APPLICATION OF BOUNDARY ELEMENT METHOD TO IDENTIFYING THE BOUNDARY CONDITIONS OF PAVEMENT – SUBGRADE SYSTEM

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Key Words: pavement-subgrade system, identifying the boundary restraints, boundary element method, optimization techniques.

ABSTRACT

In this paper, the pavement-subgrade system is modeled by a lot of rectangular thin plates on an elastic foundation, and the various joint restraints between these connected plates are simulated by generalized boundary conditions with a number of undetermined parameters. A boundary element method is developed for the free vibration problem of plates resting on an elastic foundation. The identification procedure is performed by combining the boundary element method with optimization techniques. Two examples, respectively associated with plate structures and the pavement-subgrade system, are presented to illustrate both the boundary element method and the identification procedure proposed in this paper.

I. INTRODUCTION

Research on the mechanical behavior of the pavement-subgrade system is an important category in aeronautic and transport engineering. In general the pavement-subgrade system can be modeled by a rectangular thin plate on an elastic foundation subjected to an aircraft load. Although such topics have already been reported on the literature, there are still some weak aspects in practical applications. In most investigations, one of the major drawbacks is that classical boundary conditions are assumed, which may be quite insufficient for actual cases. In fact, most real pavements are constructed of many rectangular plates, and these plates are joined together in various different ways (as shown in Fig. 1), so the load transfer capacity of these joints should be accounted for fully. In our recent papers (Zheng and Yao, 1993) it was shown that the various joint restraints for the pavement-subgrade system could be effectively simulated by elastic restraint edges with a number of undetermined parameters. In this paper, we focus our attention on developing an identification method to determine the boundary conditions of the pavement-subgrade system. In this case the Rayleigh-Ritz method is unsuitable because the constructing trial function is quite difficult for various complicated boundary conditions. The finite element method is also disadvantageous for such problems since a lot of repeated computation would be necessarily required when the boundary conditions are changed as the iteration proceeds. However the boundary element method is very convenient for such problems (Tanaka *et al.*, 1988).

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In this paper, a boundary element method is developed for the free vibration problem of plates resting on an elastic foundation. The fundamental solution of the problem is derived by Hankel transform. Based on the direct boundary element method, a complete set of boundary integral equations and a relevant numerical scheme are established. The inverse problem of identification procedure is performed by combining the boundary element method with optimization techniques. The objective function for this procedure is taken to be the difference, in the least-squares sense, between the computed and the measured natural frequencies of the system. The design variables are defined in terms of some undetermined boundary parameters. An important feature in this procedure is that the sensitivities with respect to these boundary parameters can be directly obtained from the boundary element formulation by using the differentiation approach. Finally, the conjugate gradient algorithm for unconstrained optimization is adopted for minimizing the objective function.

II. FUNDAMENTAL EQUATIONS

In the general the pavement-subgrade system can be modeled by a rectangular thin plate on an elastic foundation subjected to an aircraft load. Here, consider a thin plate resting on an elastic foundation and occupying the region Ω bounded by the boundary curves Γ. The governing differential equation dealing with the free vibration can be expressed as

$$D\nabla^4 W+(K-\rho\omega^2)W=0 \tag{1}$$

where D is the bending stiffness of the plate, K is the foundation modulus, ρ is the mass density of the plate, and ω is the frequency parameter of the system. Moreover, the deflection W must satisfy the boundary conditions and the corner conditions.

In order to simulate the various joint restraints in the pavement-subgrade system (as shown in Fig. 1), the following generalized boundary conditions and corner conditions are employed

$$\left.\begin{array}{l}\alpha_1 W+\alpha_2 Q=\alpha_3\\ \beta_1\Theta+\beta_2 M=\beta_3\end{array}\right\}\text{ on }\Gamma \tag{2}$$

$$C_{1k}W^{(k)}+C_{2k}R^{(k)}=C_{3k}\text{ at the corner point }k \tag{3}$$

In expressions (2) and (3), α_i, β_i and C_{ik} (i=1, 2, 3) stand for undetermined parameters, respectively defined on the boundary Γ and at the corner point k. W, Θ, M and Q respectively are the deflection, rotation, bending moment and Kirchhoff equivalent

