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# GREEN'S FUNCTION FOR RANDOM MEDIA

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***Key Words:*** stochastic boundary element, bioengineering application, Fourier transform of distribution, *Mathematica*.

## ABSTRACT

Random coefficients in partial differential equations and boundary conditions pose a computational challenge. The stochastic finite element formulation is involved because the Tatarski *convection like* terms must be captured via stochastic strain-displacement matrices. Once the stochastic Green's Function is obtained, standard packages for boundary element analysis, e.g., BEASY, can be employed. Here, for random constitutive properties, a stationary iteration scheme is demonstrated via Fourier transform of distributions. The *deterministic* Green's function associated with a uniform medium provides the kernel. There is no such analog for stochastic finite elements. In a current bio-engineering stress analysis program a computer algebra environment, viz. *Mathematica*, is used to approximate stochastic Green's Functions.

## I. INTRODUCTION

Constitutive properties of engineering materials cannot be described in exact terms. Fig. 1 shows a typical two-dimensional percentage-variations of the shear modulus, $\mu$, of a thin elastic plate with non-dimension thickness $h$=.01. These uncertainties influence the overall performance in design-analysis. Even for a deterministic case an exact closed-form solution is intractable due to the complexity of the geometry or constitutive inhomogeneity. Numerical techniques such as stochastic boundary element and stochastic finite element methods have been developed, as natural extensions of the conventional boundary element and finite element methods, respectively, when randomness occurs.

Strategies to consider randomness can be classified as follows:

(i) *random forces:* a deterministic system subjected to nondeterministic forces;

(ii) *system stochasticity:* a nondeterministic system under deterministic loads;

This paper focuses on the latter. A combination of the above two cases involves technical as opposed to conceptual difficulties of formulation and numerical computation in stochastic mechanics problems of solid/fluid.

It is now possible to carry out, rather inexpensively, a large scale thermomechanical boundary element simulation. Techniques of probabilistic analysis and statistical computation are invoked to model realistic engineering systems. This paper is motivated by the following historical milestones.

The perturbation scheme, *vide* (Nayfeh, 1973), occasioned the original design of computer algebra systems, utilized in the early 60's to carry out steps of lengthy algebraic expressions. With the development of the programming languages particularly suited to computer algebra, even a second order expansion can be automated rather easily with a short code.

A Monte Carlo implementation was demonstrated to be effective by Astill *et al.*, (1972) to study the failure of concrete cylinders under impact loading. Since each sample was separately discretized and independently computed, the overall statistical results were acceptable. However, the numerical efficiency was questionable.

