MOVING ELEMENT FREE PETROV-GALERKIN VISCOUS METHOD

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ABSTRACT

Moving meshless methods are new generation of numerical methods for time dependent partial differential equations that have shock or high gradient region. These methods couple the moving finite element methods (MFE) with meshless methods. Here, grid coordinates are time dependent, unknown and are found together with an approximate solution to time dependent PDE. Weak form system is an stiff ODE system and here, it will be found with Galerkin and Petrov-Galerkin method. A penalty is appended to the energy functional for preventing high velocity, colliding and collapsing of nodes and prevention of concentration of all the nodes in the shock region. It controls their motion and also causes better conditioning of the mass matrix. Numerical solution of two examples demonstrates the accuracy of the approximation.

Key Words: r-refinement, adaptive grids, moving finite element, moving least square method, diffuse element method, element free Galerkin and Petrov-Galerkin method.

I. INTRODUCTION

Numerical solution of time dependent partial differential equations with shock, boundary layer, high gradient region and high oscillatory region such as gas dynamics problems, large deformation process, explosion and underwater bubble explosion, plasticity, elasticity, crack propagation phenomenon, wave propagation and penetration are subject of research for many years. These problems have some wild small region in which the solution has not good activity and this region moves with time. Approximation of this region needs special techniques. Up to this time, there are two main r-refinement methods for numerical solution of time dependent PDE's with shock: (*a*) moving mesh methods and (*b*) moving finite element methods with time dependent or moveable nodes. In these methods, the aim is moving of a fixed number of nodes and finding their adaptive coordinates to handle the activity of the problem and then finding approximate solution at them. In these methods, one gets an optimal process with minimal work. There is no addition or deletion of nodes and there is no need to raise the computation order. Progressing the meshless methods, such as Shepard Interpolant (Shepard, 1968), Smoothed Particle Hydrodynamics (SPH) (Lucy, 1977; Gingold and Monaghan, 1982; Monaghan, 1988), Moving Least Square method (MLS) for interpolation and non-interpolation (approximation) (Lancaster and Salkauskas, 1981), Generalized Finite Difference (GFD) method (Liszka and Orkisz, 1984), Kansa's method based on radial basis functions (Kansa, 1990), Diffuse Element Method (DEM) and Diffuse Approximation Method (DAM) (Nayroles *et al.*, 1992), Element Free Galerkin (EFG) method (Belytschko *et al.*, 1995), Element Free Petrov-Galerkin method (EFPG) (Krongauz and Belytschko, 1997), Reproducing Kernel Method (RKM) and Reproducing Kernel Particle Method (RKPM) (Sukky *et al.*, 1998; Aluru, 2000), Partition of Unity Method (PUM) (Babuska and Melenk, 1997), h-p clouds and h-p meshless method (Duarte and Oden, 1996), Wavelet Galerkin method (WGM) (Glowinsky *et al.*, 1990), and the methods for problems with shocks were explored. Two main advantages of the meshless methods are: (*a*) computational efficiency by avoiding the mesh generation and remeshing process which explored high volume of computational work and (*b*) flexibility and

