Journal of the Chinese Institute of Engineers, Vol. 22, No. 6, pp. 777-783 (1999)

COMBINATION OF THE FINITE AND THE BOUNDARY ELEMENT METHODS FOR AN EXTERNAL BOUNDARY VALUE PROBLEM OF THE POISSON EQUATION

Takemi Shigeta*, Takaya Harayama, and Kenji Shirota

Graduate School of Mathematical Sciences
Ibaraki University
2-1-1, Bunkyo, Mito, Ibaraki 310-8512, Japan

Key Words: 2-D external boundary value problems, iterative FE/BE coupling scheme, Dirichlet-Neumann map.

ABSTRACT

A numerical algorithm for an external Dirichlet problem of the Poisson equation is considered. The domain Ω extending to infinity is divided into a bounded subdomain Ω_0 and the unbounded subdomain Ω_1. The finite and the boundary element methods are applied to the boundary value problems in the bounded and the unbounded subdomains, respectively. An iterative scheme using the Dirichlet-Neumann map on the interface $\partial\Omega_1$ is presented. The convergence of the scheme is mathematically guaranteed. A simple numerical example shows the effectiveness of our scheme.

I. INTRODUCTION

In practice, we often confront external problems, in which domains are extended to infinity. Let $\Omega\subset \boldsymbol{R}^2$ be an external domain with the smooth boundary Γ_0. Let $f\in L^2(\Omega)$ be given, whose support is assumed to be compact. We denote by $H^1(\Omega)$ and $H^{1/2}(\Gamma)$ the usual Sobolev spaces. Then, we consider the following external boundary value problem:

Problem 1. For given Dirichlet data $g\in H^{1/2}(\Gamma_0)$, find $u\in H^1(\Omega)$ such that

$$-\Delta u=f \quad \text{in} \quad \Omega,$$

$$u=g \quad \text{on} \quad \Gamma_0.$$

A domain decomposition method for the external problem was suggested in the middle 1980s. Gatica and Hsiao (1995) considered the method of solution that treats a problem with an unbounded domain as a problem with a bounded domain. Recently, Yu (1996) suggested a non-overlapping domain decomposition method for an external Dirichlet problem. His method is called, by himself, the Dirichlet-Neumann alternating method. The mapping from the Dirichlet data to the Neumann data is called a Dirichlet-Neumann map, and this map is expressed by a boundary integral operator. His method is based on the Dirichlet-Neumann map. This is different from the method presented by Feng and Owen (1996).

The purpose of this paper is to inquire further into the Dirichlet-Neumann alternating method for the external Dirichlet problem of the Poisson equation (Harayama *et al.*, 1998.7, 1998.10, 1998.12). The unbounded domain Ω is divided into an internal bounded subdomain and the external unbounded subdomain. We apply the finite and the boundary element methods for the internal and the external subdomains, respectively. Different from Yu's method we apply

*Correspondence addressee

the boundary element method for the external subdomain in order to cope with an arbitrary shape of the interface between the internal and the external subdomains.

II. THE DIRICHLET-NEUMANN ALTERNATING METHOD

We consider a closed curve Γ_1, which satisfies the following conditions.

- The unbounded domain Ω is divided into an internal subdomain Ω_0 and an external subdomain Ω_1 by the interface Γ_1.
- The interface Γ_1 encloses the support of the function f.
- The distance between Γ_0 and Γ_1 is always positive.

Let n_0 and n_1 be unit normal outward vectors corresponding to Ω_0 and Ω_1, respectively. Then, we consider the following method in order to solve Problem 1.

Step 1. Pick a boundary value $\lambda^{(0)} \in H^{1/2}(\Gamma_1)$ and set $k:=0$.

Step 2. Solve the Dirichlet problem in Ω_1:

$$-\Delta u_1^{(k)} = 0 \quad \text{in } \Omega_1,$$

$$u_1^{(k)} = \lambda^{(k)} \quad \text{on } \Gamma_1.$$

Step 3. Solve the mixed boundary value problem in Ω_0:

$$-\Delta u_0^{(k)} = f \quad \text{in } \Omega_0,$$

$$\frac{\partial u_0^{(k)}}{\partial n_0} = -\frac{\partial u_1^{(k)}}{\partial n_1} \quad \text{on } \Gamma_1,$$

$$u_0^{(k)} = g \quad \text{on } \Gamma_0.$$

Step 4. Modify the boundary value:

$$\lambda^{(k+1)} = \alpha_k u_0^{(k)} + (1-\alpha_k)\lambda^{(k)} \text{ on } \Gamma_1,$$

where a relaxation parameter α_k is selected as a suitable real number.

