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A GREEN'S FUNCTION NUMERICAL METHOD FOR SOME INVERSE BOUNDARY VALUE PROBLEMS

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ABSTRACT

We solve identification problems in potential theory, electrostatics, steady state heat conduction, etc. in conductive plates. A plate.with simple shape contains a hole. Either the size, shape, and location of the hole, or the flux across its boundary, is unknown. Boundary conditions are prescribed on part of the plate edge, and voltage or heat is applied to the remainder. An additional measurement is taken on part of the plate edge. From this overspecified data, we determine (1) the structure of the hole, given the flux across its boundary, or (2) the flux across the hole boundary, given the hole. Boundary elements, least squares optimization, and special boundary value properties of Green's functions are exploited in this numerical treatment of the identification problem.

I. INTRODUCTION

Suppose that a domain is occupied by an electrically or thermally conductive material which contains a hole. From electrostatic or steady state thermal measurements at the boundary of the domain, we would like to characterize the location, size, and shape of the hole, or some other unknown property such as current flux across the hole boundary.

Numerical methods for solution of inverse boundary value problems of internal cavity detection or flux reconstruction have been proposed before; see (Bryan, 1993; Das and Mitra, 1992; Kassab *et al.*, 1997; Oğuz and Han, 1998; Saigal and Zeng, 1992; Tanaka and Masuda, 1986); see also references therein. These methods use least squares optimization, coupled with an integral equation for calculation of a potential at the material boundary as part of the iteration process. Our method applies the same idea, except that Green's functions are used at the potential construction stage. Special boundary value properties of Green's functions provide a computational advantage that use of the simple logarithmic fundamental solution does not provide; namely, use of a Green's function situates the integral equation only on the hole boundary; for each updated data estimate, maximum error is confined to the boundary of the guessed cavity, so that accuracy is achieved in the simulated data at the plate boundary.

Fortunately, Green's functions or matrices for elliptic partial differential equations with mixed boundary conditions exist in closed form for a number of domains of a variety of simple shapes. Melnikov (1995) has developed techniques which make it possible to obtain formulae for these Green's functions. The Green's functions technique has already been applied to several problems in applied mechanics; see e.g. (Melnikov, 1977; 1995; Melnikor and Koshnarjova, 1994; Melnikov and Titarenko, 1995) This paper's purpose is to demonstrate their

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applicability to the solution of inverse boundary value problems; see also our previous work (Melnikov and Powell, 1998).

In section 2, we formulate the inverse problems. In section 3, we discuss some of their theoretical aspects. Section 4 contains the numerical procedure for solution of the direct and inverse problems; roughly, an initial estimate is made, and the boundary integral equation is solved for the corresponding potential at the material boundary; then this estimate is updated by means of the least squares method, with the object of minimizing the difference between measured data and calculated data corresponding to the estimated hole or flux. Numerical examples are presented in section 5. Our results are summarized in section 6.

One can find a description in (Melnikov, 1995) of the algorithm that we use to obtain compact, easily computable representations of Green's functions. It should be noted that the method for Green's function construction in (Melnikov, 1995) is not limited to the harmonic case. (Melnikov, 1995) gives a method for their construction for two dimensional Helmholtz equations, biharmonic equations, and elastic systems. Hence, their application to inverse hole determination problems can be extended to problems modeled by those equations. They could be applied as well to the related problem of determining material interfaces; see e.g. (Isakov, 1998), chapter 4, and (Schnur and Zabaras, 1992).

II. PROBLEM FORMULATION

Let Ω be a planar domain with piecewise smooth boundary

$$\Gamma = \bigcup_{i=1}^{m} \Gamma_i .$$

We assume that an electrically or thermally conductive material occupies Ω. The boundary value problem for voltage or temperature u in Ω is

$$\Delta u=0 \qquad z\in\Omega \tag{1}$$

$$\alpha_i \frac{\partial u}{\partial n_i} + \beta_i u = h_i \quad z\in\Gamma_i,\ i=1,\ \ldots,\ m \tag{2}$$

where Δ is Laplace's operator, α_i, β_i are constants not simultaneously zero on Γ_i, not all β_i=0, n_i are unit normal vectors to Γ_i exterior to Ω, and h_i are functions given only on Γ_i.