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simplicity in raising the smoothing degree of the approximation and saving some finite element properties, such as the locality and easy work on problems with complex region and without definition of type of relation between nodes. In 1968, Donald Shepard (1968) gave his famous paper, where he presented a new interpolation of irregularly spaced data points in meteorological science. Shepard interpolation was the beginning point for moving least square (MLS) interpolation method given by Lancaster and Salkauskas (1981) and Partition of Unity method given by Babuska and Melenk (1997). MLS method is generalization of Shepard interpolation and tends to meshless interpolation and meshless approximation (quasi-interpolation or non-interpolation, *i.e* without Kronecker Delta property). After introducing MLS method, Nayroles, in 1992 (Nayroles *et al.*, 1992), employed this method for numerical solution of some PDE's using local compactly supported nonsingular weight functions and got relatively good results in spite of important disadvantages in finding complete derivatives of his base functions. His method, named Diffuse Element Method (DEM), has the following properties: (1) retains locality of finite element method; (2) raises the smoothing degree of approximation; (3) avoids the mesh generation and remeshing time consuming process; (4) decreases the volume of computational work; (5) the base functions derivatives are not complete; (6) essential boundary condition can not be satisfied because of the lack of Kronecker Delta property; (7) the approximations are based on irregular distribution of nodes; (8) smoothing degree of approximation directly related to smoothing degree of weight; (9) the base functions built by this method satisfy the consistency conditions but are not integrable (Krongauz and Belytschko, 1997). In 1995, Ted Belytschko and his colleagues (Belytschko *et al.*, 1995) generalized DEM and presented EFG method. This method has high accuracy in solving of PDE's specially on active PDE's and has more computational tasks than DEM. In this paper, we linked MFE method by Keith Miller (Miller and Miller, 1981; Miller, 1981), EFG and EFPG method by Ted Belytschko (Belytschko *et al.*, 1995; Krongauz and Belytschko, 1997). In fact, instead of usual piecewise linear hat function in MFE method, we employed the base functions built by MLS and their EFG derivatives as approximations or trial functions and piecewise C^1 cubic hermite base functions as test or weight functions. By using penalty parameters, we selected Moving Element Free Petrov-Galerkin Viscous Method (MEFP-GVM) as the name of this method. The paper is presented as follows: Section I explains MFE method. Section II explains MLS method and Shepard interpolation. DEM is subject of Section III. In Section IV, EFG and EFPG methods are explained. Some properties of weight functions are explained in Section V. The next section presents our method, the hybrid of MFE and EFPG. Heat and Burger equations are solved approximately by this method in Section VII and concluding remarks conclude this paper.

II. MOVING FINITE ELEMENTS

In 1981, Keith Miller (Miller and Miller, 1981; Miller, 1981) gave his famous paper on 1-D Moving Finite Elements *i.e.* piecewise linear finite elements at unknown, unsteady, time dependent and moveable nodes. In 1983, Herbst (Herbst *et al.*, 1983) and his co-worker proved implicitly the existence and type of equi-distribution in both 1-D moving piecewise linear finite element method and MFE Petrov-Galerkin method by piecewise cubic Hermite polynomials. In 1986, Mitchell (Mitchell and Herbst, 1986) explicitly employed piecewise C^1 cubic Hermite polynomials with compact support instead of the usual approximation base functions as test functions. Here, MFE method is explained based on Keith Miller. Let

$$u_t=\boldsymbol{A}u \tag{1}$$

be the general form of unsteady PDE, where operator $\boldsymbol{A}$ contains only space derivatives. For 1-D Burger equation $\boldsymbol{A}u=-uu_x+(1/R)u_{xx}=-(u^2/2)_x+(1/R)u_{xx}$ $(R>>1)$ and for 1-D heat equation $\boldsymbol{A}u=u_{xx}$ where $0<x<1$. Here, we have the node distribution: $0=x_0(t)<x_1(t)<\cdots<x_{n-1}(t)<x_n(t)=1$. Assume we have the following approximation

$$\tilde{u}(x,t)=\sum_{j=0}^{n}u_j(t)\phi_j(x,t) \tag{2}$$

where $\{(x_j(t),\ u_j(t))\}_{j=0}^{n}$ are unknowns. In MFE method, the set $\{\phi_j(x,t)\}_{j=0}^{n}$ is an usual set of hat functions at moving nodes and, in MEFP-GVM, instead of hat functions, we use base functions built by EFG method (see Section V). u_t, the partial time derivative of u, is found in the following form:

$$\tilde{u}_t(x,t)=\sum_{j=0}^{n}(\dot{u}_j(t)\phi_j(x,t)+\dot{x}_j(t)\psi_j(x,t)) \tag{3}$$

where dot denotes time derivative and

$$\psi_j(x,\ t)=\partial\tilde{u}\,\partial x_j=-\tilde{u}_x\phi_j(x,\ t) \tag{4}$$