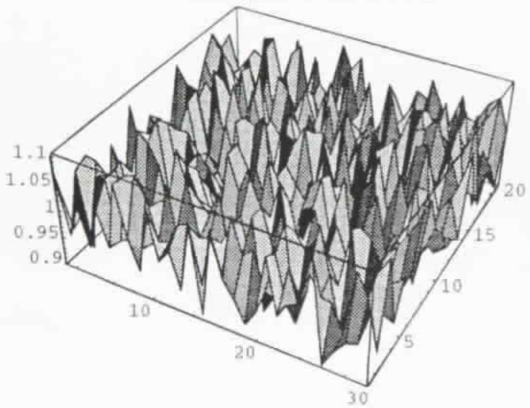


Fig. 1 Observed random variation for the shear modulus

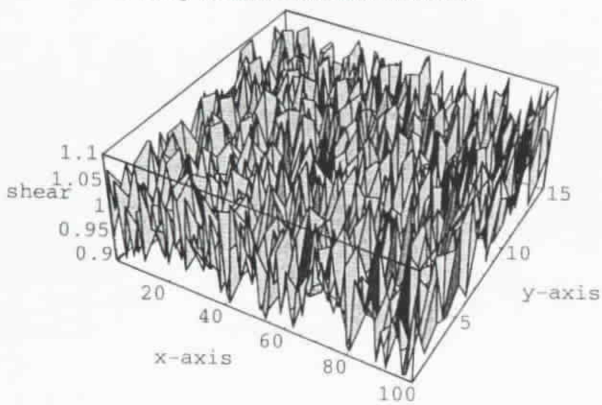


Fig. 2 Simulated random variation for the shear modulus

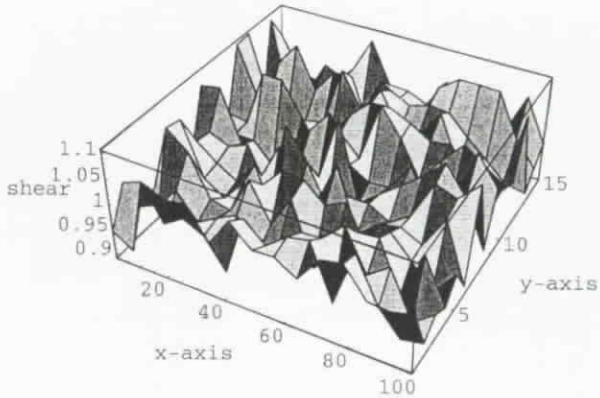


Fig. 3 Smoothed random field

Nakagiri and Hisada (1980), published perturbation finite element methods to solve problems with "small" statistical variabilities. Their technique demands factorization of the reference stiffness matrix *only once* to compute statistical moments of any order.

The pioneering work of Keller and his associates, *vide* references (Keller, 1962; 1964; Keller and Mc Kean, 1973), provides a foundation for estimating the mean stochastic operator. In addition to analytical work, they elaborated upon the Monte Carlo simulation technique, method of statistical moments and smoothing schemes. Sobczyk (1985; 1991) in two books utilized those mathematical formulations and furnished a basic methodology suitable to develop stochastic finite/boundary elements.

The principal issues of *probabilistic computation* related to the stochastic finite and boundary elements share a number of strikingly similar common grounds. In boundary elements, one needs stochastic Green's Functions to construct system matrices: $[\tilde{G}]$ and $[\tilde{H}]$ in $[\tilde{G}]\{\tilde{\tau}\} = [\tilde{H}]\{\tilde{u}\}$, where $\{\tilde{\tau}\}$ and $\{\tilde{u}\}$ are, respectively, *stochastic traction* and *stochastic displacement* vectors. Correspondingly, to capture the effects of material stochasticity, in finite elements stochasitc shape functions, vide reference (Dasgupta and Yip, 1989), must be first obtained. Subsequently, the *stochastic strain-displacement* relation $[\tilde{B}]$ for a given *stochastic constitutive* spatial variation $[\tilde{D}]$ should be computed without contaminating static equilibrium conditions.

Based on analytical results, the author in reference (Dasgupta, 1987) illustrated schemes to estimate mean, standard deviation and covariance matrix of responses when the constitutive variables form a correlated stochastic field. Foundation-structure interaction problems, in reference (Gyebi and Dasgupta, 1992), where random soil properties should be considered, initiated the research. The pointwise observed random field data is approximated by a spatial correlation function, *vide* Figs. 2 and 3. The contour plot is in Fig. 4. The interpolated function is used in the numerical simulation of random fields.

In general, random coefficients in partial differential equations, which govern the responses of stochastic thermo-mechanical systems, demand unconventional computational tools. This paper addresses system stochasticity originating from nondeterministic distribution of material properties in computer algebra environment. In particular, the randomness of constitutive variables related to *potential energy* is considered in an efficient boundary element setting.

A concurrent investigation analyzes an advanced algorithm for self-adjoint problems of mathematical physics as a special case. The required *Stochastic Green's Functions* are constructed from the *corresponding* Green's Functions related to uniform material properties. Here *stationary* iteration on the *deterministic operator* (after 'invertion') guarantee the accuracy of the boundary element method for *all*