Suppose a hole exists inside the material that occupies Ω; that the hole coincides with a simply connected, open set D_0; the boundary Γ_0 of D_0 is a smooth, simple, closed Jordan curve; and $\overline{D_0}\subset\Omega$. For the numerical solution of our inverse problem, the recovery of D_0 from measurements on Γ, we need numerical solutions u of the "direct" problem

$$\Delta u=0, \qquad z\in\Omega\backslash D_0 \tag{3}$$

with boundary data (2), and

$$\alpha_0 \frac{\partial u}{\partial n_0} + \beta_0 u = h_0\,,\ z\in\Gamma_0, \tag{4}$$

where $\alpha_0\neq 0$, and possibly h_0=0.

In case the location, size, and shape of D_0 are unknown, the inverse hole determination problem is the following: Given (2), (3), (4), and h_0=0 on the unknown boundary Γ_0, determine D_0 from one additional measurement

$$\gamma_i \frac{\partial u}{\partial n_i} + \delta_i u = f_i \tag{5}$$

on one or more parts Γ_i of the boundary Γ, where the pairs (γ_i, δ_i) are linearly independent from (α_i, β_i). Note that if D_0 were known, then this additional data would make determination of u in $\Omega\backslash D_0$ an ill-posed problem.

If D_0 is known, α_0=1, β_0=0, but $\partial u/\partial u_0=h_0$ is unknown, then the inverse flux determination problem is: Given (2), (3), plus one additional measurement (5) on some parts Γ_i, determine h_0 on Γ_0 as a function of a parameter, say arclength or a radial variable.

For example, suppose α_i=1, β_i=0, h_i=0 for all i except, say, $i=i_0$. Suppose α_{i_0}=0, β_{i_0}=1, $h_{i_0}\neq$constant. In other words, suppose all edges Γ_i, $i\neq i_0$, are insulated, and a voltage is applied to Γ_{i_0}. Then measure the induced voltage $u=f_j$ on some other Γ_j. The result is Cauchy data, i.e., Dirichlet and Neumann data, on the edge Γ_j. In case of mixed data on Γ_j, linear independence of (α_j, β_j) and (γ_j, δ_j) would also result in Cauchy data on this edge. in case $j=i_0$, current could be applied instead, and the resulting voltage measured; or mixed conditions could be applied, and mixed conditions measured. However, as will be shown, it is essential that the combination of applied and measured fields results in nonconstant u on some part of Γ.

We remark that Γ_0 and h_0 cannot be determined simultaneously from a single set (h_i, f_j) of overspecified data for any or all of i, j=1, ..., m. In fact, let $\xi^*\in\Omega$. Then it is easy to find two distinct open sets D_1, D_2 in Ω which both contain ξ^*, such that the functions $g_k(z)=\partial G(z,\ \xi^*)/\partial n$, k=1, 2, where $z\in\partial D_k$ and G is Green's function for (1), (2), satisfy $g_1\neq g_2$. Hence, the nonidentical pairs (D_k, g_k), k=1, 2, produce identical boundary data $\alpha_i\partial G/\partial n_i+\beta_i G=h_i$, $\gamma_i\partial G/\partial n_i+\delta_i G=f_i$ on all segments Γ_i.

III. THEORETICAL CONSIDERATIONS

If D_0 is a crack (Friedman and Vogelius, 1989), then a single additional measurement (5) on Γ_i is insufficient to uniquely determine D_0; see also (Brian and Vogelius, 1992). In fact (Friedman and Vogelius, 1989), it takes two separate measurement pairs (h_{ij}, f_{ij}) on Γ_i, j=1, 2 in order to uniquely determine a crack D_0. Physically, this means that a material is tested twice, with distinct input h_{ij} at each test j=1, 2. Therefore, we impose the condition of nonempty interior on D_0, the hole which is to be determined. Note that we are not attempting to solve in this paper the crack determination problem.

The following uniqueness result could be generalized to a more complicated hole boundary structure, and more general boundary conditions; our purpose is to suggest necessary boundary conditions for unique determination of D_0, so we avoid these technicalities.

Theorem 1.

Let D_k, k=1, 2 be simply connected open sets in Ω with nonempty interiors, such that the boundaries ∂D_k are simple, closed C^2 Jordan curves. Suppose u_k satisfy (2), (3), (4) in $\Omega\backslash D_k$, with $h_0\equiv 0$. If $u_k=h_{i0}$ on Γ_{i_0} for k=1, 2, for some i_0, where $h_{i0}\neq$constant, and for some i,

$$u_1=u_2,\ \frac{\partial u_1}{\partial n_i}=\frac{\partial u_2}{\partial n_i}\ \text{ on } \Gamma_i,$$

then $D_1=D_2$.