(see Miller and Miller (1981) and Miller (1981)). Under some conditions, the base functions or nonlinear manifold $\{\phi_j(x,\ t),\ \psi_j(x,\ t)\}_{j=0}^{n}$, are linearly independent (see Miller and Miller, (1981), Miller (1981)). After substitution of approximation Eq. (2) into PDE (1), the residual becomes

$$\boldsymbol{R}(x, t) \equiv \sum_{j=0}^{n} (\dot{u}_j(t)\phi_j(x, t) + \dot{x}_j(t)\psi_j(x, t)) - \boldsymbol{A}\tilde{u}(x, t) \not\equiv 0 \quad (5)$$

By the least square method or minimizing the following residual functional with penalty terms with respect to the coordinate velocities $(\dot{x}_i(t), \dot{u}_i(t))$, i=0, 1, ⋯, n,

$$\boldsymbol{J}(t) = \| \boldsymbol{R}(.\,, t) \|^2_{L^2(0,1)} + \sum_{j=0}^{n} (\varepsilon_j(t)\dot{h}_j(t) - \eta_j(t))^2 \quad (6)$$

where $h_j(t)=x_{j+1}(t)-x_j(t)$ and $\varepsilon_j(t)$, $\eta_j(t)$, j=0, 1, ⋯, n are functions of $h_j(t)$, using the Galerkin method or the following inner product:

$$(\boldsymbol{R}(., t), \phi_i(., t))=0$$

$$(\boldsymbol{R}(., t), \psi_i(., t)) - \varepsilon^2_{i-1}(t)\dot{x}_{i-1}(t) + (\varepsilon^2_{i-1}(t) + \varepsilon^2_i(t))\dot{x}_i(t) - \varepsilon^2_i(t)\dot{x}_{i+1}(t) - \varepsilon_{i-1}(t)\eta_{i-1}(t) + \varepsilon_i(t)\eta_i(t)=0, \quad i=0, 1, \cdots, n \quad (7)$$

The penalty parts are appended to the MFE residual functional $J(t)$, because of: (1) prevention of colliding and collapsing of the nodes, (2) prevention of concentration all the nodes in the shock region, (3) prevention of high velocity of nodes and (4) prevention of singularity of the mass matrix and making it to be positive definite, nonsingular with better condition number and prevention of the system to be stiff. Using the Petrov-Galerkin method and without regularizing terms, we set the following:

$$(\boldsymbol{R}(., t), S_i(., t))=0$$

$$(\boldsymbol{R}(., t), T_i(., t))=0, \quad i=0, 1, \cdots, n \quad (8)$$

and the ODE system is found. The sets of $\{S_j(x, t)\}^n_{j=0}$ and $\{T_j(x, t)\}^n_{j=0}$ are piecewise cubic Hermite base functions in the following form:

$$S_i(x, t)=l^2_i(x, t)(3-2l_i(x, t))$$

$$T_i(x, t)=l^2_i(x, t)(x-x_i(t)), \quad i=0, 1, \cdots, n \quad (9)$$

where $l_i(x, t)$'s are Lagrange basis functions(here at three or five nodes with $x_i(t)$ as its center). In applying Galerkin method, mollification is needed but for the Petrov-Galerkin, it is not. Herbst, in (Herbst *et al.*, 1983), proved the existence of the following equidistribution MFE by Galerkin method:

$$h_{i-1}(t)u_{xx}(x^-_i, t)=h_i(t)u_{xx}(x^+_i, t)+O(h^2)$$

$$i=1, 2, \cdots, n-1 \quad (10)$$

For Petrov-Galerkin method, the following equidistribution principle is satisfied:

$$h^2_{i-1}(t)\tilde{u}_{xx}(x^-_i, t)=h^2_i(t)\tilde{u}_{xx}(x^+_i, t)+O(h^3)$$

$$i=1, 2, \cdots, n-1 \quad (11)$$

The final ODE system with time derivative is given in the following form:

$$\underline{\boldsymbol{A}}(t)\,\dot{\underline{\boldsymbol{u}}}(t) = \underline{\boldsymbol{b}}(t) \quad (12)$$

where $\underline{\boldsymbol{A}}(t)$ is a block tridiagonal mass matrix, $\underline{\boldsymbol{u}}(t)=[u_1(t), x_1(t), u_2(t), x_2(t), \cdots, u_{n-1}(t), x_{n-1}(t)]^T$. This dynamical system can be solved by the method of lines or other ODE methods. (Lambert, 1991)

