x_\alpha n_\alpha] dl - \frac{S_n}{2}$$

(21)

4. Integrals with Kernels of the Type $\frac{(x_1 - y_1)(x_2 - y_2)}{r^k}$, $k>0$

Integrals of this type may be regularized using the Ostrogradskii-Gauss theorem (4) using the following representation for kernels

$$\frac{(x_1 - y_1)(x_2 - y_2)}{r^k} = \frac{1}{(k-2)(k-6)} \Delta_2 \frac{(x_1 - y_1)(x_2 - y_2)}{r^{k-2}},$$

$$k>0,\ k\neq 2,\ k\neq 6.$$

This equation is valid for every positive integer k, except k=1 and k=2.

In this case $f(\boldsymbol{x})=\frac{(x_1-y_1)(x_2-y_2)}{r^k}$ and $g(\boldsymbol{x})=\frac{(x_1-y_1)(x_2-y_2)}{(k-2)(k-6)r^{k-2}}$. Replacing these functions in the integral (6) and taking into account the Ostrogradskii-Gauss theorem (4) it is easy to find that

$$J_k^{1,1}=F.P.\int_{S_n}\frac{(x_1-y_1)(x_2-y_2)}{r^k}dS$$

$$=\frac{1}{(k-2)(k-6)}F.P.\int_{S_n}\Delta_2\frac{(x_1-y_1)(x_2-y_2)}{r^{k-2}}dS$$

$$=\frac{1}{(k-2)(k-6)}[\int_{\partial S_n}\partial_n\frac{(x_1-y_1)(x_2-y_2)}{r^{k-2}}dl$$

$$=\frac{1}{(k-2)(k-6)}\int_{\partial S_n}[\frac{r_*}{r^{k-2}}-\frac{(k-2)(x_1-y_1)(x_2-y_2)r_n}{r^k}]dl \tag{22}$$

Where $r_*=x_1n_2+x_2n_1$.

These equations may be used for the regularization of integrals with the kernels $\frac{(x_1-y_1)(x_2-y_2)}{r^k}$ for every integer, such that k>0, $k\neq2$, and $k\neq6$.

The weakly singular integral with kernel $\frac{(x_1-y_1)(x_2-y_2)}{r^3}$ is calculated using Eq. (22). Taking into account that $\frac{(x_1-y_1)(x_2-y_2)}{r^3}=-\frac{1}{3}\Delta_2\frac{(x_1-y_1)(x_2-y_2)}{r}$, it is easy to show that

$$J_3^{1,1}=W.S.\int_{S_n}\frac{(x_1-y_1)(x_2-y_2)}{r^3}dS$$

$$=-\frac{1}{3}W.S.\int_{\partial S_n}\Delta_2\frac{(x_1-y_1)(x_2-y_2)}{r}dS$$

$$=-\frac{1}{3}\int_{S_n}\partial_n\frac{(x_1-y_1)(x_2-y_2)}{r}dl$$

$$=\frac{1}{3}\int_{\partial S_n}[\frac{(x_1-y_1)(x_2-y_2)r_n}{r^3}-\frac{r_*}{r}]dl \tag{23}$$

The singular integral with kernel $\frac{(x_1-y_1)(x_2-y_2)}{r^4}$ is calculated using Eq. (22). Taking into account that $\frac{(x_1-y_1)(x_2-y_2)}{r^4}=-\frac{1}{4}\Delta_2\frac{(x_1-y_1)(x_2-y_2)}{r^2}$, it is easy to show that

$$J_4^{1,1}=P.V\int_{S_n}\frac{(x_1-y_1)(x_2-y_2)}{r}dS$$

$$=-\frac{1}{4}P.V.\int_{S_n}\Delta_2\frac{(x_1-y_1)(x_2-y_2)}{r^2}dS$$

$$=-\frac{1}{4}\int_{\partial S_n}\partial_n\frac{(x_1-y_1)(x_2-y_2)}{r^2}dl$$

$$=\frac{1}{2}\int_{\partial S_n}[\frac{(x_1-y_1)(x_2-y_2)r_n}{r^4}-\frac{r_*}{2r^2}]dl \tag{24}$$

