Journal of the Chinese Institute of Engineers, Vol. 23, No. 3, pp. 261-268 (2000)

A NOTE ON A BOUNDARY ELEMENT METHOD FOR THE NUMERICAL SOLUTION OF BOUNDARY VALUE PROBLEMS IN ISOTROPIC INHOMOGENEOUS ELASTICITY

David L. Clements* and Moh. Ivan Azis
Department of Applied Mathematics
The University of Adelaide
S.A. 5005, Australia

Key Words: boundary element, boundary value problem, inhomogeneous elasticity.

ABSTRACT

A boundary element method is derived for the solution of boundary value problems for inhomogeneous isotropic elastic materials. Some particular problems are considered to illustrate the application of the method.

I. INTRODUCTION

Following the early work of Rizzo (1967) a large number of authors have used the boundary element method to effectively obtain numerical solutions to a variety of elastic problems for homogeneous isotropic elastic materials (see Brebbia and Dominguez (1989)). In contrast the application of the method to problems for inhomogeneous isotropic elastic materials is very limited due to the difficulty in obtaining appropriate Green's functions for the kernels of the relevant boundary integral equations. Recently Manolis and Shaw (1996) obtained a suitable Green's function for the vector wave equation in a mildly heterogeneous isotropic continuum. Their Green's function was obtained for a particular variation in the material parameters and in particular is restricted to the case when the Lamé parameters λ and μ are equal. This leads to a Poisson's ratio of 0.25 which restricts the application of the method but as Manolis and Shaw (1996) point out, this particular value of Poisson's ratio is a common value for rock materials (see Turcotte and Schubert (1982)).

The present note builds on the work of Manolis and Shaw (1996) to develop a perturbation procedure for the solution of plane static problems for isotropic inhomogeneous media with Lamé parameters given by

$$\lambda(\boldsymbol{x})=\lambda^{(0)}g(\boldsymbol{x})+\epsilon\lambda^{(1)}(\boldsymbol{x}),$$

$$\mu(\boldsymbol{x})=\mu^{(0)}g(\boldsymbol{x})+\epsilon\mu^{(1)}(\boldsymbol{x}),$$

where $\boldsymbol{x}=(x_1, x_2, x_3)$ is a vector in R^3, $g(\boldsymbol{x})$ is a function which must satisfy particular constraints, $\lambda^{(0)}=\mu^{(0)}$ are constants and ϵ is a small parameter. Within these constraints these forms permit a wide choice of variations for the elastic parameters $\lambda(\boldsymbol{x})$ and $\mu(\boldsymbol{x})$. Boundary integral equations are obtained for the solution of problems for materials with Lamé parameters of this form and these integral equations are used to solve some particular boundary value problems.

II. BASIC EQUATIONS

Referred to a Cartesian frame $Ox_1x_2x_3$ the equilibrium equations in an elastic material in the absence of body force may be written in the form

$$\sigma_{ij,j}=0, \tag{1}$$

where σ_{ij} for i, j=1, 2, 3 denotes the stress tensor, the

*Correspondence addressee

indexed commas indicate partial differentiation with respect to the spatial coordinates x_j and the repeated suffix summation convention (summing from 1 to 3) is employed. The stress-displacement relations are

$$\sigma_{ij}=\lambda(\boldsymbol{x})\delta_{ij}u_{k,k}+\mu(\boldsymbol{x})(u_{i,j}+u_{j,i}), \tag{2}$$

where u_k for k=1, 2, 3 denotes the displacement and δ_{ij} the Kronecker delta. Also in (2) $\lambda(\boldsymbol{x})$ and $\mu(\boldsymbol{x})$ with $\boldsymbol{x}=(x_1, x_2, x_3)$ denote the Lamé parameters which are taken to be twice differentiable functions of the spatial variables x_1, x_2 and x_3. Substitution of (2) into (1) yields

$$[\lambda(\boldsymbol{x})\delta_{ij}\mu_{k,k}+\mu(\boldsymbol{x})(u_{i,j}+u_{j,i})]_{,j}=0. \tag{3}$$

III. STATEMENT OF THE BOUNDARY VALUE PROBLEM

An inhomogeneous isotropic elastic material occupies the region Ω in R^3 with boundary $\partial\Omega$ which consists of a finite number of piecewise smooth closed surfaces. On $\partial\Omega_1$ the displacement u_i is specified and on $\partial\Omega_2$ the stress vector $P_i=\sigma_{ij}n_j$ is specified where $\partial\Omega=\partial\Omega_1\cup\partial\Omega_2$ and $\boldsymbol{n}=(n_1, n_2, n_3)$ denotes the outward pointing normal to $\partial\Omega$. It is required to find the displacement and stress throughout the material. Thus a solution to (3) is sought which is valid in Ω and satisfies the specified boundary conditions on $\partial\Omega$.