III. MOVING LEAST SQUARE METHOD

Let u: $\Omega\times[0, T]\to I\!R$, where $\Omega\subset I\!R^d$, d=1, 2 or 3 (here d=1) be a continuous function that we don't know and try to approximate by having some data point on it. Given $x_j(t)\in\Omega$, j=0, 1, ⋯, n, a random distribution of nodes in the domain and $u_j(t)=u(x_j(t), t)$, j=0, 1, ⋯, n. Let $\boldsymbol{P}^T(x)=\{1, x, \cdots, x^{m-1}\}$ be a given m dimensional base. Define local approximation

$$\tilde{u}_y(x, t)=\boldsymbol{P}^T(x)\boldsymbol{a}(y, t) \quad (13)$$

where $y\in\Omega$ is fixed and the coefficient vector $\boldsymbol{a}(y, t)=[a_1(y, t), \cdots, a_m(y, t)]^T$ must be found. Select a suitable weight $w_i(x, t)$, i=0, 1, ⋯, n. By minimizing the weighted discrete square of the local error functional

$$\begin{aligned}\boldsymbol{J}(\boldsymbol{a}(y, t)) &= \left\| u(.\,, t) - \tilde{u}_y(.\,, t) \right\|^2_w \\ &= \sum_{j=0}^{n} w(y - x_j(t))(u_j(t) - \tilde{u}_y(x_j(t), t))^2 \\ &= \sum_{j=0}^{n} w(y - x_j(t))(u_j(t) - \boldsymbol{P}^T(x_j(t))\boldsymbol{a}(y, t))^2\end{aligned} \quad (14)$$

with respect to the coefficient vector $\boldsymbol{a}(y, t)$, we will have the following system:

$$\boldsymbol{A}(y, t)\boldsymbol{a}(y, t)=\boldsymbol{F}(y, t)\boldsymbol{U}(t) \quad (15)$$

where

$$\boldsymbol{A}(y, t)=\boldsymbol{B}(t)\boldsymbol{W}(y, t)\boldsymbol{B}(t)^T$$

$$\boldsymbol{F}(y, t)=\boldsymbol{B}(t)\boldsymbol{W}(y, t)$$

$$\boldsymbol{B}(t)=\{x^{i-1}_j(t)\}, \; i=1, 2, \cdots, m, \; j=0, 1, \cdots, n$$

$$\boldsymbol{W}(y, t)=diag(w(y-x_0(t)), w(y-x_1(t)), \cdots, w(y-x_n(t)))$$

$$\boldsymbol{U}(t)=[u(x_0(t)), u(x_1(t)), \cdots, u(x_n(t))]^T$$

Then the local approximation (13) becomes

$$\tilde{u}_y(x,\ t)=\Phi_y^T(x,\ t)\boldsymbol{U}(t) \tag{16}$$

and the global approximation becomes

$$\tilde{u}(x,\ t) = \Phi^T(x,\ t)\boldsymbol{U}(t) = \sum_{j=0}^{n} \Phi_j(x,\ t)u_j(t) \tag{17}$$

where the vector base function in the global form is

$$\Phi^T(x,\ t)=\boldsymbol{P}^T(x)\boldsymbol{A}^{-1}(x,\ t)\boldsymbol{F}(x,\ t) \tag{18}$$

and $\phi_j(x,\ t)$, $j=0,\ 1,\ \cdots,\ n$ are its elements.