The hypersingular integral with kernel $\frac{(x_1-y_1)(x_2-y_2)}{r^5}$ is calculated using Eq. (22). Taking into account that $\frac{(x_1-y_1)(x_2-y_2)}{r^5}=-\frac{1}{3}\Delta_2\frac{(x_1-y_1)(x_2-y_2)}{r^3}$, it is easy to show that

$$J_5^{1,1}=F.P.\int_{S_n}\frac{(x_1-y_1)(x_2-y_2)}{r^5}dS$$

$$=-\frac{1}{3}F.P.\int_{S_n}\Delta_2\frac{(x_1-y_1)(x_2-y_2)}{r^3}dS$$

$$=-\frac{1}{3}\int_{\partial S_n}\partial_n\frac{(x_1-y_1)(x_2-y_2)}{r^3}dl$$

$$=\frac{1}{3}\int_{\partial S_n}[\frac{3(x_1-y_1)(x_2-y_2)r_n}{r^3}-\frac{r_*}{r^3}]dl \tag{25}$$

For k=2 and k=6 the general Eq. (22) is not valid. The corresponding integrals can be calculated using the Green theorem (5). For example, the integral with kernel $\frac{(x_1-y_1)(x_2-y_2)}{r^2}$ is in reality a regular one, but it may also be transformed into the regular contour integral, which is easier to calculate. In this case $f(\boldsymbol{x})=\frac{1}{r}$ and $\varphi(\boldsymbol{x})=r(x_1-y_1)(x_2-y_2)$. And taking into account that $\Delta_2 r(x_1-y_1)(x_2-y_2)=\frac{5(x_1-y_1)(x_2-y_2)}{r}$ it is easy to show that

$$\int_{S_n} \frac{(x_1-y_1)(x_2-y_2)}{r^2} dS$$

$$= -\frac{1}{4}\int_{\partial S_n} [r(x_1-y_1)(x_2-y_2)\partial_n \frac{1}{r} - \frac{1}{r}\partial_n r(x_1-y_1)(x_2-y_2)]dl$$

and that

$$J_2^{1,1} = \int_{S_n} \frac{(x_1-y_1)(x_2-y_2)}{r^2} dS$$

$$= \frac{1}{2}\int_{\partial S_n} [\frac{(x_1-y_1)(x_2-y_2)r_n}{r^2} + \frac{r_*}{2}]dl \qquad (26)$$

V. CALCULATION OF THE DIVERGENT INTEGRALS OVER ANY POLYGONAL ELEMENT

Divergent integrals of type (6) have been transformed into regular integrals and may be easily calculated. For example, the integral (8) for a circular area with the point y located in the center of the circle leads to the following result

$$J_k^{0,0} = \frac{1}{(k-2)^2}\int_{\partial S_n} \partial_n \frac{1}{r^{k-2}} dl = \frac{1}{(k-2)^2}\int_0^{2\pi} \frac{d}{dr}(\frac{1}{r^{k-2}}) r d\varphi$$

$$= -\frac{2\pi}{(k-2)r^{k-2}} \qquad (27)$$

Where polar coordinates are used and r is the circle radius.

In the application of the divergent integrals in the BEM, it is necessary to calculate the above integrals over any triangular, rectangular or polygonal elements. For that purpose these integrals must be transformed into a more convenient form for the calculation (Zozulya and Lukin, 1998; Zozulya and Menshicol, 1999).

Let us consider the contour ∂V_n as a polygon with Q angles. To calculate the divergent integrals of type (6) the approach developed in (Zozulya and Lukin, 1998; Zozulya and Menshicol, 1999) will be used. All the calculations will be done using the local rectangular coordinate system with its origin located in the point y, the x_1 and x_2 axis located in the plane of the polygon and the x_3 axis perpendicular to this plane.