IV. REDUCTION TO CONSTANT COEFFICIENT EQUATIONS

In this section the procedure developed in Manolis and Shaw (1996) is used to obtain a boundary element method for particular classes of coefficients $\lambda(\boldsymbol{x})$ and $\mu(\boldsymbol{x})$. This derivation is achieved by introducing a transformation of the dependent variable $u_i(\boldsymbol{x})$ to transform (3) to a constant coefficients equation. The coefficients $\lambda(\boldsymbol{x})$ and $\mu(\boldsymbol{x})$ are required to take the form

$$\lambda(\boldsymbol{x})=\lambda^{(0)}g(\boldsymbol{x}),\ \mu(\boldsymbol{x})=\mu^{(0)}g(\boldsymbol{x}), \tag{4}$$

where $\lambda^{(0)}$ and $\mu^{(0)}$ are constants. Use of (4) in (3) yields

$$[g(\boldsymbol{x})\{\lambda^{(0)}\delta_{ij}u_{k,k}+\mu^{(0)}(u_{i,j}+u_{j,i})\}]_{,j}=0. \tag{5}$$

Let

$$\psi_i(\boldsymbol{x})=g^{1/2}(\boldsymbol{x})u_i(\boldsymbol{x}) \tag{6}$$

so that (5) may be written in the form

$$[g(\boldsymbol{x})\{\lambda^{(0)}\delta_{ij}(g^{-1/2}\psi_k)_{,k}+\mu^{(0)}((g^{-1/2}\psi_i)_{,j}+(g^{-1/2}\psi_j)_{,i})\}]_{,j}=0.$$

Thus

$$\lambda^{(0)}[g(g^{-1/2}\psi_k)_{,k}]_{,i}+\mu^{(0)}[g(g^{-1/2}\psi_i)_{,j}]_{,j}$$
$$+\mu^{(0)}[g(g^{-1/2}\psi_j)_{,i}]_{,j}=0. \tag{7}$$

Now

$$[g(g^{-1/2}\psi_k)_{,k}]_{,i}$$
$$=\frac{1}{4}g^{-3/2}g_{,i}g_{,k}\psi_k-\frac{1}{2}g^{-1/2}g_{,ki}\psi_k-\frac{1}{2}g^{-1/2}g_{,k}\psi_{k,i}$$
$$+\frac{1}{2}g^{-1/2}g_{,i}\psi_{k,k}+g^{1/2}\psi_{k,ki}$$
$$=-g^{1/2}_{,ki}\psi_k+g^{1/2}\psi_{k,ki}-\frac{1}{2}g^{-1/2}g_{,k}\psi_{k,i}+\frac{1}{2}g^{-1/2}g_{,i}\psi_{k,k}. \tag{8}$$

Similarly

$$[g(g^{-1/2}\psi_i)_{,j}]_{,j}=-g^{1/2}_{,jj}\psi_i+g^{1/2}\psi_{i,jj}, \tag{9}$$

$$[g(g^{-1/2}\psi_j)_{,i}]_{,j}=-g^{1/2}_{,ij}\psi_j+g^{1/2}\psi_{j,ij}$$
$$-\frac{1}{2}g^{-1/2}g_{,i}\psi_{j,j}+\frac{1}{2}g^{-1/2}g_{,j}\psi_{j,i}. \tag{10}$$

Substitution of (8), (9) and (10) into (7) yields

$$g^{1/2}[\lambda^{(0)}\delta_{ij}\psi_{k,k}+\mu^{(0)}(\psi_{i,j}+\psi_{j,i})]_{,j}$$
$$-[\lambda^{(0)}\psi_k g^{1/2}_{,ki}+\mu^{(0)}\psi_i g^{1/2}_{,jj}+\mu^{(0)}\psi_j g^{1/2}_{,ij}]$$
$$-(\lambda^{(0)}-\mu^{(0)})[\frac{1}{2}g^{-1/2}][g_{,k}\psi_{k,i}-g_{,i}\psi_{k,k}]=0. \tag{11}$$

If $g(\boldsymbol{x})$ assumes the form

$$g(\boldsymbol{x})=(\alpha x_1+\beta x_2+\gamma x_3+\delta)^2, \tag{12}$$

where α, β, γ and δ are constants and also

$$\lambda^{(0)}=\mu^{(0)}, \tag{13}$$

so that $\lambda(\boldsymbol{x})=\mu^{(0)}(\alpha x_1+\beta x_2+\gamma x_3+\delta)^2=\mu(\boldsymbol{x})$ then (4.8) reduces to

$$[\lambda^{(0)}\delta_{ij}\psi_{k,k}+\mu^{(0)}(\psi_{i,j}+\psi_{j,i})]_{,j}=0 \text{ (with } \lambda^{(0)}=\mu^{(0)}). \tag{14}$$