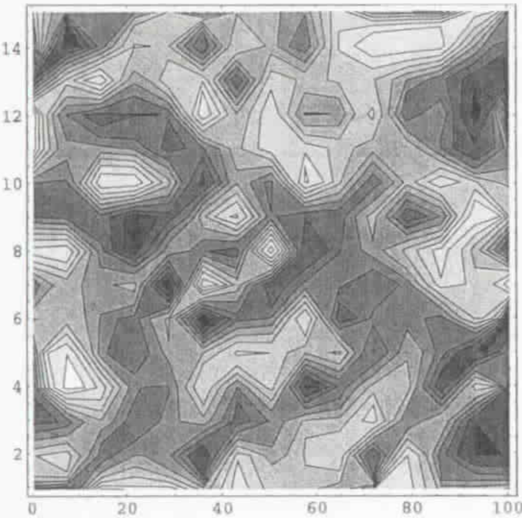


Fig. 4 Contours of smoothed simulated field

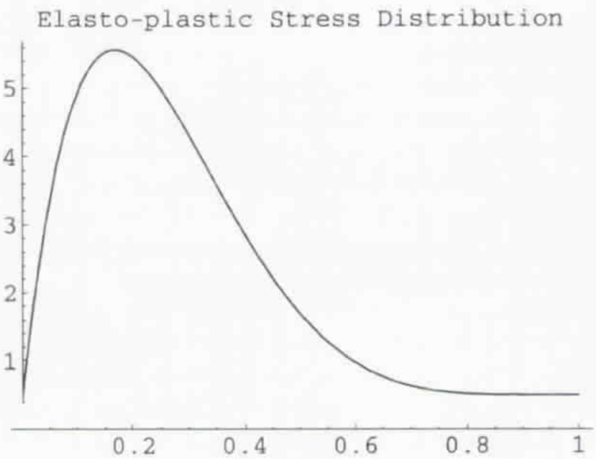


Fig. 5 Beta distribution of responses

samples. The resulting stochastic Green's Functions can be easily incorporated in computer codes, e.g., BEASY, reference (BEASY, 1998). Use of a symbolic program, viz., *Mathematica*, enhanced the proposed formulation.

## 1. Green's Function vis a vis Shape Function

Finite element formulations incur various second order and, in many cases, even first order numerical error. For example, incompressible cases and failure in simultaneous patch and zero-locking test can be cited. Additionally in a stochastic finite element formulation the *convection like* terms, following Tatarski's wave consideration, *vide* references (Molyneux, 1968; Tatarski, 1961; Whitham, 1974), must be captured via stochastic strain-displacement matrices arising out of stochastic shape functions. The boundary element procedure, assisted by symbolic formulation using computer algebra, circumvents such deficiencies.

The relative merit of the stochastic boundary element formulation via stochastic Green's Function in comparison with the stochastic finite element method needs to be examined. An estimation of the stochastic strain-displacement transformation matrix $[\tilde{B}]$ via stochastic shape functions in reference (Dasgupta and Yip, 1989) is computationally expensive. This is economized in a fixed-mesh finite element by essentially capturing the *secondary* order effects of randomness.

In a Monte Carlo computation, *all* samples should have the same order of error out of $[\tilde{B}]$ to preserve the quality of the stress field. The conventional *ad hoc* approximate procedure of using the deterministic *element*-SUBROUTINE, without considering *stochastic shape functions*, leads to the approximate stochastic stiffness matrix $[\mathcal{K}^{app}]=\int_\Omega [B]^T[\tilde{D}][B]d\Omega$ in terms of the sample constitutive field $[\tilde{D}]$. On the other hand, the (hybrid) stress equilibrium statement, *vide* reference (Dasgupta, 1987), yields $[\tilde{D}][\tilde{B}]=[D][B]$ and leads to $[\tilde{\mathcal{K}}]=\int_\Omega [B]^T([D][\tilde{D}]^{-1}[D])[B]d\Omega$.

## 2. Current Application

The stochastic boundary element method is employed for analyzing biological objects with random material properties. In his on-going research, the author is using an "operator expansion" technique to analyze thin elastic plate, refer to (Camp and Gipson, 1990), and shell problems with random moduli. Algebraic formulation is suitable for such cases, particularly when lengthy governing differential equations are encountered. For severe fluctuations of the random field, a collection of one hundred Monte Carlo boundary element solutions will be curve fitted to construct $\beta$-like distributions as shown in Fig. 5.

In this paper we focus on *randomness in material properties*; not shown here are the cases of stochastic boundary geometry, which can be formulated very conventiently by stochastic boundary elements as well. A number of available methods in the literature dealing with *random vibration* can analyze effects of random loading. In practical applications, stochastic computation not only furnishes a design tool but also paves the way to develop future computer soft/hardware technology. Currently, large scale personal computing is viable in everyday design-analysis using the boundary element method.