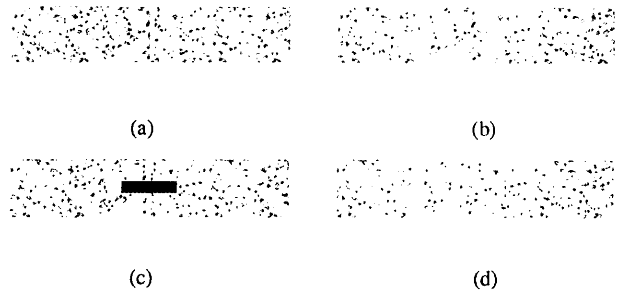

Fig. 1 Different connecting forms of the plates in the pavement-subgrade system

shear force along the plate boundary, $W^{(k)}$ and $R^{(k)}$ are the corner deflections and corner concentrated forces, which can be expressed as

$$\left.\begin{array}{l}\Theta=\dfrac{\partial W}{\partial n}\\ M=-D[\mu\nabla^2 W+(1-\mu)\dfrac{\partial^2 W}{\partial n^2}]\\ Q=-D\{\dfrac{\partial}{\partial n}[\nabla^2 W+(1-\mu)\dfrac{\partial^2 W}{\partial s^2}]-(1-\mu)\dfrac{\partial}{\partial s}(\dfrac{1}{\rho}\dfrac{\partial W}{\partial s})\}\\ R^{(k)}=-(1-\mu)D(\dfrac{\partial^2 W}{\partial n\partial s}-\dfrac{1}{\rho}\dfrac{\partial W}{\partial s})^{[k]}\end{array}\right\} \tag{4}$$

where n and s respectively stand for the outward normal and tangent of the boundary Γ; $(\cdots)^{[k]}$ means the discontinuous jump at the point k.

The boundary conditions (2) and the corner conditions (3) are more suitable for the practical problem, from which many kinds of conventional boundary conditions, including the mixed boundary conditions, can be derived.

III. BOUNDARY ELEMENT FORMULATION

We define the fundamental solution W^* of the Eq. (1) by

$$D\nabla^4 W^*+(K-\rho\omega^2)W^*=\delta(r) \tag{5}$$

The fundamental solution W^* of the problem can be derived by Hankel transform (Puttonen and Varpasuo, 1986). By using Kelvin functions of the second kind, the fundamental solution W^* can be expressed in the following form

$$W^*(r)=-\frac{1}{2\pi D}\sqrt{\frac{D}{K-\rho\omega^2}}\,Kei\left(\sqrt[4]{\frac{K-\rho\omega^2}{D}}\,r\right) \tag{6}$$

By using the general reciprocal theorem (Bezine, 1988), the boundary integral equation for the direct method can be expressed in the following form

$$CW(P) = \int_T [W^*Q - \Theta^*M + M^*\Theta - Q^*W]d\Gamma$$

$$+\sum_k [(W^*)^{(k)}R^{(k)} - (R^*)^{(k)}W^{(k)}] \quad (7)$$

where the value of a coefficient C depends on the position of the point P. When $P\rightarrow\Gamma$, C is the inner angle between the tangents.

From Eq. (7) all other variables can be computed when the values of W, Θ, M, Q and $R^{(k)}$ are known. For the solution of these unknown variables, two boundary integral equations are needed at every boundary point. One of these can be obtained by $P\rightarrow\Gamma$ in Eq. (7) and the other can be obtained by differentiating Eq. (7). By using the notation $W_1^*=W^*$ and $W_2^*=\partial W^*/\partial n$, these two boundary integral equations can be written as follows

$$CW(P) = \int_\Gamma [W_i^*Q - \Theta_i^*M + M_i^*\Theta - Q_i^*W]d\Gamma$$

$$+\sum_k [(W_i^*)^{(k)}R^{(k)} - (R_i^*)^{(k)}W^{(k)}] \quad (i=1, 2) \quad (8)$$

The boundary integral Eq. (8), together with the boundary conditions (2) and corner conditions (3) constitute a set of simultaneous equations, which can be solved to yield the solution of the problem.