Thus if ψ_i is any solution of the equations of equilibrium in displacement form for a homogeneous isotropic elastic material with Lamé constants $\lambda^{(0)}$ and $\mu^{(0)}$ then a corresponding solution of the equations of equilibrium for an inhomogeneous isotropic elastic material with Lamé parameters $\lambda(\boldsymbol{x})$ and $\mu(\boldsymbol{x})$ given by the multiparameter form (4) may be written, from (6), in the form

$$u_i(\boldsymbol{x})=g^{-1/2}(\boldsymbol{x})\,\psi_i(\boldsymbol{x})$$
$$=(\alpha x_1+\beta x_2+\gamma x_3+\delta)^{-1}\psi_i(\boldsymbol{x}).$$

The corresponding stresses obtained from (2) are given by

$$\sigma_{ij}=-\,\psi_k\sigma_{ijk}^{[g]}+g^{1/2}\sigma_{ij}^{[\psi]}$$

where

$$\sigma_{ijk}^{[g]}=\lambda^{(0)}\delta_{ij}(g^{1/2})_{,k}+\mu^{(0)}[\delta_{ki}(g^{1/2})_{,j}+\delta_{kj}(g^{1/2})_{,i}],$$

$$\sigma_{ij}^{[\psi]}=\lambda^{(0)}\delta_{ij}\psi_{k,k}+\mu^{(0)}(\psi_{i,j}+\psi_{j,i})$$

and the stress vector

$$P_i=-\psi_kP_{ik}^{[g]}+g^{1/2}P_i^{[\psi]}, \tag{15}$$

where

$$P_{ik}^{[g]}=\sigma_{ijk}^{[g]}n_j\,,\quad P_i^{[\psi]}=\sigma_{ij}^{[\psi]}n_j\,. \tag{16}$$

A boundary integral equation for the solution of (14) with ψ_i given on $\partial\Omega_1$ and $P_i^{[\psi]}$ given on $\partial\Omega_2$ is given in Brebbia and Dominguez (1989) in the form

$$\eta\psi_j(\boldsymbol{x}_0)=\int_{\partial\Omega}[\Phi_{ij}P_i^{[\psi]}-\Gamma_{ij}\psi_i]ds\,. \tag{17}$$

where $\boldsymbol{x}_0$ is the source point, $\eta=0$ if $\boldsymbol{x}_0\notin\Omega$, $\eta=1$ if $\boldsymbol{x}_0\in\Omega$ and $\eta=\frac{1}{2}$ if $\boldsymbol{x}_0\in\partial\Omega$ and $\partial\Omega$ has a continuously turning tangent at $\boldsymbol{x}_0$. Also for the three dimensional case

$$\Phi_{ij}=\frac{1}{16\pi\mu^{(0)}(1-\nu)d}[(3-4\nu)\delta_{ij}+d_{,i}d_{,j}]\,, \tag{18}$$

$$\Gamma_{ij}=-\frac{1}{8\pi(1-\nu)d^2}[\frac{\partial d}{\partial n}\{(1-2\nu)\delta_{ij}+3d_{,i}d_{,j}\}$$
$$+(1-2\nu)(n_id_{,j}-n_jd_{,i})] \tag{19}$$

and for two dimensional case

$$\Phi_{ij}=\frac{1}{8\pi\mu^{(0)}(1-\nu)}[(3-4\nu)\log\frac{1}{d}\delta_{ij}+d_{,i}d_{,j}]\,, \tag{20}$$

$$\Gamma_{ij}=-\frac{1}{4\pi(1-\nu)d}[\frac{\partial d}{\partial n}\{(1-2\nu)\delta_{ij}+2d_{,i}d_{,j}\}$$
$$+(1-2\nu)(n_id_{,j}-n_jd_{,i})]\,, \tag{21}$$

where $d=||\boldsymbol{x}-\boldsymbol{x}_0||$, $\nu=\lambda^{(0)}/(2(\mu^{(0)}+\lambda^{(0)}))$ and $\partial d/\partial n=d_{,k}n_k$.
Use of (6) and (15) in (17) yields

$$\eta g^{1/2}(\boldsymbol{x}_0)u_j(\boldsymbol{x}_0)$$
$$=\int_{\partial\Omega}[(g^{-1/2}\Phi_{ij})P_i-(g^{1/2}\Gamma_{kj}-P_{ik}^{[g]}\Phi_{ij})u_k]ds. \tag{22}$$

This equation provides a boundary integral equation for determining u_i and σ_{ij} at all points of Ω.