Observe that the normal derivative $h_0\equiv 0$ implies that the conjugate periods

$$\int_\Gamma \frac{\partial u_k}{\partial n}(t)d\Gamma(t)=0\,,\quad \int_{\Gamma_0} \frac{\partial u_k}{\partial n_0}(t)d\Gamma(t)=0\,.$$

Hence (Henrici, 1986), u_k has a single valued harmonic conjugate function v_k defined on all of $\Omega\backslash D_k$. Furthermore, by the Cauchy-Riemann conditions, $h_0\equiv 0$ implies that the tangential derivative $\partial v_k/\partial\tau\equiv 0$ on ∂D_k, so that $v_k\equiv$constant on ∂D_k.

Proof of Theorem 1:

Note that u_1, u_2 have identical Cauchy data on Γ_i, so that by uniqueness of continuation of harmonic functions, $u_1=u_2$ on $\Omega\backslash D_1\cup D_2$. We suppose $D_1\neq D_2$ and without loss of generality that D_1 is not contained in D_2. We consider two cases: (i) $D_1\cap D_2=\emptyset$, (ii) $D_1\cap D_2\neq\emptyset$. For (i), the function u defined by

$$u=\begin{cases} u_1 & z\in\Omega\backslash D_1 \\ u_2 & z\in D_1 \end{cases}$$

is a harmonic continuation of u_1 into all of D_1. In particular, u is harmonic inside D_1, and ∂_u/∂_n=0 on ∂D_1 implies that $u\equiv$ constant on D_1. Therefore, by uniqueness of continuation, $u\equiv$ constant on Ω, which contradicts the assumption that u_1 is nonconstant on Γ_{i_0}.

In case (ii), either (a) $\overline{D_2}\subset D_1$, or (b) there exists a simply connected component C of $D_1\backslash D_2$ such that $C\subset\Omega\backslash D_2$. If (a), then the function $u=u_2$ is a harmonic continuation of u_1 into the set $D_1\backslash D_2$, and u has zero normal derivative on the boundary of this set. Therefore, the harmonic function u cannot attain its maximum in $D_1\backslash D_2$ on the boundary of $D_1\backslash D_2$, unless $u\equiv$ constant; see (Miranta, 1970). This contradicts the assumption that $u_1\neq$ constant on Γ_{i_0}. If (b), then the boundary ∂C is composed of two sets $\Lambda_k=\partial C\cap\partial D_k$, k=1, 2, and may have a complicated, piecewise smooth geometry. Let $u=u_1$ in $\Omega\backslash D_1$, $u=u_2$ in C. Then u is a harmonic continuation of u_1 from $\Omega\backslash D_1$ into C; and $\partial u/\partial n$=0 on ∂C, where n is the exterior unit normal vector to ∂C, except at points where ∂C is not smooth. In general, it is not true that a function harmonic inside a domain C with zero Neumann data except at nonsmooth boundary points is constant; consider even the function $u(z)=\ln|z+1|-\ln|z-1|$, which has zero Neumann data on the upper and lower semicircles of the unit disk. However, in our situation, the harmonic conjugates v_k of u_k, k=1, 2, differ only by a constant, say $v_2=v_1+\lambda$ on $\Omega\backslash D_1\cup D_2$. Hence, $v=v_2-\lambda$ is a harmonic continuation of v_1 into C which is constant, say $v=\kappa$, at all points of ∂C. Since v is continuous on the compact set $\overline{C}$, it obtains its maximum at some point $z_0\in\overline{C}$. We claim $v\equiv\kappa$ in C. If not, suppose that v attains its maximum at z_0 in the interior of C, and $v(z_0)=\xi>\kappa$. Then z_0 is contained inside a region R which is bounded by a closed level curve $v(z)=\chi$, $\kappa<\chi<\xi$; if not, then a path P in the interior of C from z_0 to ∂C exists such that $v(z)\neq\chi$ on P, which contradicts the intermediate value theorem. The closed curve $v(z)=\kappa$ is smooth. Therefore, the maximum principle implies that v is constant on R, hence on C by uniqueness of continuation. The Cauchy-Riemann conditions now show that $u\equiv$ constant on C. Thus, u is a constant harmonic continuation of u_1 to all of Ω. The theorem is proved.