IV. DIFFUSE ELEMENT METHOD

We want to solve Eq. (1) by DEM. To do this, we substitute $u(x,\ t)$ in the PDE by $\tilde{u}(x,\ t)$ of Eq. (17) built by MLS method. The nature of this method is locality or retaining locality view until getting the derivative of the approximation or the base functions, so the derivative of the vector base function in this method is

$$(\Phi^T)'(x,\ t)=(\boldsymbol{P}^T)'(x)\boldsymbol{A}^{-1}(x,\ t)\boldsymbol{F}(x,\ t) \tag{19}$$

For elliptic PDE's, the first derivatives are sufficient. This method is one of the first bridges from meshless interpolation to its application for numerical solution of PDE's. DEM has acceptable accuracy when the support of the weight is small or the number of nodes is large, but its derivatives are not complete (see Section V), *i.e* not integrable, satisfy in the consistency conditions (Krongauz and Belytschko, 1997), the base functions don't have the Kronecker Delta property and have difficulties in exertion of the boundary conditions.

V. ELEMENT FREE GALERKIN AND PETROV-GALERKIN METHOD

In this method, the first derivative of vector base function is found in the following form:

$$\begin{aligned}&\Phi'(x,\ t)\\&=(\boldsymbol{P}^T)'(x)\boldsymbol{A}^{-1}(x,\ t)\boldsymbol{F}(x,\ t)+\boldsymbol{P}^T(x)\boldsymbol{A}^{-1}(x,\ t)(\boldsymbol{F}'(x,\ t)\\&\quad-\boldsymbol{A}'(x,\ t)\boldsymbol{A}^{-1}(x,\ t)\boldsymbol{F}(x,\ t))\end{aligned} \tag{20}$$

(see Belytschko *et al*. (1995), Duarte and Oden (1995)). Kronecker Delta property doesn't hold, so in this method, usually a penalty with Lagrange multipliers and Lagrange interpolation is appended on the boundary as a remedy for this difficulty. In MEFP-GVM, we employ the MLS base functions in the approximation Eq. (18) as trial functions and their complete or EFG derivatives are found in Eq. (20) in the the problem and are piecewise C^1 cubic Hermite (Eq. 9) as test functions.

VI. WEIGHT FUNCTIONS

The weight functions give higher value to each node $x_0(t)$, $x_1(t)$, $\cdots$, $x_n(t)$ and lower value to each neighborhood points. The cardinal number of nodes in support of weight functions must have lower bound and upper bound. Each point in the domain must be in the intersection of at least m weight functions ($m < n$) for non-singularity of matrix $\boldsymbol{A}(x,\ t)$. Support of all of the weight functions must cover the domain. Selected weight functions must have smoothing degree proportional to the problem. The weight functions have variable radius of support and this radius must be selected carefully (usually experimentally, or maybe analytically by using of PDE or multiplication of a parameter named dilation parameter by radius). The weight functions are positive and are applied or approximated equivalent to Dirac Delta Distribution. Some of the singular weight functions in standard or radial form are

$$w(x)=1/|x|^{2k} \tag{21}$$

where k is an integer.

$$w(x) = \begin{cases} (r/x)^2\cos^2(\pi x/2r) & |x| \le r \\ 0 & |x| > r \end{cases} \tag{22}$$

$$w(x) = r/x^2(1-|x|/r)_+^2 \tag{23}$$

Some of the nonsingular weight functions in standard form are

$$w(x) = \begin{cases} (1-|x/r|^{2k_1})^{2k_2} & |x| \le r \\ 0 & |x| > r \end{cases} \tag{24}$$

where k_1 and k_2 are integers and r is the radius of support.

$$w(x)=\exp(-c|x|^{2k}) \tag{25}$$

where $c>0$ is a constant, and k is a positive integer.

$$w(x) = \begin{cases} 2/3-4(|x|/r)^2+4(|x|/r)^3 & |x| \le r/2 \\ 4/3-4|x|/r+4(|x|/r)^2 & r/2 < |x| \le r \\ \quad -4/3(|x|/r)^3 & \\ 0 & |x| > r \end{cases} \tag{26}$$