Now we discretize the boundary Γ into boundary elements and require that all of the boundary corners are contained in the nodes. Inside the element a boundary variable is defined as a linear function of its nodal values. By using the standard boundary element discretization for the above set of equations, a linear system of algebraic equations with respect to the unknown nodal values $\boldsymbol{a}$ on the boundary can be expressed in an abbreviated form

$$\boldsymbol{A}(\lambda)\boldsymbol{a}=\boldsymbol{0} \quad (9)$$

where λ is the non-dimensional frequency parameter associated with the frequency parameter ω of the system. Since the singularity of the fundamental solution W_i^* in Eq. (8) is strong, an auxiliary boundary technique has been employed to calculate the influence matrices $\boldsymbol{A}(\lambda)$.

In order to determine the natural frequency, the determinant of the coefficient matrix $\boldsymbol{A}(\lambda)$ is zero, which can be solved by the step-by-step search technique (Kitahara, 1985).

IV. THE IDENTIFICATION PROCEDURE

The identification of the boundary condition may be viewed as an optimization problem. The objective function to be minimized is written as a least-squares difference between the computed frequencies $\lambda_i(z)$ and the measured natural frequencies $\overline{\lambda}_i$ of the system.

$$f(z) = \sum_{i=1}^{L} \phi_i \left[\frac{\lambda_i(z) - \overline{\lambda}_i}{\overline{\lambda}_i}\right]^2 \quad (10)$$

where ϕ_i is an arbitrary weighting parameter to change the sensitivity in the minimization process; and the design vector z is defined in terms of some undetermined parameters that can completely describe the boundary environment of the system. For the minimization of the objective function $f(z)$, a conjugate gradient technique of optimization was adopted. The method starts with an initial guess z_0, and generates the improved value as

$$z_{k+1}=z_k+l_k\theta_k \quad (11)$$

where k is the iteration number, θ_k is the search direction that is modified at each step by the conjugate gradient technique, and l_k is the step-length along the search direction. In the present study, the Golden section method is employed for determining the step-length, which requires the scalar l_k satisfy

$$f(z_{k+1})=\min\{f(z_k+l\theta_{k+1}|_{l\geq 0})\} \quad (12)$$

The algorithm for the minimization of the function $f(z_k)$ is considered to have converged when the successive evaluations are such that $\|f(z)\|\leq\varepsilon$, where ε is a prescribed tolerance.

Moreover, the gradient of the objective function $f(z)$ in Eq. (10) may be written as

$$\frac{\partial f(z)}{\partial z} = 2\sum_{i=1}^{L} \phi_i \frac{\lambda_i(z) - \overline{\lambda}_i}{\overline{\lambda}_i^2}\frac{\partial\lambda_i}{\partial z} \quad (13)$$

where $\partial\lambda_i/\partial z$ are the natural frequency sensitivities that can be conveniently obtained from the boundary element formulation by using the implicit-differentiation approach in (Saigal, 1989). An important feature of these derivations is that they do not require the computation of the inverse with respect to the Hessian matrix of the objective function.

V. NUMERICAL EXAMPLES

Two examples, respectively associated with the plate structures and the pavement-subgrade system, are presented to demonstrate the effectiveness of the formulations developed in this paper. All of these

Table 1 Identifying the boundary conditions of square plates

Case 1 (S-C-S-C)				Case 2 (S-F-S-F)			
k	α_1	β_1	$f(z_k)$	k	α_1	β_1	$f(z_k)$
1	0.5000×10^{-1}	0.5000×10^{-1}	0.1293×10^{1}	1	0.5000×10^{-1}	0.5000×10^{-1}	0.3536
2	0.7253×10^{-2}	0.7136×10^{-2}	0.7447×10^{-1}	2	0.9756	0.1426×10^{1}	0.6108×10^{-1}
3	0.9443×10^{-3}	0.4855×10^{-2}	0.3609×10^{-1}	3	0.4517×10^{2}	0.4201×10^{1}	0.3693×10^{-1}
4	0.7251×10^{-4}	0.1843×10^{-3}	0.9623×10^{-2}	4	0.1650×10^{3}	0.6731×10^{1}	0.3382×10^{-2}
5	0.5790×10^{-4}	0.8938×10^{-4}	0.2543×10^{-2}	5	0.8405×10^{3}	0.6917×10^{2}	0.2088×10^{-2}
6	0.3254×10^{-4}	0.7051×10^{-4}	0.1721×10^{-3}	6	0.1373×10^{4}	0.2639×10^{3}	0.5219×10^{-3}
7	0.9827×10^{-5}	0.2677×10^{-4}	0.9354×10^{-4}	7	0.3428×10^{4}	0.8087×10^{4}	0.1078×10^{-3}
8	0.8618×10^{-5}	0.5109×10^{-5}	0.8407×10^{-4}	8	0.7807×10^{4}	0.6878×10^{5}	0.9525×10^{-4}