V. A PERTURBATION METHOD

The boundary element procedure described in the previous section provides an effective numerical method for determining $u_i(\boldsymbol{x})$ when $g(\boldsymbol{x})$ takes the form (12) and the parameters $\lambda^{(0)}$ and $\mu^{(0)}$ satisfy the relation (13). In this section a procedure is obtained for the case when the coefficients $\lambda(\boldsymbol{x})$ and $\mu(\boldsymbol{x})$ are perturbed about $\lambda^{(0)}g(\boldsymbol{x})$ and $\mu^{(0)}g(\boldsymbol{x})$ respectively while retaining Eqs. (12) and (13).

The coefficients $\lambda(\boldsymbol{x})$ and $\mu(\boldsymbol{x})$ are required to take the form

$$\lambda(\boldsymbol{x})=\lambda^{(0)}g(\boldsymbol{x})+\epsilon\lambda^{(1)}(\boldsymbol{x}), \tag{23}$$

$$\mu(\boldsymbol{x})=\mu^{(0)}g(\boldsymbol{x})+\epsilon\lambda^{(1)}(\boldsymbol{x}), \tag{24}$$

with

$$\lambda^{(0)}=\mu^{(0)}\text{ and }g^{1/2}_{,ij}=0$$

and where $\lambda^{(1)}$ and $\mu^{(1)}$ are twice differentiable functions. Therefore from (3)

$$[g\{\lambda^{(0)}\delta_{ij}u_{k,k}+\mu^{(0)}(u_{i,j}+u_{j,i})\}]_{,j}$$
$$=-\epsilon[\lambda^{(1)}\delta_{ij}u_{k,k}+\mu^{(1)}(u_{i,j}+u_{j,i})_{,j}. \tag{25}$$

Now use of the transformation (6) and following the analysis used to derive (11) from (5) gives

$$g^{1/2}[\lambda^{(0)}\delta_{ij}\psi_{k,k}+\mu^{(0)}(\psi_{i,j}+\psi_{j,i})]_{,j}$$
$$=-\epsilon[\lambda^{(1)}\delta_{ij}(g^{-1/2}\psi_k)_{,k}+\mu^{(1)}\{(g^{-1/2}\psi_i)_{,j}+(g^{-1/2}\psi_j)_{,i}\}]_{,j}.$$

This equation may be written in the form

$$[\lambda^{(0)}\delta_{ij}\psi_{k,k}+\mu^{(0)}(\psi_{i,j}+\psi_{j,i})]_{,j}$$
$$=-\epsilon g^{-1/2}[(A_{j,i}+B_{i,j}\}\psi_j+B_{j,j}\psi_i+(A_j+D_{,j})\psi_{j,i}$$
$$+(B_j+D_{,j})\psi_{i,j}+(B_i+C_{,i})\psi_{j,j}+(C+D)\psi_{j,ij}+D\psi_{i,jj}], \tag{26}$$

where

$$A_i(x)=\lambda^{(1)}g_{,i}^{-1/2},\ B_i(x)=\mu^{(1)}g_{,i}^{-1/2},$$

$$C(x)=\lambda^{(1)}g^{-1/2},\ D(x)=\mu^{(1)}g^{-1/2}.$$

A solution to equation (26) is sought in the form

$$\psi_i(x)=\sum_{r=0}^{\infty}\epsilon^r\psi_i^{(r)}(x). \tag{27}$$

Substitution of (27) into (26) and equating the coefficients of powers of ϵ yields

$$[\lambda^{(0)}\delta_{ij}\psi_{k,k}^{(0)}+\mu^{(0)}(\psi_{i,j}^{(0)}+\psi_{j,i}^{(0)})]_{,j}=0, \tag{28}$$

$$\begin{aligned}&[\lambda^{(0)}\delta_{ij}\psi_{k,k}^{(r)}+\mu^{(0)}(\psi_{i,j}^{(r)}+\psi_{j,i}^{(r)})]_{,j}\\&=-g^{-1/2}[(A_{j,i}+B_{i,j})\psi_j^{(r-1)}+B_{j,j}\psi_i^{(r-1)}\\&+(A_j+D_{,j})\psi_{j,i}^{(r-1)}+(B_j+D_{,j})\psi_{i,j}^{(r-1)}+(B_i+D_{,i})\psi_{j,j}^{(r-1)}\\&+(C+D)\psi_{j,ij}^{(r-1)}+D\psi_{i,jj}^{(r-1)}],\ \text{for } r=1, 2, ...\end{aligned} \tag{29}$$

The integral equations for (28) and (29) are respectively

$$\eta\psi_j^{(0)}(x_0)=\int_{\partial\Omega}[\Phi_{ij}P_i^{[\psi^{(0)}]}-\Gamma_{ij}\psi_i^{(0)}]ds, \tag{30}$$

$$\eta\psi_j^{(r)}(x_0)=\int_{\partial\Omega}[\Phi_{ij}P_i^{[\psi^{(r)}]}-\Gamma_{ij}\psi_i^{(r)}]ds+\int_{\Omega}h_i^{(r)}\Phi_{ij}dS, \tag{31}$$

where $h_i^{(r)}$ is the right hand side of (29) for r=1, 2, ...