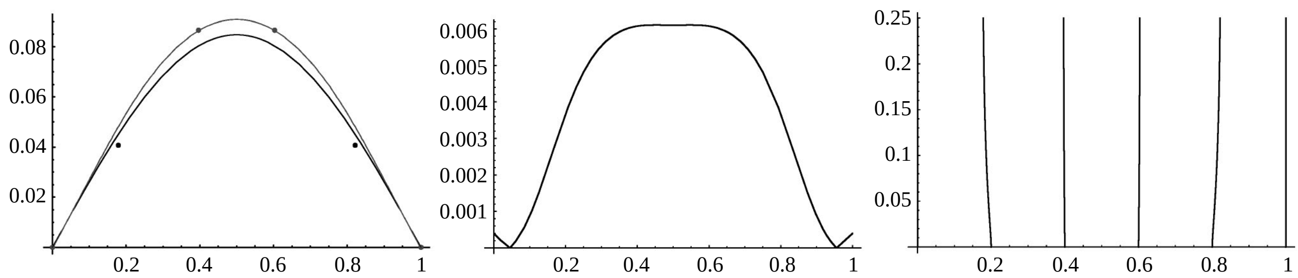

Fig. 1 Approximation, error, and x motion of heat equation

This weight is named cubic spline.

$$w(x) = \begin{cases} c\exp(r^2/(x^2 - r^2)) & |x| \le r \\ 0 & |x| \ge r \end{cases} \tag{27}$$

In MLS, the singular weight function is good for interpolation, but there are difficulties in computational work with it. Nonsingular weight functions haven't interpolation property, but can build good approximation. However, lack of interpolation property by using nonsingular weights shows itself on exertion of boundary condition and needs some methods for handling and correcting it such as Lagrange multiplier (Belytschko *et al.*, 1995) or penalty methods (Belytschko *et al.*, 1995). One of the main sources of meshless methods' accuracy is related to selected weight function.

VII. MOVING ELEMENT FREE PETROV-GALERKIN VISCOUS METHOD

Combination of the previous two methods, MFE and EFG, with piecewise C^1 cubic Hermite as the test functions make new powerful and flexible generation of numerical solutions of time dependent PDE's. Let (17) be the approximation of u in PDE (1). Then the $L^2(0, 1)$ norm residual functional by EFG base function with penalty and regularizing term for node movement will be built such as Eq. (6) in which $\{\phi_j(x, t)\}_{j=0}^{n}$ were built by EFG method and are elements of (18) and $\psi_j(x, t) \equiv \partial\tilde{u}(x, t)/\partial x_j$ without local support, $\varepsilon_j(t)$ and $\eta_j(t)$ are (open) experimental parameters that cover some slits that exists in mass matrix $\underline{\boldsymbol{A}}(t)$ to make it well conditioned and positive definite. $\varepsilon_j(t)$ are viscosity parameters and $\eta_j(t)$ take care of the first term of the penalty near it (see Miller and Miller (1981), Miller (1981), Herbst *et al.* (1983)). Minimizing this functional with respect to $\dot{x}_j(t)$, $\dot{u}_j(t)$, $j=0, 1, \cdots, n$, gives us a system of ODEs with time derivative similar to (12). This method is MEFGVM and the weighted residual form of this functional by cubic Hermite weights with regularizing terms lead us to MEFP-GVM.

VIII. NUMERICAL EXAMPLES

We want to show the power of the method by approximating the following two 1-D examples:

Example 1. Approximation of 1-D Heat Equation

$u_t = u_{xx}$, $0<x<1$,

$u(0, t)=u(1, t)=0$, $u(x, 0)=\sin(\pi x)$

with its exact solution $u(x, t)=\sin(\pi x)\exp(-\pi^2 t)$ by MEFP-GVM shown in Fig. 1. Here, polynomial base for MLS interpolation is $\boldsymbol{P}^T(x)=\{1, x, x^2\}$, selected weight function is Spline (26), Lagrange polynomial functions are quadratic, the penalty parameters are constants $\varepsilon_i(t)=0.2$, $\eta_i(t)=0$, for $i=0, 1, \cdots, n$, $n=5$, the numerical quadrature method is Simpson rule, ODE method is implicit, the time step is $\Delta t=0.0025$ and the final time is $T=0.25$ seconds. The initial distribution of nodes is uniform {0, 1/5, 2/5, 3/5, 4/5, 1} and the final coordinates of nodes are {0, 0.178553, 0.396684, 0.603316, 0.8214465, 1}. One can see a smooth motion of nodes and a smooth approximation by a small number of nodes. The approximation curve doesn't pass throgh the nodes, because the base functions do not have Kronecker Delta property.