The coordinates of an arbitrary point on the contour ∂V_n may be represented in the form

$$x_1(t)=x_1(q)-tn_2 \text{ and } x_2(t)=x_1(q)+tn_1$$

where $x_1(q)$ and $x_2(q)$ are the coordinates of the q-th side of the contour, $\boldsymbol{n}(n_1, n_2)$ is a unit vector normal to the contour and $t\in[-\Delta_q, \Delta_q]$ is a parameter of integration along the q-th side, $2\Delta_q$ is the length of a q-th side.

These are some useful notations

$$r^2(t)=t^2+2tr_\tau(q)+r_\tau^2(q),\ r_\tau(q)=-x_1(q)n_2(q)+x_2(q)n_1(q),$$

$$r^2(q)=x_1^2(q)+x_2^2(q),\ r_n(q)=x_\alpha(q)n_\alpha(q),$$

$$r_*(q)=x_1(q)n_2(q)+x_2(q)n_1(q)$$

$$I_{m,l} = \int_{-\Delta_q}^{\Delta_q} \frac{t^l}{r^m(t)} dt = \int_{-\Delta_q}^{\Delta_q} \frac{t^l}{(r^2+2tr_\tau(q)+r_\tau^2(q))^{m/2}} dt,$$

$$I_{2,\ln} = \int_{-\Delta_q}^{\Delta_q} \frac{\ln(r(t))}{r^2(t)} dt$$

Using these notations the integrals under consideration may be represented in a convenient form for the calculation.

1. Integrals with Kernels of the Type r^{-k}, $k>0$

$$J_k^{0,0} = -\frac{1}{(k-2)}\sum_{q=1}^{Q} r_n(q)I_{k,0},\ J_1^{0,0} = \sum_{q=1}^{Q} r_n(q)I_{1,0},$$

$$J_3^{0,0} = -\sum_{q=1}^{Q} r_n(q)I_{3,0},\ J_2^{0,0} = -\sum_{q=1}^{Q} r_n(q)I_{2,\ln} \qquad (28)$$

2. Integrals with Kernels of the Type $\frac{x_\alpha - y_\alpha}{r^k}$, $k>0$

$$J_k^{1,0} = \frac{1}{(k-2)(k-4)}\sum_{q=1}^{Q} (n_\alpha(q)I_{k-2,0}$$
$$-(k-2)r_n(q)(x_\alpha(q)I_{k,0} - n_\beta(q)I_{k,1})),$$

$$J_1^{1,0} = \sum_{q=1}^{Q} (n_\alpha(q)I_{-1,0} - r_n(q)(x_\alpha(q)I_{1,0} - n_\beta(q)I_{1,1}))$$

$$J_2^{1,0} = \sum_{q=1}^{Q} (n_\alpha(q)\Delta_q + r_n(q)(x_\alpha(q)I_{2,0} - n_\beta(q)I_{2,1}))$$

$$J_3^{1,0} = -\frac{1}{3}\sum_{q=1}^{Q} (n_\alpha(q)I_{1,0} - r_n(q)(x_\alpha(q)I_{3,0} - n_\beta(q)I_{3,1}))$$

$$J_4^{1,0} = -\frac{1}{4}\sum_{q=1}^{Q} (n_\alpha(q)I_{2,0} - 2r_n(q)(x_\alpha(q)I_{4,0} - n_\beta(q)I_{4,1}))$$

(29)

3. Integrals with Kernels of the Type $\frac{(x_\alpha - y_\alpha)^2}{r^k}$, $k>0$

$$J_k^{2,0} = -\frac{1}{(k-2)(k-6)}\sum_{q=1}^{Q}((k-2)r_n(q)(n_\beta^2(q)I_{k,2}$$
$$-2n_\beta(q)x_\alpha(q)I_{k,1} + x_\alpha^2(q)I_{k,0})$$
$$+2n_\alpha(q)(n_\beta(q)I_{k-2,1} - x_\alpha(q)I_{k-2,0})$$
$$-2r_n(q)I_{k-2,0}/(k-4))$$