The highly parallel nature of *convolutions* of Green's functions with loading distributions encourages the development of computers with massive parallel processing capabilities. At the level of concept development, FORTRAN and C are not suitable for rapid prototyping. Symbolic computing, also known as *computer algebra*, proves to be handy and practical. Here the major concern is that all such languages are cryptic and require considerable effort to produce results. However, the long term research benefit of learning a symbolic computing language is extremely valuable.

Material stochasticity gives rise to a second order effect. This demands a better computing environment than that of the deterministic counterpart, where the interest is to capture the dominant first order responses. computer algebra can be employed in all steps to guarantee a uniform accuracy in evaluating the statistical moments for skewness and curtosis. A typical bioengineering result from a probabilistic analysis of a thin shell to depict a skull is cited in Fig. 5.

## II. DEVIATOR FORM

The stochastic operator $\tilde{\mathcal{L}}$ is a deviation from the deterministic datum $\mathcal{L}^o$:

$$\tilde{\mathcal{L}} = \mathcal{L}^o + (\tilde{\mathcal{L}} - \mathcal{L}^o) \tag{1}$$

The solution for $\mathcal{L}^o$, the operator for a uniform material property, is first obtained as $\mathcal{G}^o$ according to a standard method. Hence,

$$[\mathcal{L}^o]\mathcal{G}^o = \delta \text{ and } [\tilde{\mathcal{L}}]\tilde{\mathcal{G}} = \delta \tag{2}$$

With an initial guess $\tilde{\mathcal{G}}^{(i)}$, a correction term $\Delta\tilde{\mathcal{G}}^{(i)}$ emerges from:

$$[\tilde{\mathcal{L}}]\tilde{\mathcal{G}}^{(i+1)} = [\mathcal{L}^o]\mathcal{G}^o \tag{3}$$

which is written in the incremental form:

$$[\mathcal{L}^o + (\tilde{\mathcal{L}} - \mathcal{L}^o)](\tilde{\mathcal{G}}^{(i)} + \Delta\tilde{\mathcal{G}}^{(i)}) = [\mathcal{L}^o]\mathcal{G}^o \tag{4}$$

so that

$$\tilde{\mathcal{G}}^{(i+1)} = \tilde{\mathcal{G}}^{(i)} + \Delta\tilde{\mathcal{G}}^{(i)} \text{ is an improvement on } \tilde{\mathcal{G}}^{(i)} \tag{5}$$

A main restriction in the proposed formulation is that the term: $(\tilde{\mathcal{L}} - \mathcal{L}^o)\Delta\tilde{\mathcal{G}}^{(i)}$ contributes only to second order effects.

The first order approximation from the expansion of Eq. (4):

$$[\mathcal{L}^o]\tilde{\mathcal{G}}^{(i)} + [(\tilde{\mathcal{L}} - \mathcal{L}^o)]\tilde{\mathcal{G}}^{(i)} + [\mathcal{L}^o]\Delta\tilde{\mathcal{G}}^{(i)} + [(\tilde{\mathcal{L}} - \mathcal{L}^o)]\Delta\tilde{\mathcal{G}}^{(i)} = [\mathcal{L}^o]\mathcal{G}^o \tag{6}$$

becomes: $[\mathcal{L}^o]\tilde{\mathcal{G}}^{(i)} + [(\tilde{\mathcal{L}} - \mathcal{L}^o)]\tilde{\mathcal{G}}^{(i)} + [\mathcal{L}^o]\Delta\tilde{\mathcal{G}}^{(i)} = [\mathcal{L}^o]\mathcal{G}^o$ (7)

Consequently, the corrector $\Delta\tilde{\mathcal{G}}^{(i)}$ is estimated to be:

$$\Delta\tilde{\mathcal{G}}^{(i)} = -(\tilde{\mathcal{G}}^{(i)} - \mathcal{G}^o) - [\mathcal{L}^o]^{-1}[(\tilde{\mathcal{L}} - \mathcal{L}^o)]\tilde{\mathcal{G}}^{(i)} \tag{8}$$

This convolution is accelerated by Fourier transformation of *distributions*. *Mathematica* can carry out the required closed-form analytical calculations. An analytical example in one spatial variable is included in this paper. Application of the results from the aforementioned iterations for thermo-mechanical boundary elements is summarized below.