From (6) and (27) the displacement u_k may be written in the series form

$$u_k(x)=\sum_{r=0}^{\infty}\epsilon^r u_k^{(r)}(x), \tag{32}$$

where $u_k^{(r)}$ corresponds to $\psi_k^{(r)}$ according to the relationship

$$\psi_k^{(r)}(x)=g^{1/2}u_k^{(r)}(x),\ \text{for } r=0, 1, ... \tag{33}$$

Also

$$P_i^{[\psi^{(r)}]}=g^{1/2}P_i^{(r)}+u_k^{(r)}P_{ik}^{[g]}\ \text{for } r=0, 1, ..., \tag{34}$$

where

$$P_i^{(r)}=[\lambda^{(0)}\delta_{ij}u_{k,k}^{(r)}+\mu^{(0)}(u_{i,j}^{(r)}+u_{j,i}^{(r)})]n_j,\quad \text{for } r=1, 2, ..., \tag{35}$$

and $P_{ik}^{[g]}$ is given by (16). Thus the integral Eqs. (30) and (31) may be written in the form

$$\eta g^{1/2}(x_0)u_j^{(0)}(x_0)=\int_{\partial\Omega}[(g^{1/2}\Phi_{ij})P_i^{(0)}-(g^{1/2}\Gamma_{kj}-P_{ik}^{[g]}\Phi_{ij})u_k^{(0)}]ds. \tag{36}$$

$$\eta g^{1/2}(x_0)u_j^{(r)}(x_0)=\int_{\partial\Omega}[(g^{1/2}\Phi_{ij})P_i^{(r)}-(g^{1/2}\Gamma_{kj}-P_{ik}^{[g]}\Phi_{ij})u_k^{(r)}]ds+\int_{\Omega}h_i^{(r)}\Phi_{ij}dS$$

$$\text{for } r=1, 2, ... \tag{37}$$

where the function $h_i^{(r)}$ is given by

$$\begin{aligned}h_i^{(r)}=&-g^{-1/2}[u_i^{(r-1)}\{g^{1/2}B_{j,j}+g_{,j}^{1/2}(B_j+D_{,j})\}\\&+u_j^{(r-1)}\{g^{1/2}(A_{j,i}+B_{i,j})+g_{,i}^{1/2}(A_j+D_{,j})+g_{,j}^{1/2}(B_i+C_{,i})\}\\&+u_{i,j}^{(r-1)}\{g^{1/2}(B_j+D_{,j})+2g_{,j}^{1/2}D\}\\&+u_{j,i}^{(r-1)}\{g^{1/2}(A_j+D_{,j})+g_{,j}^{1/2}(C+D)\}\\&+u_{j,j}^{(r-1)}\{g^{1/2}(B_i+C_{,i})+g_{,i}^{1/2}(C+D)\}\\&+u_{i,jj}^{(r-1)}g^{1/2}D+u_{j,ij}^{(r-1)}g^{1/2}(C+D)],\\&\text{for } r=1, 2, ...\end{aligned} \tag{38}$$

Now the corresponding value of P_i may be written as

$$P_i=gP_i^{(0)}+\sum_{r=1}^{\infty}\epsilon^r(gP_i^{(r)}+G_i^{(r)}), \tag{39}$$

where

$$G_i^{(r)}=[\lambda^{(1)}\delta_{ij}u_{k,k}^{(r-1)}+\mu^{(1)}(u_{i,j}^{(r-1)}+u_{j,i}^{(r-1)})]n_j. \tag{40}$$

To satisfy the boundary conditions in Section 3 it is required that $u_i^{(0)}=u_i$ on $\partial\Omega_1$ where u_i takes on its specified value on $\partial\Omega_1$. Also it is required that on $\partial\Omega_2$ $P_i^{(0)}=g^{-1}P_i$ where P_i takes on its specified value on $\partial\Omega_2$. It then follows from (32) and (39) that $u_i^{(r)}=0$ on $\partial\Omega_1$ for r=1, 2, 3 ... and $P_i^{(r)}=-g^{-1}G_i^{(r)}$ on $\partial\Omega_2$ for r=1, 2, 3, ...