$$J_3^{2,0} = \frac{1}{3}\sum_{q=1}^{Q}(r_n(q)(n_\beta^2(q)I_{3,2} - 2n_\beta(q)x_\alpha(q)I_{3,1} + x_\alpha^2(q)I_{3,0})$$
$$-2n_\alpha(q)(n_\beta(q)I_{1,1} - x_\alpha(q)I_{1,0}) + 2r_n(q)I_{1,0})$$

$$J_4^{2,0} = \frac{1}{2}\sum_{q=1}^{Q}(r_n(q)(n_\beta^2(q)I_{4,2} - 2n_\beta(q)x_\alpha(q)I_{4,1} + x_\alpha^2(q)I_{4,0})$$
$$-2n_\alpha(q)(n_\beta(q)I_{2,1} - x_\alpha(q)I_{2,0}) + r_n(q)I_{2,1n}/2)$$

$$J_5^{2,0} = \frac{1}{3}\sum_{q=1}^{Q}(r_n(q)(n_\beta^2(q)I_{5,2} - 2n_\beta(q)x_\alpha(q)I_{5,1} + x_\alpha^2(q)I_{5,0})$$
$$-2n_\alpha(q)(n_\beta(q)I_{3,1} - x_\alpha(q)I_{3,0}) - 2r_n(q)I_{3,0})$$

$$J_2^{2,0} = \frac{1}{2}\sum_{q=1}^{Q}(r_n(q)(n_\beta^2(q)I_{2,2} - 2n_\beta(q)x_\alpha(q)I_{2,1} + x_\alpha^2(q)I_{2,0})$$
$$+2n_\alpha(q)(n_\beta(q)\Delta_q^2 - 2x_1(q)\Delta_q)) - S_n \quad (30)$$

4. Integrals with Kernels of the Type $\frac{(x_1-y_1)(x_2-y_2)}{r^k}$, $k>0$

$$J_k^{1,1} = \frac{1}{(k-2)(k-6)}\sum_{q=1}^{Q}((k-2)r_n(q)(n_1(q)n_2(q)I_{k,2}$$
$$-r^-(q)I_{k,1} - x_1(q)x_2(q)I_{k,0}) + (n_1^2(q) - n_2^2(q))I_{k-2,1}$$
$$+r_*(q)I_{k-2,0}))$$

$$J_2^{1,1} = \frac{1}{2}\sum_{q=1}^{Q}(r_n(q)(-n_1(q)n_2(q)I_{2,2} + r^-(q)I_{2,1}$$
$$+x_1(q)x_2(q)I_{2,0}) - ((n_1^2(q) - n_2^2(q))\Delta_q^2/2 + r_*(q)\Delta_q)/2)$$

$$J_3^{1,1} = \frac{1}{3}\sum_{q=1}^{Q}(r_n(q)(-n_1(q)n_2(q)I_{3,2} + r^-(q)I_{3,1}$$
$$+x_1(q)x_2(q)I_{3,0}) - (n_1^2(q) - n_2^2(q))I_{1,1} - r_*(q)I_{1,0}))$$

$$J_4^{1,1} = \frac{1}{2}\sum_{q=1}^{Q}(r_n(q)(-n_1(q)n_2(q)I_{4,2} + r^-(q)I_{4,1}$$
$$+x_1(q)x_2(q)I_{4,0}) - ((n_1^2(q) - n_2^2(q))I_{2,1} + r_*(q)I_{2,0})/2)$$

$$J_5^{1,1} = \frac{1}{3}\sum_{q=1}^{Q}(3r_n(q)(-n_1(q)n_2(q)I_{5,2} + r^-(q)I_{5,1}$$
$$+x_1(q)x_2(q)I_{5,0}) - (n_1^2(q) - n_2^2(q))I_{3,1} + r_*(q)I_{3,0}))$$
$$(31)$$

VI. CONCLUSION

In the present publication weakly singular, singular and hypersingular integrals, which arise when the BIE are solved using the BEM have been considered. The approach based on the theory of distribution, (Zozulya, 1991; Guz' and Zozulya, 1993; Zozulya and Lukin, 1998; Zozulya and Menshizov, 1999; Guz' and Zozulya, 1995) has been used. The equations to calculate the divergent integrals with various singularities have been given. This approach may be used to calculate various multidimensional divergent integrals.