The authors are unaware of as general a proof of uniqueness in the hole determination problem. This proof is a simple extension of the method for proving uniquess via harmonic continuation which was used in a related inverse problem of determining a piecewise constant conductivity (Isakov, 1998). For this problem, uniqueness is known to hold in special cases, namely when the discontinuous piece is in the form of a disk, a polygon, or a ball; for these cases, a hole could be construed as a region of zero conductivity. In (Sylvester and Uhlmann, 1987), a more general inverse conductivity problem was examined; but the data set consisted of the infinite set of all possible Dirichlet-Neumann measurement pairs, whereas this hole determination problem requires only a single measurement pair.

For flux determination, we have

Theorem 2.

Let D_0 be an open, simply connected set, with $\overline{D}_0 \subset \Omega$, and boundary Γ_0 a simple, closed Jordan curve. Suppose that u_1, u_2 satisfy (2), (3), and that

$$\alpha_0 \frac{\partial u_k}{\partial n} + \beta_0 u_k = h_{0k} \text{ on } \Gamma_0,\ k=1,\ 2.$$

If for some i,

$$u_1 = u_2,\ \frac{\partial u_1}{\partial n_i} = \frac{\partial u_2}{\partial n_i} \text{ on } \Gamma_i,$$

then $h_{01}=h_{02}$.

Proof of Theorem 2:

Let $u=u_1-u_2$. Then u has zero Cauchy data on Γ_i, hence $u\equiv 0$ on $\Omega\backslash D_0$. Therefore, $u_1=u_2$ on $\Omega\backslash D_0$. It follows that $h_{01}=h_{02}$, which concludes the theorem.

IV. NUMERICAL ALGORITHM

Let $G(z, t)$ be the Green's function for problem (1), (2) in Ω. That is, G satisfies the boundary conditions (2) on Γ with $h_i\equiv 0$ for all $i=1, \ldots, m$;

$$-\Delta_z G(z, t) = \delta(z-t), \qquad t, z \in \Omega;$$

and G has the property that as z tends to t, $G(z, t)$ tends to $(-1/2\pi)\ln|z-t|$. Here, the complex variables z and t denote the field and source point, respectively.

The solution u of (2), (3), (4) may be constructed by means of a Fredholm boundary value problem. First, the Green's function for (1), (2) in Ω is used to obtain the solution w to (1), (2) in Ω without the hole,

$$w(z) = \sum_{i=1}^{m} \int_{\Gamma_i} G_i(z, t) h_i(t) d\Gamma(t)$$

where the G_i are defined on Γ_i by

$$G_i(z, t) = \begin{cases} G(z, t)/\alpha_i \\ -\dfrac{\partial}{\partial n_i} G(z, t)/\beta_i \end{cases}$$

If α_i and β_i are both nonzero, then either definition is valid. u is found in the form $u=v+w$, where v is the harmonic function in $\Omega\backslash D_0$ with homogeneous data on Γ and

$$\alpha_0 \frac{\partial v}{\partial n_0} + \beta_0 v = h_0 - (\alpha_0 \frac{\partial w}{\partial n_0} + \beta_0 w) = g \text{ on } \Gamma_0.$$

We write

$$v(z) = \int_{\Gamma_0} G(z, t)\mu(t) d\Gamma(t), \tag{6}$$

diffferentiate, and solve for the density μ in the equation which then results from the jump condition, (Muskhelishvili, 1992).

$$-\frac{1}{2}\alpha_0\mu(z) + \int_{\Gamma_0} [\alpha_0 \frac{\partial}{\partial n_0} G(z, t) + \beta_0 G(z, t)]\mu(t) d\Gamma(t) = g(z) \tag{7}$$

on Γ_0, where the normal derivative is with respect to $z\in\Gamma_0$. The numerical solution can be obtained by means of a matrix equation for μ at points on Γ_0.

The following numerical method for solution of the inverse problems combines the boundary element method for solution of the direct problem with a least squares optimization procedure (Marquardt, 1963) for updating approximations.

We make an initial estimate $\tilde{D}_0$ for D_0, with boundary $\tilde{\Gamma}_0$. In the case of flux determination, we make an initial estimate $\tilde{h}_0$ for unknown h_0. The integral Eq. (7) is solved for $\tilde{\mu}$ on $\tilde{\Gamma}_0$, which is inserted in (6) with Γ_0 replaced by $\tilde{\Gamma}_0$. This results in calculated data $\tilde{f}_i$ on Γ_i. For notational economy, we will omit the subscript i on calculated data.