Recall that the deterministic Green's function $\mathcal{G}^o$ solves the homogeneous continua governed by $\mathcal{L}^o$. In the convolution form the iterated stochastic Green's function for the stochastic operator $\tilde{\mathcal{L}}$, from Eq. (8), becomes:

$$\Delta\tilde{\mathcal{G}}^{(i)}(\eta) = -(\tilde{\mathcal{G}}^{(i)}(\eta) - \mathcal{G}^o(\eta)) - \int_\Omega \mathcal{G}^o(\eta|\zeta)[(\tilde{\mathcal{L}}(\zeta) - \mathcal{L}^o(\zeta))]\tilde{\mathcal{G}}^{(i)}(\zeta)d\zeta \tag{9}$$

This strategy of *stationary iteration* with $\mathcal{G}^o$ has been applied successfully for inhomogeneous continua in references (Dasgupta, 1989; 1992). In thermo-mechanical problems the response vector $\{\tilde{v}\}$ houses the displacement vector $\tilde{u}$ and the temperature $\tilde{T}$, and is governed by:

$$\tilde{\mathcal{L}}\{\tilde{v}\} = \{\delta\} \text{ where } \tilde{\mathcal{L}} = \begin{bmatrix} \tilde{\mathcal{L}}_{11} & \tilde{\mathcal{L}}_{12} \\ \tilde{\mathcal{L}}_{21} & \tilde{\mathcal{L}}_{22} \end{bmatrix} \text{ and } \{\tilde{v}\} = \begin{Bmatrix} \tilde{u} \\ \tilde{T} \end{Bmatrix} \tag{10}$$

The nondeterministic field operator is decomposed such that the stochastic part $\mathcal{L}^*$ is isolated according to the following scheme:

$$\tilde{\mathcal{L}} = \mathcal{L}^o + \mathcal{L}^* \text{ where } \mathcal{L}^o = \begin{pmatrix} \mathcal{L}^o_{11} & 0 \\ 0 & \mathcal{L}^o_{22} \end{pmatrix} \text{ and}$$

$$\mathcal{L}^* = \begin{pmatrix} \mathcal{L}^*_{11} & \tilde{\mathcal{L}}_{12} \\ \tilde{\mathcal{L}}_{21} & \mathcal{L}^*_{22} \end{pmatrix} \tag{11}$$

An existing boundary element code, which solves the steady vibration problems, and the temperature problems *separately* will have the modules to generate and use the boundary element system matrices related to $\mathcal{L}^o_{11}$ and $\mathcal{L}^o_{22}$. The aforementioned decomposition in Eqs. (10 and 11) accelerates the stationary iteration in a stochastic boundary element formulation as indicated in the bibliography of reference (Dasgupta, 1992).

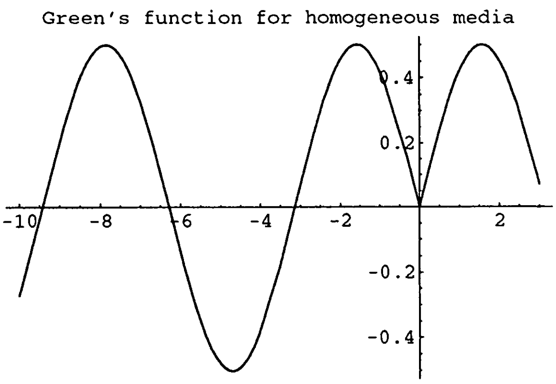


Fig. 6 Original green's function

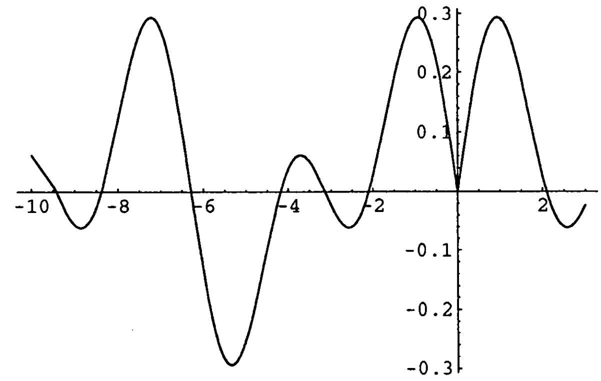


Fig. 7 Second term-(coefficient of $\tilde{\epsilon}$)