The integral Eqs. (36) and (37) may now be used to find the numerical values of the unknowns on the boundary $\partial\Omega$ and the numerical values of $u_i^{(r)}$ and derivatives in the domain Ω for r=0, 1, ... At each stage in using (37) to determine $u_i^{(r)}$ the $P_i^{(r)}=-g^{-1}G_i^{(r)}$ occuring in (37) may be obtained from (40) which may be evaluated from the previous iteration. Equations (32) and (39) then provide the values of u_i and P_i throughout the domain Ω.

Table 1 Numerical displacement u_i results for Problem 1

Position	BEM 40 segments		BEM 80 segments		BEM 160 segments	
(x_1',x_2')	$u_1/\overline{u}$	$u_2/\overline{u}$	$u_1/\overline{u}$	$u_2/\overline{u}$	$u_1/\overline{u}$	$u_2/\overline{u}$
(0.1,0.5)	0.0289	0.0000	0.0299	0.0000	0.0304	0.0000
(0.3,0.5)	0.0881	-0.0002	0.0896	-0.0001	0.0903	0.0000
(0.5,0.5)	0.1451	-0.0002	0.1470	0.0000	0.1479	0.0000
(0.7,0.5)	0.2002	-0.0001	0.2023	0.0000	0.2033	0.0000
(0.9,0.5)	0.2530	-0.0001	0.2557	0.0000	0.2568	0.0000

Table 2 Analytical displacement u_i solutions for Problem 1

Position	Analytical solutions	
(x_1',x_2')	$u_1/\overline{u}$	$u_2/\overline{u}$
(0.1,0.5)	0.0309	0.0000
(0.3,0.5)	0.0910	0.0000
(0.5,0.5)	0.1488	0.0000
(0.7,0.5)	0.2044	0.0000
(0.9,0.5)	0.2580	0.0000

VI. NUMERICAL RESULTS

In this section some particular boundary value problems in plane strain are solved numerically by employing the integral equations obtained in section 5. In implementing this method to obtain numerical results standard boundary element procedures are employed (see for example Clements (1981)). For the chosen variations in the elastic parameters the right hand side of (29) is small so that it is only necessary to retain two terms in the expression (32).

For all the problems considered the domain Ω is taken to be a square of side l and each side of the square is divided into a number of M (a multiplication of 5) segments of equal length. Simpson's 3/8 rule is applied to evaluate the line integrals on each segment. For the domain integral in (37) the domain is divided into M^2 equal square cells and the integrand is assumed to be constant taking on its value at the mid point of each cell. However, the values of the previous iteration solutions $u_i^{(r-1)}$ and derivatives in (38) are evaluated only at a number of 5x5 mid-points of sub-squares of side $l/5$ and are assumed to be constant over each sub-square. Specifically the values of $u_i^{(r-1)}$ and derivatives in a particular sub-square are the same over a number of $(M/5)^2$ cells contained in the sub-square.

1. Problem 1

Consider the boundary value problem given in Fig. 1 for a material with elastic coefficients

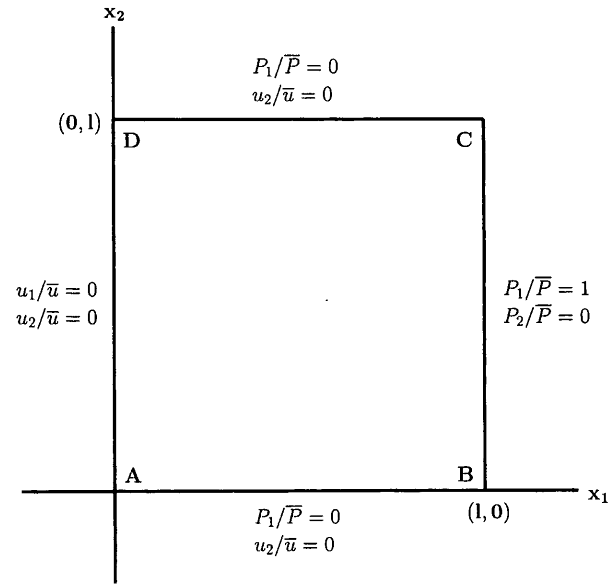

Fig. 1 The geometry for Problem 1

$$\lambda(x)=1.2\lambda_0(1+0.1x_1')^2, \tag{41}$$

$$\mu(x)=\lambda_0(1+0.1x_1')^2, \tag{42}$$

where λ_0 is a reference elastic modulus and $x_1'=x_1/l$. The elastic coefficients (41) and (42) take the form (23) and (24) with $g(x)=(1+0.1x_1')^2$, $\mu^{(0)}=\lambda^{(0)}$, $\lambda^{(1)}=\lambda_0(1+0.1x_1')^2$, $\mu^{(1)}=0$ and $\epsilon=0.2$.