Example 2. Approximation of 1-D Burger Equation

$u_t+(u^2/2)_x=(1/R)u_{xx}$, $0<x<1$,

with its exact solution $u(x, t)=(\mu+\lambda+(\mu-\lambda)\exp(\lambda\xi/\varepsilon)/(1+\exp(\lambda\xi/\varepsilon))$, where $\xi=x-\mu t-\beta$, and the related parameters are $\lambda= 0.4$, $\beta=0.16$, $\mu=0.5$, $R=15$ shown in Fig. 2. In this figure, there are Burger approximation together with its exact solution, the error ($|\tilde{u}(x, T)-u(x, T)|$) and the motion of nodes respectively. The polynomial base, weight function, Lagrange polynomial functions, numerical integration, ODE method and time step are similar to example (1), $T=0.51$, $n=7$

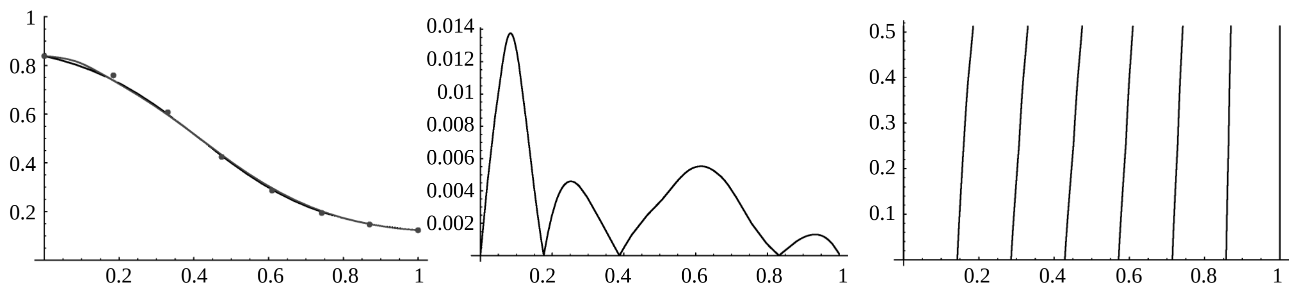

Fig. 2 Approximation, error, and x motion of Burger equation

seconds. In this example, the approximation is very smooth and one can not make such an approximation by fixed piecewise finite elements. The initial distribution is uniform {0, 1/7, 2/7, 3/7, 4/7, 5/7, 6/7, 1} and the approximate final distribution is {0, 0.185026, 0.330253, 0.474687, 0.609325, 0.742019, 0.869880, 1}. The third figure shows smooth node motions with respect to time. The nodes try to move into the shock region and move together with it by proceeding time step.

IX. CONCLUSIONS

1. The choice of a finite subset polynomial base in MLS interpolation, computation of matrix inverse in Eq. (18) is time consuming with errors, numerical quadrature, approximate solution of system in each time level iteration, and use of a finite difference method for ODE system are some sources of errors. One of the best method for estimating the error at a point for MEFP-GVM can be seen in (Gavete *et al.*, 2002). In this paper, by minimizing a functional that is weighted error between two methods GFD and EFG, error for a point and a neighborhood of it is estimated.
2. This method can be extended to another meshless method.
3. There isn't mathematical theory such as existence, uniqueness and convergence theory for this method.
4. In this method, equi-distribution principle maybe holds.
5. The computational volume is high, but can be optimized.
6. Use of C^1 piecewise cubic Hermite needs a typical mesh of elements in the domain. This is a disadvantage.
7. The exertion of essential boundary conditions needs another technique for handling.
8. This method has satisfactory accuracy with small number of nodes.
9. MFE method mass matrix is tridiagonal but full in MEFPGM.
10. The initial distribution can be uniform, Chebyshev, roots of orthogonal polynomials and can be found by equi-distribution principles (for example see Coyle *et al.* (1986)).

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