The objective functional is

$$F(\tilde{f}(\mathbf{c}), f) = \int_{\Gamma_i} \left|\tilde{f}(t; \mathbf{c}) - f(t)\right|^2 d\Gamma(t)$$

where f is data on Γ_i for the inverse problem, and $\mathbf{c}=(c_1, \ldots, c_n)$ is the set of parameters which characterizes $\tilde{\Gamma}_0$ or $\tilde{h}_0$. We would like to minimize the difference F with respect to $\tilde{D}_0$ or $\tilde{h}_0$. We parameterize the estimate $\tilde{\Gamma}_0$ with respect to arclength or a radial variable σ by $t(\sigma)=t(\sigma;\ \mathbf{c})$, so that σ will become the variable of integration in (6), (7). We sometimes use $z(\theta)=z(\theta;\ \mathbf{c})$ to denote the same parameterization. In the flux determination problem, $\tilde{h}_0(\sigma)=\tilde{h}_0(\sigma;\ \mathbf{c})$, with parameterization $t(\sigma)$ given.

We assume that $\mathbf{c}+\delta$, $\delta=(\delta_1, \ldots, \delta_n)$ to be determined, gives the closest parameterization of the unknown, where the first order Taylor approximation of F is

$$\Phi(\mathbf{c}+\delta) = \int_{\Gamma_i} [\tilde{f}(t; \mathbf{c}) + \sum_{j=1}^{n} \frac{\partial \tilde{f}}{\partial c_j}(t; \mathbf{c})\delta_j - f(t)]^2 d\Gamma(t)$$

The minimality assumption

$$\frac{\partial \Phi}{\partial \delta_k}(c+\delta) = 2\int_{\Gamma_i}[\tilde{f}(t;c) + \sum_{j=1}^{n}\frac{\partial \tilde{f}}{\partial c_j}(t;c)\delta_j - f(t)]\frac{\partial \tilde{f}}{\partial c_k}(t;c)d\Gamma(t)$$
$$=0$$

for k=1, ..., n results in the matrix equation

$$A\delta = q$$

for δ. The $n\times n$ matrix $A=(a_{kj})$ and $q=(q_1, \ldots, q_n)$ have entries

$$a_{kj} = \int_{\Gamma_i}\frac{\partial \tilde{f}}{\partial c_j}(t;c)\frac{\partial \tilde{f}}{\partial c_k}(t;c)d\Gamma(t)$$

and

$$q_k = \int_{\Gamma_i}[\tilde{f}(t;c) - f(t)]\frac{\partial \tilde{f}}{\partial c_k}(t;c)d\Gamma(t) \qquad (8)$$

respectively.

The derivatives $\partial\tilde{f}/\partial c_j$, for $z\in\Gamma_i$, are calculated from $\hat{u} = \tilde{v} + w$ on the parameterization of the integral (6), with t given in case h_0 is to determined,

$$\tilde{v}(z) = \int_0^{2\pi} G(z, t(\sigma))\tilde{\mu}(\sigma)\left|t'(\sigma)\right|d\sigma$$

as follows:

$$\frac{\partial \hat{u}}{\partial c_j}(z) = \frac{\partial \tilde{v}}{\partial c_j}(z) = \int_0^{2\pi} G(z, t(\sigma))\frac{\partial \tilde{\mu}}{\partial c_j}(\sigma)\left|t'(\sigma)\right|d\sigma$$
$$+\int_0^{2\pi}\frac{\partial}{\partial c_j}[G(z, t(\sigma))\left|t'(\sigma)\right|]\tilde{\mu}(\sigma)d\sigma$$

where the derivatives $\partial\tilde{\mu}/\partial c_j$ are calculated from the parameterized integral equations (7), for $0\le\theta\le2\pi$

$$-\frac{1}{2}\alpha_0\frac{\partial \tilde{\mu}}{\partial c_j}(\theta) + \int_0^{2\pi}[\alpha_0\frac{\partial}{\partial n_0}G(z(\theta), t(\sigma))$$
$$+\beta_0 G(z(\theta), t(\sigma))]\frac{\partial \tilde{\mu}}{\partial c_j}(\sigma)\left|t'(\sigma)\right|d\sigma$$
$$= -\int_0^{2\pi}\frac{\partial}{\partial c_j}[\alpha_0\frac{\partial}{\partial n_0}G(z(\theta), t(\sigma))$$
$$+\beta_0 G(z(\theta), t(\sigma))\left|t'(\sigma)\right|]\tilde{\mu}(\sigma)d\sigma$$
$$+\frac{\partial}{\partial c_j}[\tilde{h}_0(z(\theta)) - (\alpha_0\frac{\partial}{\partial n_0}w(z(\theta)) + \beta_0 w(z(\theta)))],$$

The $\partial\tilde{h}_0/\partial c_j$ terms are zero if D_0 is to be determined. The parameter derivatives of $\partial w/\partial n_0$ are zero if D_0 is known and h_0 is to be determined.