The boundary conditions (see Fig. 1) are

$$P_1/\overline{P}=0,\ u_2/\overline{u}=0,\ \text{on } AB,$$

$$P_1/\overline{P}=1,\ P_2/\overline{P}=0,\ \text{on } BC,$$

$$P_1/\overline{P}=0,\ u_2/\overline{u}=0,\ \text{on } CD,$$

$$u_1/\overline{u}=1,\ u_2/\overline{u}=0,\ \text{on } AD,$$

where $\overline{u}$ is a reference displacement and $\overline{P}=\overline{\lambda}\,\overline{u}/l$. This problem admits the analytical solution $u_1/\overline{u}=x_1'/[3.2(1+0.1x_1')]$, $u_2=0$ with the stress given by $\sigma_{11}/\overline{P}=1$, $\sigma_{12}/\overline{P}=0$ and $\sigma_{22}/\overline{P}=0.375$.

Tables 1-4 show the analytical and BEM results

Table 3 Numerical stress σ_{ij} results for Problem 1

Position	BEM 40 segments			BEM 80 segments		
(x'_1,x'_2)	$\sigma_{11}/\overline{P}$	$\sigma_{12}/\overline{P}$	$\sigma_{22}/\overline{P}$	$\sigma_{11}/\overline{P}$	$\sigma_{12}/\overline{P}$	$\sigma_{22}/\overline{P}$
(0.1,0.5)	0.9900	0.0002	0.3767	0.9956	0.0000	0.3756
(0.3,0.5)	0.9907	0.0002	0.3822	0.9953	0.0001	0.3782
(0.5,0.5)	0.9926	0.0005	0.3822	0.9960	0.0001	0.3784
(0.7,0.5)	0.9952	0.0003	0.3809	0.9972	0.0001	0.3778
(0.9,0.5)	0.9467	0.0000	0.4195	0.9978	0.0001	0.3787

Table 4 Numerical and analytical stress σ_{ij} results for Problem 1

Position	BEM 160 segments			Analytical		
(x'_1,x'_2)	$\sigma_{11}/\overline{P}$	$\sigma_{12}/\overline{P}$	$\sigma_{22}/\overline{P}$	$\sigma_{11}/\overline{P}$	$\sigma_{12}/\overline{P}$	$\sigma_{22}/\overline{P}$
(0.1,0.5)	0.9978	0.0000	0.3751	1.0000	0.0000	0.3750
(0.3,0.5)	0.9975	0.0000	0.3764	1.0000	0.0000	0.3750
(0.5,0.5)	0.9978	0.0000	0.3766	1.0000	0.0000	0.3750
(0.7,0.5)	0.9984	0.0000	0.3763	1.0000	0.0000	0.3750
(0.9,0.5)	0.9992	-0.0002	0.3764	1.0000	0.0000	0.3750

Table 5 Numerical displacement u_i results for Problem 2

Position	BEM 40 segments		BEM 80 segments		BEM 160 segments	
(x'_1,x'_2)	$u_1/\overline{u}$	$u_2/\overline{u}$	$u_1/\overline{u}$	$u_2/\overline{u}$	$u_1/\overline{u}$	$u_2/\overline{u}$
(0.1,0.5)	0.1036	0.0000	0.1058	0.0000	0.1069	0.0000
(0.3,0.5)	0.3159	-0.0008	0.3175	-0.0002	0.3182	-0.0001
(0.5,0.5)	0.5210	-0.0008	0.5218	-0.0002	0.5220	0.0000
(0.7,0.5)	0.7188	-0.0006	0.7189	-0.0001	0.7185	0.0000
(0.9,0.5)	0.9091	-0.0004	0.9096	0.0000	0.9084	-0.0001

for some points in the domain Ω and for the cases when the boundary $\partial\Omega$ is divided into 40, 80 and 160 segments.

The results converge to the known solution as the number of segments increases. The displacement displays fourth figure and the stress third figure accuracy when 160 boundary segments are used.

2. Problem 2

Now consider the boundary value problem given in Fig. 2 with the coefficients $\lambda(\mathbf{x})$ and $\mu(\mathbf{x})$ again given by (41) and (42).