Step 5. Set $k:=k+1$ and go to Step 2.

The Dirichlet-Neumann map $\mathcal{K}_1$ for Ω_1 is defined by

$$\mathcal{K}_1 \lambda := \frac{\partial u_1}{\partial n_1},$$

where u_1 is a solution to the Dirichlet problem of the Laplace equation in Ω_1:

$$-\Delta u_1 = 0 \quad \text{in } \Omega_1,$$

$$u_1 = \lambda \quad \text{on } \Gamma_1.$$

Then, we notice that the equations in Step 2 and 3 are equivalent to the following equations:

$$-\Delta u_0^{(k)} = f \quad \text{in } \Omega_0,$$

$$\frac{\partial u_0^{(k)}}{\partial n_0} = -\mathcal{K}_1 \lambda^{(k)} \quad \text{on } \Gamma_1,$$

$$u_0^{(k)} = g \quad \text{on } \Gamma_0.$$

III. DISCRETISATION

We adopt the boundary element method to solve the external Dirichlet problem in Step 2. We start with the problem mentioned in Step 2:

$$-\Delta u_1 = 0 \quad \text{in } \Omega_1,$$

$$u_1 = \lambda \quad \text{on } \Gamma_1.$$

We consider the fundamental solution of the Laplace equation

$$G(x;\xi) = \frac{1}{2\pi} \ln \frac{1}{\|x-\xi\|_2},$$

which satisfies

$$-\Delta G(x;\xi) = \delta(x-\xi)$$

with the Dirac measure on the right-hand side.

At the point ξ on the boundary, the following boundary integral equation holds:

$$\frac{1}{2} u_1(\xi) + \int_{\Gamma_1} u_1(x) \frac{\partial G}{\partial n_1}(x;\xi) d\Gamma(x) = \int_{\Gamma_1} q_1(x) G(x;\xi) d\Gamma(x)$$

with $\Gamma_1 = \partial\Omega_1$ and $q_1(x) = \partial u_1(x)/\partial n_1$.

Problem 2. For given u_1 on the boundary Γ_1, find q_1 such that

$$\int_{\Gamma_1} q_1(x) G(x;\xi) d\Gamma(x) = \frac{1}{2} u_1(\xi) + \int_{\Gamma_1} u_1(x) \frac{\partial G}{\partial n_1}(x;\xi) d\Gamma(x).$$

We shall describe a discretisation procedure for the boundary integral equation by introducing finite elements on the boundary. To begin with , we approximate Γ_1 by a polygon consisting of n_1 small line segments called elements as $\Gamma_1 = \cup_{j=1}^{n_1} \Gamma_1^{(j)}$. By using the finite element base functions $\varphi^{(j)}(x)$ corresponding to the subdivision of Γ_1, we approximate u_1 and q_1 in the form:

$$u_1(x) \simeq \sum_{j=1}^{n_1} u_1^{(j)} \varphi^{(j)}(x), \quad q_1(x) \simeq \sum_{j=1}^{n_1} q_1^{(j)} \varphi^{(j)}(x),$$

where $u_1^{(j)}$ and $q_1^{(j)}$ are, respectively, nodal values of the functions u_1 and q_1 at the jth node $x^{(j)}$ on the boundary. Then, we take n_1 points of collocation $x^{(i)}$ (i=1, 2, ..., n_1) on the boundary. After replacing the exact u_1 and q_1 by the above approximations, we can obtain the following linear system of equations:

$$\boldsymbol{Hu}=\boldsymbol{Gq},$$

where

$$\boldsymbol{u} = \begin{pmatrix} u_1^{(1)} \\ u_1^{(2)} \\ \vdots \\ u_1^{(n_1)} \end{pmatrix}, \quad \boldsymbol{q} = \begin{pmatrix} q_1^{(1)} \\ q_1^{(2)} \\ \vdots \\ q_1^{(n_1)} \end{pmatrix}.$$

Since the Dirichlet data λ is given on the boundary Γ_1 in Step 2, the linear system of equations can determine unknown $\boldsymbol{q}$ from

$$\boldsymbol{Gq} = \boldsymbol{H\lambda}$$

with given $\boldsymbol{\lambda}$ such that

$$\boldsymbol{\lambda} = \begin{pmatrix} \lambda_1^{(1)} \\ \lambda_1^{(2)} \\ \vdots \\ \lambda_1^{(n_1)} \end{pmatrix}.$$