REFERENCES

1. Chen J.T., and Hong H.-K, 1999, "Review of Dual Boundary Element Methods with Emphasis on Hypersingular Integrals and Divergent Series," *Applied Mechanic Review*, Vol. 52, pp. 17-33.
2. Gunter N.M., 1953, *Potential theory and its application to basic problems of mathematical physics*, Moscow, GITTL, , [English translation, Fredric Ungar, New York (1967)].
3. Gel'fand I.M., and Shilov G.E., 1962, Generalized functions, Vol.1, Moscow, Fismatgiz, English translation, Academic Press, New York (1964).
4. Guz' A.N., and Zozulya V.V., 1993, *Brittle fracture of constructive materials under dynamic loading*, Kiev, Naukova Dumka, (in Russian).
5. Guz' A.N., and Zozulya V.V. 1995, "Dynamic Problems of Fracture Mechanic with Account of the Contact Interaction of the Crack Edges," *International Applied Mechanics*, Vol. 31, No. 1, pp. 1-31.
6. Hadamard J., 1932, *Le probleme de cauchy et les eguations aux devivees partielles lineaires hyperboliques*, Herman, Paris.
7. Kupradze V.D., Gegelia T.G., Bashelishvili M. O., and Burkhuladze T.V., 1976, *Three-dimensional problems of the mathematical theory of elasticity and thermoelasticity*, Moscow, Nauka,

English translation, North-Holland, Amsterdam. (1979).
8. Krishnasamy G., Rizzo F.J., and Rudolphi T.J., 1992, "Hypersingular Boundary Integral Equations: Their Occurrence, Interpretation, Regularization and Computation," In: *Development in Boundary Element Methods-7*, Banerjee P.K. and Kobayashi S. (eds), Elsevier Applied Science Publisher, Londin, Chap. 7, pp. 207-252.
9. Michlin S.G., 1962, *Multidimensional Singular Integrals and Integral Equations*, Moscow, Fismatgiz, English translation, Pergamon Press, Oxford (1965).
10. Muskhelishvili N.I. 1968, *Singular integral equations*, Moscow, Nauka, English Translation, Nordhoff, Groningen. (1953).
11. Tanaka M., Sladek V., and Sladek J., 1994, "Regularization Techniques Applied to Boundary Element Methods. *Applied Mechanics Review*, Vol. 47, pp. 457-499.
12. Zozulya V.V., 1991, "Integrals of Hadamard Type in Dynamic Problem of the Crack Theory," *Dokl. Acad. Nauk. UkrSSR, Ser. A. Physical Mathematics. and Technical Science*, 2, pp. 19-22, (in Russian).
13. Zozulya V.V., and Lukin A.N., 1998, "Solution of Three Dimensional Fracture Mechanics Problems with Boundary Integral Equation Method, *International Applied Mechanics*, Vol. 34, 7, pp. 45-53.
14. Zozulya V.V., and. Menshicov W.A, 1999, "Hypersingular Integrals in the Three-dimensional Elastodynamic Problems for Bodies with Cracks," *International Applied Mechanics*, Vol. 35, in press.

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三維彈性力學及破壞力學上之弱奇異、奇異和超強奇異積分

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本文探討使用邊界積分方程解決彈性力學及破壞力學問題時所發生的弱奇異、奇異和超強奇異積分。利用 Gauss-Ostrogradskii 及格林定理進行正規化。同時建構一簡易之計算式進行任意凸多邊形之弱奇異、奇異和超強奇異積分。本法可推廣應用至各種奇異性質之多維積分。

關鍵詞：超強奇異積分、邊界積分方程、彈性力學、破壞力學。