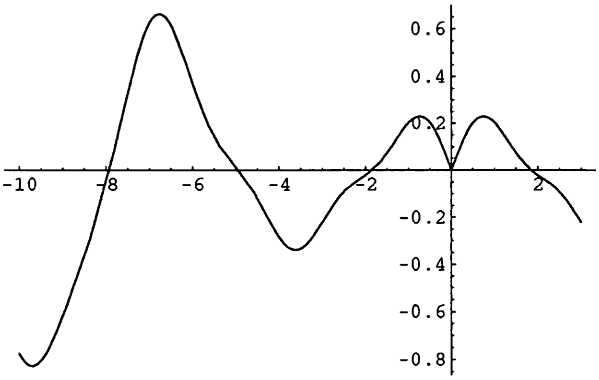


Fig. 8 Third term-(coefficient of $\tilde{\epsilon}^2$)

## III. AN ANALYTICAL EXAMPLE

The computer algebra program *Mathematica* was employed to obtain closed-form expressions presented in this section. Fourier Transform, an add-on package, was extensively used. The transformations of *distributions* in the weak sense were carried out analytically without approximation. A canonical one-dimensional problem of spatial variability highlighting material randomness is illustrated below.

Consider a canonical Helmholtz operator $\mathcal{L}^o$:

$$\mathcal{L}^o = \frac{d^2}{dx^2} + 1 \tag{12}$$

The Green's function $\mathcal{G}^o$, such that:

$$[\mathcal{L}^o(x)]\mathcal{G}^o(x) = \delta(x) \tag{13}$$

is, *vide* a standard reference (Stackgold, 1979), in the *singular* distribution form, see Fig. 6:

$$\mathcal{G}^o(x) = \frac{1}{2}\sin(|x|) \tag{14}$$

Let us consider a single stochastic variable $\tilde{\epsilon}$ associated with a spatial variability function $a(x)$. The inhomogeneous (self-adjoint Helmholtz) equation becomes:

$$\tilde{\mathcal{L}} = \frac{d}{dx}((1 + \tilde{\epsilon}a(x))\frac{d}{dx}) + 1 \tag{15}$$

An example of a cosine-spatial variability for $a(x)$ leads to:

$$\tilde{\mathcal{L}}(x) = 1 - \tilde{\epsilon}\sin(x)\frac{d}{dx} + (1 + \tilde{\epsilon}\cos(x))\frac{d^2}{dx^2} \tag{16}$$

Using Eq. (14) as the starting function according to Eq. (9) the first correction term for the Green's function, shown in Fig. 7 becomes:

$$\Delta\tilde{\mathcal{G}}^{(1)} = -\frac{\tilde{\epsilon}}{6}(\sin(|x|) + \sin(2|x|)) \tag{17}$$

*Mathematica* in an Apple 233MHz machine took '47.5833 Second' for the result. In the second iteration the starting value was:

$$\tilde{\mathcal{G}}^{(1)} = \frac{1}{2}\sin(|x|) - \frac{\tilde{\epsilon}}{6}(\sin(|x|) + \sin(2|x|)) \tag{18}$$

Now the second deviator from Eqs. (9) and (18), shown in Fig. 8 was calculated to be:

$$\Delta\tilde{\mathcal{G}}^{(2)} = \frac{\tilde{\epsilon}^2}{144}(12|x|\cos(x) + 17\sin(|x|) + 8\sin(2|x|) + 9\sin(3|x|)) \tag{19}$$

*Mathematica* took '98.2667 Second' (at 233MHz) to implement an analytical code based on The FourierTransform *add-on* package.

The *Mathematica* calculation was carried out to the third iteration to capture $\tilde{\epsilon}^3$ terms. This step was initialized with:

$$\tilde{\mathcal{G}}^{(2)} = \tilde{\mathcal{G}}^{(1)} + \Delta\tilde{\mathcal{G}}^{(2)} = \frac{1}{2}\sin(|x|) - \frac{\tilde{\epsilon}}{6}(\sin(|x|) + \sin(2|x|)) + \frac{\tilde{\epsilon}^2}{144}(12|x|\cos(x) + 17\sin(|x|) + 8\sin(2|x|) + 9\sin(3|x|)) \tag{20}$$