We adopt the finite element method to solve numerically the mixed boundary value problem in Step 3. For the sake of convenience in mathematical discussion, we take the boundary value g of Problem 1 as 0 without losing generality. We define a functional space $\tilde{H}^1(\Omega_0)$ such that

$$\tilde{H}^1(\Omega_0) := \{ v \in H^1(\Omega_0),\ v=0 \text{ on } \Gamma_0 \}.$$

Problem 3. Find $u \in \tilde{H}^1(\Omega_0)$ such that

$$\int_{\Omega_0} \nabla u \cdot \nabla v d\Omega + \int_{\Gamma_1} (\mathcal{K}_1 u) v d\Gamma = \int_{\Omega_0} f v d\Omega,$$

$$\forall v \in \tilde{H}^1(\Omega_0),$$

where $\mathcal{K}_1$ is the Dirichlet-Neumann map for the domain Ω_1.

We divide the domain Ω_0 into a set of triangular elements. We write down each triangle of Ω_0 as τ, and let τ be an open set. We denote the aggregate of triangulation by T^h. For each subdivision T^h, the symbol h is the positive integer such that $h = \max_{\tau \in T^h} d(\tau)$, where $d(\cdot)$ expresses the diameter of the set.

Let $S^h \subset \tilde{H}^1(\Omega_0)$ be a finite element functional space such that

$$S_h = \{ v_h \in C(\bar{\Omega});\ v_h|_\tau = \alpha_1 + \alpha_2 x + \alpha_3 y,\ \tau \in T^h \},$$

where α_1, α_2, α_3 are coefficients to be determined.

Problem 4. Find $u_h \in S_h$ such that

$$\int_{\Omega_0} \nabla u_h \cdot \nabla v_h d\Omega + \int_{\Gamma_1} (\mathcal{K}_1 u_h) v_h d\Gamma = \int_{\Omega_0} f v_h d\Omega,$$

$$\forall v_h \in S_h. \tag{1}$$

Let N be the number of the vertices P_j in T^h. We notice that dim $S_h = N$. Let $\{P_j\}_{j=1}^{K}$ and $\{\hat{P}_l\}_{l=1}^{N-K}$ be the vertices of $\bar{\Omega}_0 \backslash \Gamma_1$ and Γ_1, respectively. We denote by $\{\phi_i\}_{i=1}^{K}$ and $\{\hat{\phi}_k\}_{k=1}^{N-K}$ the sets of the following piecewise linear functions:

$$\phi_i(P_j) = \delta_{ij}, \quad \phi_i(\hat{P}_l) = 0,$$

$$\hat{\phi}_k(P_j) = 0, \quad \hat{\phi}_k(\hat{P}_l) = \delta_{kl}.$$

Since the functions ϕ_1, ϕ_2, ..., ϕ_K, $\hat{\phi}_1$, ..., $\hat{\phi}_{N-K}$ are the basis of S_h and $\phi_i|_{\Gamma_1}=0$, Eq. (1) is equivalent to the following system:

$$\int_{\Omega_0} \nabla u_h \cdot \nabla \phi_i d\Omega = \int_{\Omega_0} f \phi_i d\Omega, \quad i=1, 2, ..., K,$$

$$\int_{\Omega_0} \nabla u_h \cdot \nabla \hat{\phi}_k d\Omega + \int_{\Gamma_1} (\mathcal{K}_1 u_h) \hat{\phi}_k d\Gamma = \int_{\Omega_0} f \hat{\phi}_k d\Omega,$$

$$k=1, 2, ..., N-K. \tag{2}$$

Denoting nodal values by $u_j = u_h(P_j)$ and $\hat{u}_l = u_h(\hat{P}_l)$, we can write

$$u_h = \sum_{j=1}^{K} u_j \phi_j + \sum_{l=1}^{N-K} \hat{u}_l \hat{\phi}_l.$$

Then the linear system (2) can be written in the matrix form:

$$\begin{bmatrix} \tilde{\boldsymbol{A}}_{11} & \tilde{\boldsymbol{A}}_{12} \\ \tilde{\boldsymbol{A}}_{12}^T & \boldsymbol{A}_{22} + \boldsymbol{K} \end{bmatrix} \begin{Bmatrix} \boldsymbol{U} \\ \boldsymbol{V} \end{Bmatrix} = \begin{Bmatrix} \tilde{\boldsymbol{b}}_1 \\ \boldsymbol{b}_2 \end{Bmatrix} \tag{3}$$