We solve for δ by means of the Marquardt procedure (Marquardt, 1963). The parameterization **c** is updated by $c=c+\delta$. The procedure iterates until a convergence criterion is met, say, difference of successive approximations ≈ 0. The final output c will be the approximate parameterization of the solution to the inverse problem.

V. NUNMERICAL EXAMPLES

In each numerical experiment, exact data for the inverse problem was simulated by means of the integral equation procedure (6), (7). This procedure was carefully validated by comparing its result with exact solutions, that is, the closed form Green's functions for the mixed boundary value problem in Ω, with source point fixed in the interior of D_0; this resulted in exact solutions with nonzero flux on Γ_0 and homogeneous mixed conditions on Γ.

The least squares iteration was terminated upon satisfaction of the following convergence criterion: Let N be the euclidean norm of differences between successive approximations of domain parameters. Convergence was considered obtained when $N<$ 0.0001.

Estimates for solutions of the inverse problems were either: disks $z(\theta)=c_0+re^{i\theta}$, $0\le\theta\le2\pi$, where the three parameters $c_0=c_1+ic_2$ and $c_3=r>0$ are the center points and radius, respectively; or ellipses $z(\theta)=\rho(\theta)e^{i\theta}$, where ρ depends on five parameters $c_1, \ldots, c_5$.

Example 1:

$\Omega=\{z=x+iy : 0\le y\le\pi/2\}$, an infinite strip. Here, $\Gamma=\Gamma_1\cup\Gamma_2$, where Γ_1 is the line y=0, and Γ_2 is the line $y=\pi/2$. Let

$$E_1(p)=|e^p+1|, \qquad E_2(p)=|e^p-1|$$

The Green's function for Ω with boundary conditions

$$u(x,0) = \frac{\partial u}{\partial y}(x, \frac{\pi}{2}) = 0, \quad u(\pm\infty, y)<\infty$$

is

$$G(z,t) = \frac{1}{2\pi}\ln\frac{E_1(z-t)E_2(z-\bar{t})}{E_2(z-t)E_1(z-\bar{t})}$$

For hole determination, we suppose $\partial u\backslash\partial y=0$ on Γ_2 and apply nonconstant heat or voltage with compact support $u=h_1(x)=1-\cos(\pi x/2)$ in $[0, 4]\subset\Gamma_1$. Voltage or heat measurement $u=f$ is simulated on Γ_2 for $0\le x\le4$. Results of application of the algorithm to obtain circular approximations of a circular hole D_0 in Ω are depicted in Fig. 1. The boundary Γ_0 is the thick circle centered at (.2, .7) with radius r=.3. The

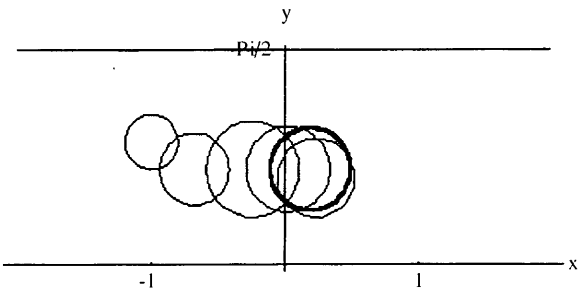

Fig. 1. Approximation of a disk in an infinite strip.

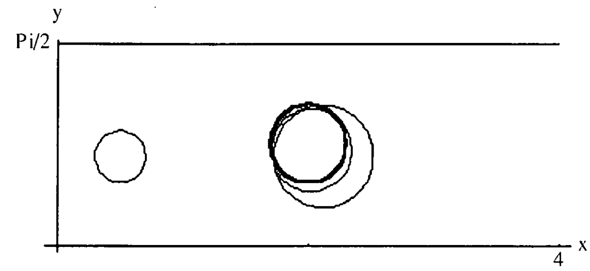

Fig. 2. Approximation of a disk in a semistrip.