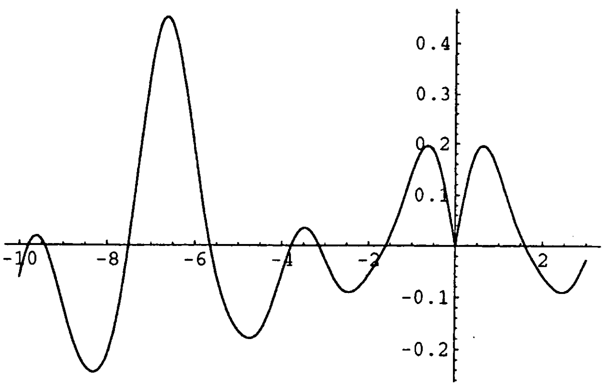

Fig. 9 Fourth term-(coefficient of $\tilde{\epsilon}^3$)

The third order improvement for the Green's function, which is associated with the $\tilde{\epsilon}^3$ (the $n^{th}$ correction contains $\tilde{\epsilon}^n$), shown in Fig. 9 is obtained as:

$$\Delta\tilde{G}^{(3)} = \frac{\tilde{\epsilon}^3}{2160}(60|x|\cos(x) + 60|x|\cos(2x) + 149\sin(|x|)$$

$$+ 230\sin(2|x|) + 45\sin(3|x|) + 54\sin(4|x|)) \qquad (21)$$

This computation took '240.667 Second' (at 233MHz). Finally, the three term approximation $\tilde{G}^{(3)}$ in:

$$[\frac{d}{dx}((1 + \tilde{\epsilon}a(x))\frac{d}{dx}) + 1]\tilde{G}^{(3)}(x) \cong \delta(x) \qquad (22)$$

becomes:

$$\tilde{G}^{(3)}(x) \cong \frac{1}{2}\sin(|x|) - \frac{\tilde{\epsilon}^3}{6}(\sin(|x|) + \sin(2|x|))$$

$$+ \frac{\tilde{\epsilon}^2}{144}(12|x|\cos(x) + 17\sin(|x|) + 8\sin(2|x|) + 9\sin(3|x|))$$

$$+ \frac{\tilde{\epsilon}^3}{2160}(60|x|\cos(x) + 60|x|\cos(2x) + 149\sin(|x|)$$

$$+ 230\sin(2|x|) + 45\sin(3|x|) + 54\sin(4|x|)) \qquad (23)$$

The results are summarized in Figs. 10 and 11.

## IV. CONCLUSIONS

Stochastic computations have been designed to capture higher order effects for accurate depiction of covariance features of responses. A natural definition of 'stochastic nonlinearity' emerged where 'large variability' causes the perturbation method to diverge. Monte Carlo procedure remains as the only choice for stochastic nonlinear systems, where the material variability results in substantial standard deviation in solutions. The stochastic Green's Function developed in this paper is useful for the entire spectrum of randomness, where only the stochastic boundary

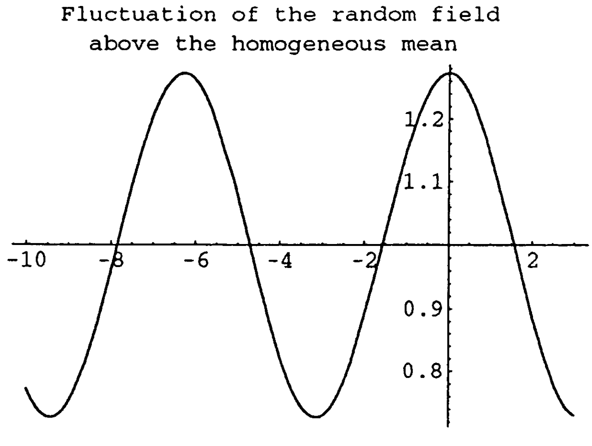


Fig. 10 Spatially correlated randon field

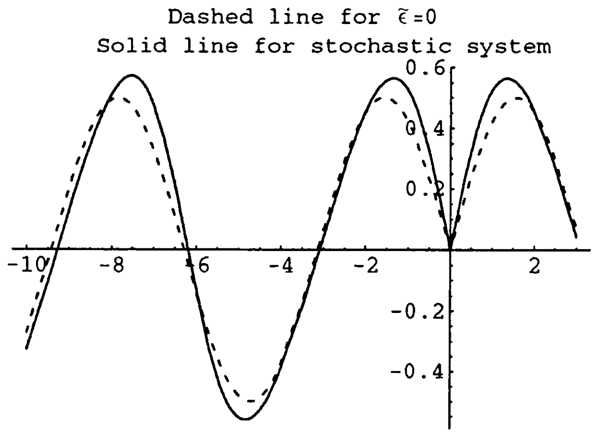


Fig. 11 Effects of stochasticity for green's function

element method furnishes a consistent solution procedure.