We substitute the Dirichlet boundary condition

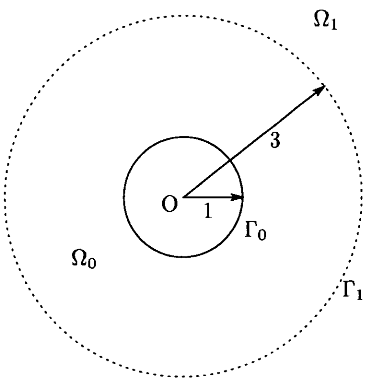

Fig. 1. Domain decomposition.

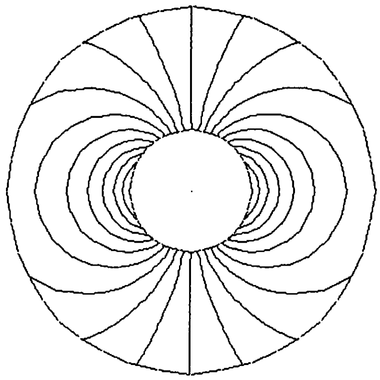
Fig. 3. Exact u.

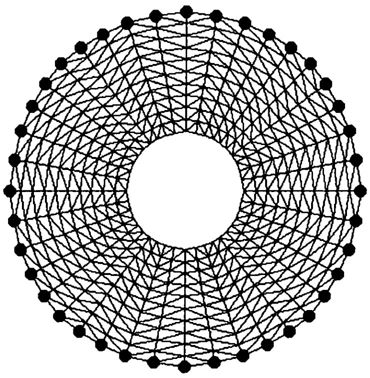
Fig. 2. Finite elements and boundary elements (800 finite elements, 40 boundary elements)

$u_h=g=0$. Let λ be the Dirichlet data prescribed in Step 2. Then, we notice that $\boldsymbol{KV}=\boldsymbol{K\Lambda}$, where $\boldsymbol{\Lambda}=(\lambda(\hat{P}_1), \lambda(\hat{P}_2), \ldots, \lambda(\hat{P}_{N-K}))^T$. Hence, the linear system can be written as

$$\begin{bmatrix} \boldsymbol{A}_{11} \boldsymbol{A}_{12} \\ \boldsymbol{A}_{12}^T \boldsymbol{A}_{22} \end{bmatrix} \begin{Bmatrix} \boldsymbol{U} \\ \boldsymbol{V} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 - \boldsymbol{K\Lambda} \end{Bmatrix}.$$

Therefore, we can get the following recurrence formula:

$$\begin{bmatrix} \boldsymbol{A}_{11} \boldsymbol{A}_{12} \\ \boldsymbol{A}_{12}^T \boldsymbol{A}_{22} \end{bmatrix} \begin{Bmatrix} \boldsymbol{U}_k \\ \boldsymbol{V}_k \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 - \boldsymbol{K\Lambda}_k \end{Bmatrix}, \quad (4)$$

$$\boldsymbol{\Lambda}_{k+1} = \alpha_k \boldsymbol{V}_k + (1-\alpha_k)\boldsymbol{\Lambda}_k \quad (k=0, 1, 2, \ldots)$$

with the relaxation parameter α_k. The coefficient matrix of Eq. (3) is partially asymmetric and full. On the other hand, the coefficient matrix of Eq. (4) is symmetric and sparse. This is the advantage of our method.

IV. NUMERICAL ALGORITHM

From the methods of approximation described in the previous section, our numerical algorithm can be summarized as follows:

Step 1. Pick an initial value $\boldsymbol{\Lambda}_0$ and set k:=0.
Step 2. Solve the Dirichlet problem in Ω_1 using the boundary element method to find $\boldsymbol{K\Lambda}_k$.
Step 3. Solve the mixed boundary value problem in Ω_0 using the finite element method to find $\boldsymbol{V}_k$.
Step 4. Update the boundary value:

$$\boldsymbol{\Lambda}_{k+1} = \alpha_k \boldsymbol{V}_k + (1-\alpha_k)\boldsymbol{\Lambda}_k .$$

Step 5. Set k:=k+1 and go to Step 2.

We obtain the following theorem about this algorithm.