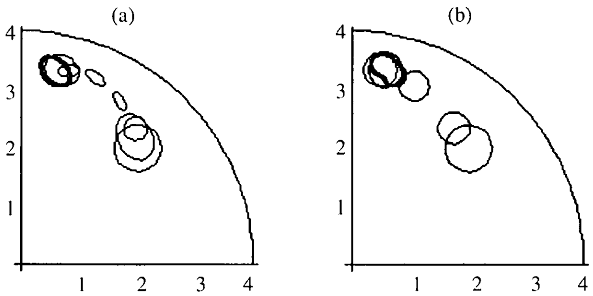

Fig. 3. Approximation of (a) an ellipse, (b) a nonconvex domain.

initial estimate is the disk with parameters $c_1=-1$, $c_2=.9$, $c_3=.2$. The convergence criterion was met in six iterations, the last of which is not visibly distinguishable from the true hole.

Example 2:

$\Omega=\{z=x+iy : 0\le x,\ 0\le y\le \pi/2\}$, a semistrip with $\Gamma=\Gamma_1\cup\Gamma_2\cup\Gamma_3$, where Γ_1, Γ_2, Γ_3 are the lower, left, and upper boundaries, respectively. Prescribe

$$\left(\frac{\partial u}{\partial y}-\beta u\right)\Big|_{\Gamma_2}=u\,\big|_{\Gamma_1}=\frac{\partial u}{\partial y}\Big|_{\Gamma_3}=0\,,$$

$$\beta\ge 0,\ u|_{x=\infty}<\infty$$

Then the Green's function for Ω that satisfies these boundary conditions is

$$G(z,t)=\frac{1}{2\pi}\ln\frac{E_1(z+\bar{t})E_1(z-t)E_2(z+t)E_2(z-\bar{t})}{E_1(z+t)E_1(z-\bar{t})E_2(z-t)E_2(z+\bar{t})}$$

$$-\frac{4\beta}{\pi}\sum_{n=1}^{\infty}\frac{e^{-\nu(x+\xi)}}{\nu(\nu+\beta)}\sin(\nu y)\sin(\nu\eta)$$

where $\nu=(2n-1)$, $z=x+iy$, and $t=\xi+i\eta$. For the numerical experiment, we assume that D_0 is a disk-shaped hole centered at (2, .8) with radius $r=.3$, let $\beta=0$, prescribe $h_j=0$, $j=1, 2, 3$ and flux $h_0(\theta)=-2$ on Γ_0, and simulate measurement $u=f$ on boundary Γ_3 for $0\le x\le 4$. Figure 2 shows the result for circular initial guess centered at $(c_1, c_2)=(.5, .7)$ with radius $c_3=.2$; the thick circle is unknown D_0. Four updates were required to meet the convergence criterion; the third is visually indistinguishable from the true hole.

Example 3:

$\Omega=\{z=re^{i\theta} : 0\le r\le 4,\ 0\le\theta\le\pi/2\}$, the right circular sector of radius 4. For the boundary $\Gamma=\Gamma_1\cup\Gamma_2\cup\Gamma_3$, Γ_1 is the circular part, Γ_2 is the interval [0, 4] on the x-axis, Γ_3 is the interval [0, 4] on the y-axis. The Green's function for Ω with mixed boundary conditions

$$\left(\frac{\partial u}{\partial r}+\beta u\right)\Big|_{\Gamma_1}=u\,\big|_{\Gamma_2}=\frac{\partial u}{\partial\theta}\Big|_{\Gamma_3}=0\,,\ \beta\ge 0,$$

is

$$G(z,t)=\frac{1}{2\pi}\ln\frac{|z-\bar{t}|\,|z+t|\,|R^2-zt|\,|R^2+z\bar{t}|}{|z-t|\,|z+\bar{t}|\,|R^2-z\bar{t}|\,|R^2+zt|}$$

$$-\frac{4\beta R}{\pi}\sum_{n=1}^{\infty}\frac{1}{\nu(\nu+\beta R)}\left(\frac{r\rho}{R^2}\right)^{\nu}\sin(\nu\theta)\sin(\nu\sigma)\,,$$

where $z=re^{i\theta}$ and $t=\rho e^{i\sigma}$ are the field and source points, respectively, and $\nu=2n+1$. Here, $R=4$.