Table 2 Identifying the boundary conditions of the pavement-subgrade system

k	α_1	α_2	β_1	β_2	$f(z_k)$
1	1.00000	1.0000	1.0000	1.0000	0.0812
2	15.4592	9.2355	1.9525	2.4142	0.1501
3	20.4155	8.8859	1.5709	1.9041	0.1390
4	27.5261	6.5993	2.8655	1.5860	0.8235
5	29.0520	6.0573	3.5747	0.8452	0.4547
6	28.2797	5.3056	3.4757	0.3897	0.1598
7	27.0568	4.8842	3.1787	0.4685	0.0925
8	26.8147	5.0823	3.0244	0.4345	0.0602
*	29.4118	5.2360	2.7443	0.4051	

*) Zheng and Yao 1994

problems are concerned with the rectangular plates, which are assumed to have symmetric elastic restraints respectively along one pair of opposite edges Γ_1 and the other pair of opposite edges Γ_2 as follows

$$\left.\begin{array}{l} W_i = \alpha_i Q_i \\ \Theta_i = \beta_i M_i \end{array}\right\} \text{ on } \Gamma_i \ (i=1,\ 2) \tag{14}$$

where α_i and β_i are the identified parameters.

1. Identifying the Boundary Conditions of Square Plates (without foundation)

In this case, the square plates are simply supported along one pair of opposite edges Γ_2. On the other pair of opposite edges Γ_1, they have symmetric elastic restraints as in Eq. (14). Poisson's ratio is chosen to be 0.3. Sixteen linear boundary elements of the same size are employed for the computation. The actual boundary conditions along Γ_1 are given in the following two cases:

(1) Clamped edges (α_1=0 and β_1=0)

(2) Free edges ($\alpha_1=\infty$ and $\beta_1=\infty$)

The first four actual frequencies are obtained from Leissa's exact values (Leissa, 1973). The convergence histories for the design variables are shown in Table 1.

2. Identifying the Boundary Conditions of the Pavement-subgrade System

The following properties for an actual pavement-subgrade system are given:

length of the pavement a=5m
width of the pavement b=4m
pavement thickness h=0.24m
mass density ρ=552.03 kg/m^2
Young's modulus of pavement material E= 3.5×10^3 MPa
Poisson's ratio of pavement material μ=0.167
Winkler modules of the foundation $K=6.35\times10^6$ kg/m^3

The first six frequencies are obtained by measuring an actual pavement-subgrade system. The convergence histories for the design variables are shown in Table 2. A comparison between the results in this paper and the results obtained by other methods (Zheng and Yao, 1994), is also listed in Table 2, which shows good agreement.

VI. CONCLUSIONS

In this paper, we focus our attention on developing an identification method to determine the boundary conditions of the pavement-subgrade system. The various joint restraints for the pavement-subgrade system are simulated by elastic restraint edges with a number of undetermined parameters. The identification procedure is performed by combining the boundary element method with optimization techniques. From this investigation we can see that boundary element methods seem to be superior to some domain methods in the boundary identification of certain structures.

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Discussions of this paper may appear in the discussion section of a future issue. All discussions should be submitted to the Editor-in-Chief.

Manuscript Received: Dec. 15, 1999
Revision Received: Jan. 23, 2000
and Accepted: Feb. 21, 2000

應用邊界元法識別路面－地基系統的邊界條件

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摘　要

本文將路面ー地基系統簡化爲彈性地基矩形板模型，相鄰板塊之間的約束採用包含待定參數的廣義邊界條件進行處理。然後採用邊界元法求解彈性地基板的自由振動問題。在此基礎上，將邊界元法與優化技術相結合，實現了路面ー地基系統的邊界條件識別。爲了說明邊界元法和參數識別過程的應用，本文分別給出了一個薄板結構數値算例和一個路面ー地基系統實際算例。

關鍵詞：路面ー地基系統，邊界條件識別，邊界元法， 優化技術。