Theorem (Yu, 1996) If the relaxation parameter α_k satisfies the inequality $0<\alpha_k<1$, then the iteration using Step 2 through Step 4 is convergent.

The convergence of our discrete iterative scheme with an arbitrary initial value $\boldsymbol{\Lambda}_0$ is thus guaranteed from this theorem.

V. NUMERICAL EXPERIMENTS

In this section, we demonstrate the effectiveness of our numerical method through numerical experiments.

We notice that the function $u=\cos\theta/r$ is a solution of the Laplace equation, where (r, θ) denotes the polar coordinates. Suppose that the function u is unknown, and consider the Laplace equation in the external domain $\Omega=\{(r, \theta); r>1, 0\le\theta<2\pi\}$ with the boundary condition $u=\cos\theta$ on the boundary $\Gamma_0=\{(1, \theta); 0\le\theta<2\pi\}$.

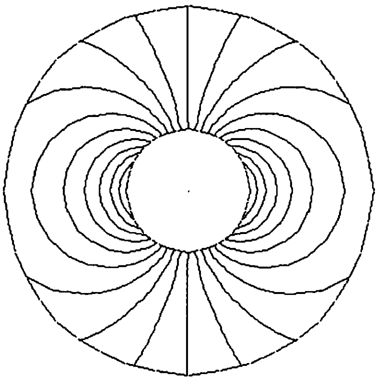

Fig. 4. Calculated $u_0^{(3)}$.

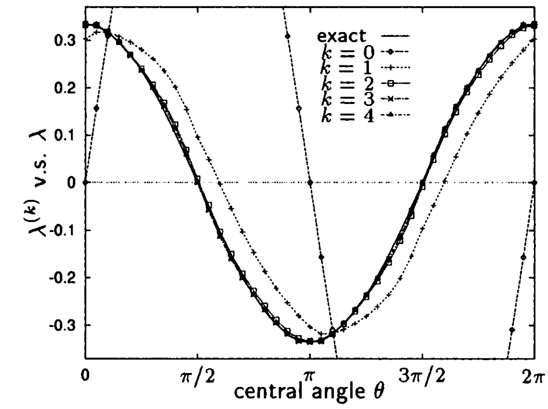

Fig. 6. Calculated $\lambda^{(k)}$ v.s. exact λ ($\lambda^{(0)}$=sinθ).

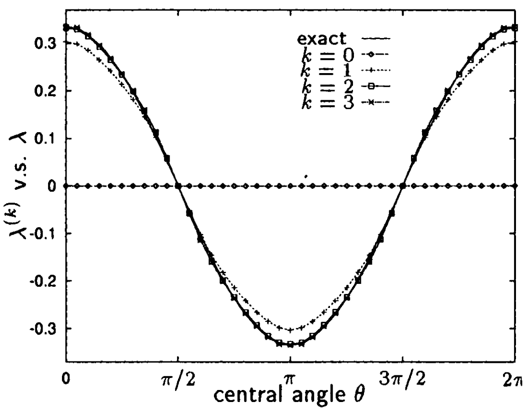

Fig. 5. Calculated $\lambda^{(k)}$ v.s. exact λ ($\lambda^{(0)}$=0).

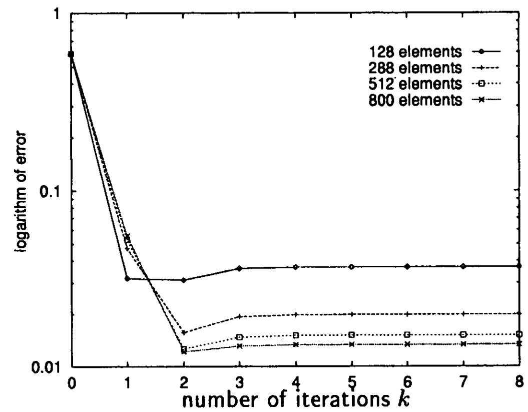

Fig. 7. Errors for each mesh size.

The external domain Ω is decomposed into the bounded subdomain $\Omega_0=\{(r, \theta);\ 1<r<3,\ 0\leq\theta<2\pi\}$ and the unbounded subdomain $\Omega_1=\{(r, \theta);\ r>3,\ 0\leq\theta<2\pi\}$ by the interface $\Gamma_1=\{(3, \theta);\ 0\leq\theta<2\pi\}$ (see Fig. 1).