For the numerical experiment, we set $\beta=0$, and impose boundary conditions $\partial u/\partial r=0$ on Γ_1, $\partial u/\partial\theta=0$ on Γ_3, and $u=h_2(r)=1-\cos(\pi r/4)$ on Γ_2, which satisfies matching conditions at endpoints $r=0$ and $r=R$. First, we assume the existence of an elliptic hole D_0 near the upper left part of Ω, with zero flux on Γ_0. Fig. 3(a) shows D_0 approximated by ellipses depending on five parameters from a circular initial estimate near the center of Ω in 32 iterations of the least squares updating procedure; pictured are updates 2, 6, 14, 18, 22, and 28. In Fig. 3(b), we show an asymmetric domain approximated by disks in six iterations; using the final disk update as initial estimate for the five parameter procedure resulted in approximation of the same hole by an ellipse in another six iterations. The latter procedure therefore entails updates for fewer parameters, as well as fewer iterations.

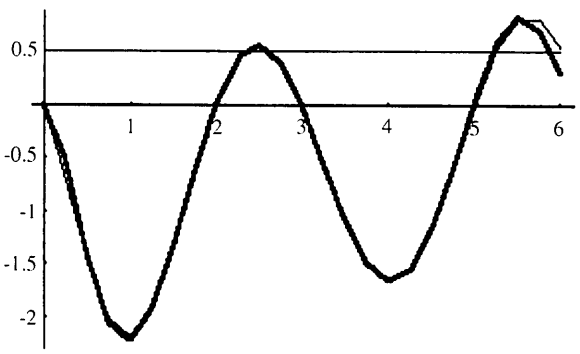

Fig. 4. Approximation of flux.

For flux determination in the wedge, we apply homogeneous conditions on Γ, and simulate measurement $u=f$ on Γ_2, for $0 \leq x \leq 4$. D_0 is assumed to be the disk of radius 1/2 centered at (3/2, 3/2). In Fig. 4, the thick line is the actual flux, the polynomial $h_0(\theta) =(.01)\theta^2(\theta-2)(\theta-3)(\theta-5)(\theta-2\pi)^2$. The initial estimate is $\tilde{h}_0(\theta)=.5$. We try to approximate h_0 with the trigonometric sum

$$\tilde{g}(\theta) = \frac{1}{2}a_0 + \sum_{k=1}^{5} [a_k\cos(k\theta) + b_k\sin(k\theta)]$$

with respect to the parameters a_k, b_k. The thin line shows the approximation after 12 iterations of the procedure. The result of this test is depicted in Fig. 4.

VI. CONCLUSIONS

Considering the form of (8), the optimization can be expected to show sensitivity to noise, which indeed is the case. However, by calculating the lowest order components of the Fourier series for f and $\tilde{f}$, and using these quantities for comparison in the objective functional F, instead of the full data, we were able to filter out high spatial frequency effects of additive random noise to obtain reliable information about the vicinity of D_0 from data with low levels of noise. Work remains to be done with respect to regularization of this problem.

Constraints were included in the optimization procedure because otherwise, estimates for D_0 were produced which fell outside of Ω; for example, in the infinite strip, when the program attempted to calculate the potential for an estimate $c+\delta$ that represented a disk which exceeded the strip boundary, the estimate was revised by centering it at $c_2=.7$ inside the strip. Constraints were also required in the flux determination problem; otherwise, updates of the higher order coefficients a_k, b_k tended to have large magnitude.

Use of several parameters is unnecessary at the early stages of approximation; example 3 showed it is sufficient to approximate an asymmetric domain with disks, then to resolve the approximation by means of domains which depend on a larger number of parameters, as appropriate.

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格林函數數值方法解邊界反算問題

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摘 要

本文求解傳導板上勢能、靜電或穩態熱傳等物理現象的孔洞形狀確認問題。一個含孔洞之簡單形狀的板子，一則大小、形狀、孔洞位置已知，一則邊界上的熱通量已知，邊界條件一端給定一次邊界量（如電壓，溫度）另一端給定二次邊界量（如熱通量），在量得邊界的另一邊界量後構成過定邊界問題。從這種過定資料，本文所提供的方法可決定(1)板子的孔洞位置（孔洞邊界上給定熱通量），(2)孔洞邊界上熱通量（孔洞位置給定）。本文將邊界元素法、最佳化最小平方法與格林函數在邊界值上特有的性質，應用在孔洞形狀確認問題的數值處理上。一個有效率建構格林函數的技巧也被簡短的回顧。

關鍵字：孔洞形狀確認問題，過定資料，格林函數，邊界元素法，最佳化最小平方法與反算問題。