The stochastic strain-displacement transformation, i.e., $[\tilde{B}] = [\tilde{D}]^{-1}[D][B]$ must be used in high accuracy stochastic finite element computer programs. The computer code development chore is quite substantial. The proposed methodology of using stochastic Green's Function in a stochastic boundary element is devoid of this additional effort.

Using *Mathematica* random field and random responses are constructed by smoothing algebraic interpolants acting upon discrete field obtained from random number generator, *vide* Figs. 2, 3 and 4. The most challenging task is to obtain a stochastic Green's function such that the accuracy does not vary from sample to sample in a Monte Carlo simulation. Algebraic, not FORTRAN or C style numeric, computer programs thus become very practical, as demonstrated here. Incompressible formulation with the Poisson's ratio $\nu = \frac{1}{2}$ entails lengthy algebraic expressions. The additional consideration of the stochastic shear modulus $\tilde{\mu}$ does not add significant coding chores. Implementation of computer algebra in *stochastic computation* can thus be observed to be a very powerful tool.

The stationary iteration, demonstrated in this paper with an example, exhibits the usefulness of computer algebra programs in formulating stochastic Green's Functions. The implementation of Fourier transform in the *weak* sense of *distribution* is straight forward. Especially, in the computer mathematics environment *Mathematica* a package FourierTransform can elegantly carry out the required steps.

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## NOMENCLATURE

| | |
|---|---|
| $h$ | plate thickness |
| $x$, $y$ | coordinate axes |
| $[\tilde{B}]$ | stochastic strain-displacement matrix |
| $[\tilde{D}]$ | stochastic constitutive matrix |
| $[\tilde{G}]$, $[\tilde{H}]$ | stochastic boundary element matrices |
| $\mathcal{G}^o$ | Green's function for homogeneous media |
| $[\mathcal{K}]$ | finite element stiffness matrix |
| $\mathcal{L}$, $[\mathcal{L}]$ | linear operator |
| $\tilde{\mathcal{L}}$, $[\tilde{\mathcal{L}}]$ | stochastic linear operator |
| $\mathcal{L}^o$ | linear operator for homogeneous media |
| $\mathcal{T}$ | matrix transpose indicator |
| $\tilde{\mu}$ | stochastic boundary element displacement |

**Greek Symbols**

| | |
|---|---|
| $\beta$ | probability distribution |
| $\delta$ | Dirac's delta |
| $\Delta$ | increment indicator |
| $\tilde{\epsilon}$ | stochastic parameter |
| $\mu$ | shear modulus |
| $\nu$ | Poisson's ratio |
| $\eta$, $\zeta$ | integration variables |
| $\Omega$ | domain |
| $\tilde{\tau}$ | stochastic boundary element traction |

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Discussions of this paper may appear in the discussion section of a future issue. All discussions should be submitted to the Editor-in-Chief.

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# 隨機介質之格林函數

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## 摘 要

偏微分方程中隨機係數及隨機邊界條件都將造成計算上的困難。因爲需考慮在介質的位移應變矩陣關係中類似 Tatarski 傳導項效應，需涉及有限元素推導。一但隨機格林函數決定，即可利用邊界元素分析的套裝軟體如 BEASY加以分析。對於隨機介質組成之特性，可利用分佈函數之傅利葉轉換驗證穩態的疊代過程。均勻介質之定率格林函數在本文中亦有提及。對於隨機有限元素不可做此類比。現今利用 Mathematica 的計算環境發展的生物工程應力分析程式可用來近似隨機格林函數。

關鍵詞：隨機邊界元素，生物工程應用，分佈函數之傅利葉轉換，Mathematica 軟體。