The domain Ω_0 and the boundary Γ_1 are divided into triangular finite elements and boundary elements respectively as shown in Fig. 2. We set α_k=0.5 (k=0, 1, 2, ...). As an initial guess, we take $\lambda^{(0)}$=0 along the circle Γ_1. Fig. 3 shows the contour lines for the exact solution, and Fig. 4 the calculated contour lines at the number of iterations k=3. We can see by comparing these two figures that the numerical solution is in good agreement with the exact one. Calculated boundary values $\lambda^{(k)}(\theta)$ with two initial values $\lambda^{(0)}$=0 and sinθ are plotted against central angles θ with reference to the exact $\lambda(\theta)=u(3, \theta)=\cos\theta/3$ in Figs. 5 and 6 respectively. It is independent of the choice of initial values that calculated boundary values $\lambda^{(k)}$ converge to the exact λ, which yields that it is possible to pick arbitrary initial boundary values. The errors $\|\lambda-\lambda^{(k)}\|_{L^2(\Gamma_1)}\approx\{2\pi\sum_{i=1}^{m}|\lambda(\theta_i)-\lambda^{(k)}(\theta_i)|^2/m\}^{1/2}$ for each mesh size are plotted in Fig. 7, where we take $\lambda^{(0)}$=0 and set $\theta_i=2\pi(i-1)/m$ with m boundary nodes. We can see that the convergence rate is independent of the mesh size of finite and boundary elements.

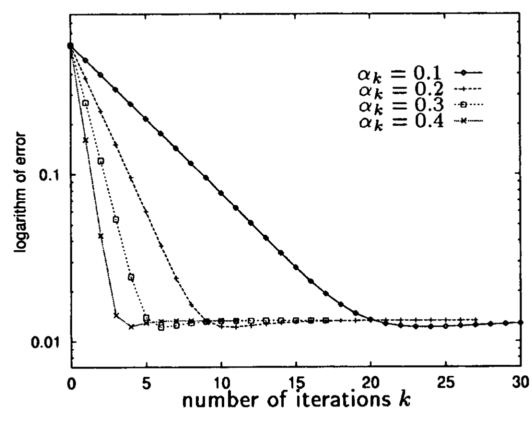

Fig. 8. Errors for each α_k (1).

Figures 8 and 9 show the errors $\{2\pi\sum_{i=1}^{m}|\lambda(\theta_i)-\lambda^{(k)}(\theta_i)|^2/m\}^{1/2}$ for each α_k with $\lambda^{(0)}$=0 and m=40. The convergence is oscillatory as α_k tends to 1. On the other hand, the convergence is not oscillatory as α_k tends to 0. It is clear that our scheme with α_k=0 is not convergent. When α_k is near 0.5, the convergence is very rapid. In this problem, we observed from Fig. 10 that the optimal α_k is 0.5 or 0.6.

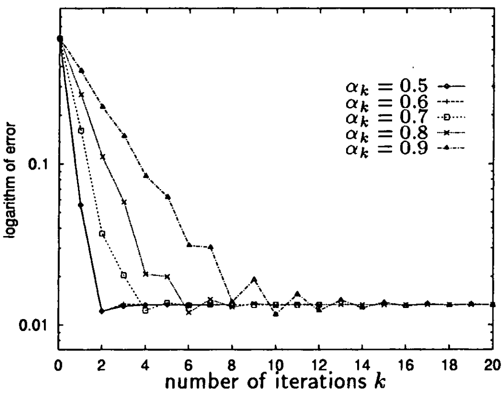

Fig. 9. Errors for each α_k (2).

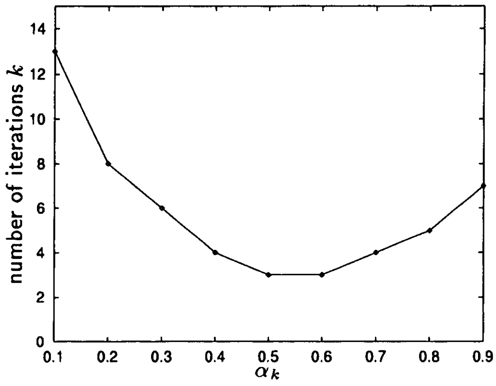

Fig. 10. Number of iterations for each α_k.