The boundary conditions (see Fig. 2) are

$$u_1/\overline{u}=x'_1\,,\quad u_2/\overline{u}=0\,,\ \text{on } AB,$$

$$u_1/\overline{u}=1\,,\quad u_2/\overline{u}=0\,,\ \text{on } BC,$$

$$u_1/\overline{u}=x'_1\,,\quad u_2/\overline{u}=0\,,\ \text{on } CD,$$

$$u_1/\overline{u}=0\,,\quad u_2/\overline{u}=0\,,\ \text{on } AD,$$

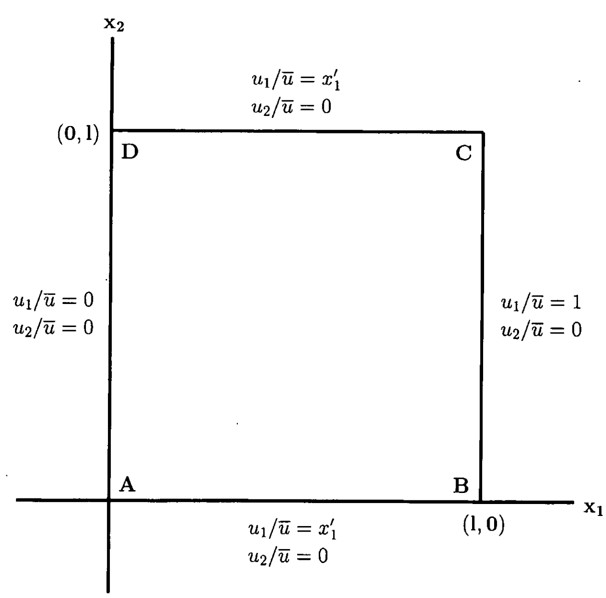

Fig. 2 The geometry for Problem 2

Table 6 Numerical stress σ_{ij} results for Problem 2

Position	BEM 40 segments			BEM 80 segments		
(x'_1,x'_2)	$\sigma_{11}/\overline{P}$	$\sigma_{12}/\overline{P}$	$\sigma_{22}/\overline{P}$	$\sigma_{11}/\overline{P}$	$\sigma_{12}/\overline{P}$	$\sigma_{22}/\overline{P}$
(0.1,0.5)	3.4965	0.0000	1.3159	3.4827	0.0000	1.2940
(0.3,0.5)	3.4481	0.0009	1.3084	3.4259	0.0003	1.2793
(0.5,0.5)	3.3983	0.0024	1.3066	3.3763	0.0005	1.2807
(0.7,0.5)	3.3489	0.0014	1.3045	3.3285	0.0002	1.2827
(0.9,0.5)	3.1544	0.0005	1.3932	3.2825	0.0006	1.2601

Table 7 Numerical stress σ_{ij} results for Problem 2

Position	BEM 160 segments		
(x'_1,x'_2)	$\sigma_{11}/\overline{P}$	$\sigma_{12}/\overline{P}$	$\sigma_{22}/\overline{P}$
(0.1,0.5)	3.4699	0.0000	1.2852
(0.3,0.5)	3.4143	0.0001	1.2670
(0.5,0.5)	3.3643	0.0002	1.2688
(0.7,0.5)	3.3166	0.0002	1.2715
(0.9,0.5)	3.2698	0.0000	1.2498

There is no explicit analytical solution to this particular problem. Tables 5-7 show the BEM results for some points in the domain Ω and for the cases when the boundary $\partial\Omega$ is divided into 40, 80 and 160 segments. As for problem 1 the results converge as the number of boundary segments increases.

VII. SUMMARY

A boundary element method for the solution of certain classes of elastic boundary value problems for isotropic inhomogeneous media has been derived. The methods are generally easy to implement to obtain numerical values for particular problems. They can be applied to a wide class of important practical problems for inhomogeneous isotropic materials. The numerical results obtained using the methods indicate that they can provide accurate numerical solutions.

REFEREBCES

1. Brebbia, C.A., and Dominguez, J. (1989) *Boundary Elements An Introductory Course*. Computational Mechanics Publications, Boston.
2. Clements, D.L. (1981) *Boundary Value Problems Governed by Second Order Elliptic Systems*. Pitman.
3. Manolis, R.P., and Shaw, R.P. (1996) Green's function for the vector wave equation in a mildly heterogeneous continuum. *Wave Motion*. Vol. 24, pp. 59-83.
4. Rizzo, F.J. (1967) An Integral Equation Approach to Boundary Value Problems of Classical Elastostatics. *Quarterly of Applied Mathematics*. Vol. 25, pp. 83-95.
5. Turcotte, D.L., and Schubert, P. (1982) *Geodynamic Applications of Continuum Physics to Geological Problems*. Wiley, New York.

Discussions of this paper may appear in the discussion section of a future issue. All discussions should be submitted to the Editor-in-Chief.

Manuscript Received: Nov. 28, 1999
Revision Received: Jan. 10, 2000
and Accepted: Feb. 27, 2000

等向非均質彈力邊界值問題之邊界元素法分析

David L. Clements and Moh. Ivan Azis
Department of Applied Mathematics
The University of Adelaide
S.A. 5005, Australia

摘　要

本文以邊界元素法可解決等向非均質彈性體邊界值問題，並以幾個特定的問題說明此法之可行性。

關鍵詞：邊界元素法，邊界值問題，非均質彈性力學。