V. CONCLUSIONS

We considered an iterative numerical algorithm for an external Dirichlet problem of the Poisson equation. The Dirichlet-Neumann alternating method proposed by Yu consists of the following steps: 1) The domain of the problem is decomposed into a bounded subdomain and an unbounded subdomain by an interface. 2) For arbitrarily given Dirichlet data on the interface, the boundary value problem in the unbounded subdomain is solved. 3) By using the solution in 2), the boundary value problem in the bounded subdomain is solved by the finite element method. For a circular interface, the solution of the boundary value problem in the unbounded subdomain can be given by the Poisson integral. In order to use this integral, we need to treat numerically the hyper-singular integration.

In our algorithm, applying the boundary element method, we can solve, numerically, the boundary value problem in the unbounded subdomain without treatment of the hyper-singular integration. We demonstrated effectiveness of our algorithm by the numerical experiments.

NOMENCLATURE

A_{ij}	coefficient matrix in FEM
f	inhomogeneous term of the Poisson equation
$G(x;\xi)$	fundamental solution of Δ
g	Dirichlet data
$\boldsymbol{H}, \boldsymbol{G}$	coefficient matrices in BEM
$\boldsymbol{K}$	coefficient matrix corresponding to $\mathcal{K}_1$
S_h	finite element space
$\boldsymbol{U}, \boldsymbol{V}$	nodal column vectors in FEM
u	solution of the Laplace equation
$\boldsymbol{u}, \boldsymbol{q}$	nodal column vectors in BEM
α_k	relaxation parameter
Γ_0	$\partial\Omega$, the boundary of Ω
Γ_1	interface
Δ	Laplacian
θ	central angle in radian
$\boldsymbol{\Lambda}$	nodal column vector corresponding to λ
λ	unknown value of u on Γ_1
$\varphi^{(j)}$, ϕ_i	finite element base functions
Ω	unbounded domain of the problem
$\mathcal{K}_1$	Dirichlet-Neumann map

REFERENCES

1. Feng, Y.T. and Owen, D.R.J., 1996, "Iterative Solution of Coupled FE/BE Discretizations for Plate-Foundation Interaction Problems", *Internatioal Journal of Numerical Methods in Engineering*, Vol. 39, pp. 1889-1901.
2. Gatica, G.N. and Hsiao, G.C., 1995, "Boundary-field Equation Methods for a Class of Nonlinear Problems", Pitman Research Notes in Mathematical Sciences 331, Longman, Essex, England.
3. Harayama, T., Shigeta, T., and Onishi, K., 1998, "Solution to an External Problem of the Poisson Equation Using the Dirichlet-Neumann Map", *Proceedings of the 8th BEM Technology Conference*, pp. 21-22, Tokyo, Japan.
4. Harayama, T., Shigeta, T., and Onishi, K., 1998, "Combination of Finite and Boundary Elements for External Problem of the Poisson Equation", *The 76th JSME Fall Annual Meeting*, Vol. II, No. 98-3, pp. 169-170, Sendai, Japan.
5. Harayama, T., Shigeta, T., and Onishi, K., 1998, "Domain Decomposition Method for an External Mixed Boundary Value Problem of the 2-dimensional Poisson Equation", *Proceedings of the 15th Japan National Symposium on Boundary Element Methods*, pp. 41-46, Toyko, Japan.
6. Yu, D., 1996, "Discretization of Non-overlapping Domain Decomposition Method for Unbounded Domains and its Convergence", *Chinese Journal of Numerical Mathematics and Applications*, Vol. 18, No. 4, pp. 93-102, China.

Discussions of this paper may appear in the discussion section of a future issue. All discussions should be submitted to the Editor-in-Chief.

Manuscript Received: May 28, 1999
Revision Received: July 20, 1999
and Accepted: Aug. 18, 1999

以有限元素法與邊界元素法求解柏松方程式之外域邊界值問題

繁田岳美　原山卓也　代田健二

茨城大學大學院理工學研究所

摘 要

本文考慮柏松方程式Dirichlet外域問題之數值算法。延伸到無限大的定義域Ω域可區分爲有界域Ω_0和無界域Ω_1。而有限元素法和邊界元素法可分別應用在有界域邊界值問題和無界域邊界值問題。本文利用在界面$\partial\Omega_1$上Dirichlet-Neumann映射圖，提出疊代的技巧。此種疊代的技巧可證明在數值上收斂。由一個簡單的例子可顯示出我們的架構是非常有效率的。

關鍵詞：二維邊界值問題，反覆有限元素／邊界元素對偶架構，Dirichlet-Neumann 圖。