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A BOUNDARY METHOD OF THE TREFFTZ TYPE FOR HYDRODYNAMIC APPLICATION

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ABSTRACT

The aim of the work is to present a new boundary technique for solving hydrodynamic problems in arbitrary domains. It is based on the use of the method of fundamental solutions with approximate trial functions. In particular, two dimensional Stokes and Navier-Stokes systems are considered.

Key Words: Trefftz method, trial functions, stokes problem.

I. INTRODUCTION

The method presented belongs to the group of the Trefftz type methods. Recall, that according to this approach, an approximate solution of a partial differential equation (PDE) $L[u]=f$ is looked for in the form of linear combination:

$$u(\boldsymbol{x}\,|q_1, ..., q_k) = v(\boldsymbol{x}) + \sum_{k=1}^{K} q_k \Phi_k(\boldsymbol{x}) \tag{1}$$

Here, $v(\boldsymbol{x})$ is the particular integral and the trial functions $\Phi_k(\boldsymbol{x})$ satisfy the homogeneous PDE $(L(\boldsymbol{x})-p)[\Phi_k]=0$ exactly, but do not necessarily satisfy boundary conditions. They are used to determine the unknown q_k.

The Trefftz type methods can be divided into two groups depending on the trial functions which are used. The first method applies analytically derived nonsingular trial functions, sometimes called T-functions, identically fulfilling certain governing PDEs inside a solution domain Ω (Herrera, 2000). Methods of the second group employ the fundamental solutions of the PDE with singular points situated outside the investigated region. This kind of trial functions is suggested and investigated by V. Kupradze (Kupradze and Aleksidze, 1967). An example of this technique is the method of fundamental solutions (MFS) where the singular solutions are used (see review (Golberg *et al*., 1999) and bibliography presented there). Recently this method has been extended to time-dependent problems (Golberg and Chen, 1998).

Following the general scheme of the MFS, we suggested to use the trial functions $\Phi(\boldsymbol{x}|\xi)$ which satisfy the governing PDE only approximately (Reutskiy, 2002; 2004). More precisely, $\Phi(\boldsymbol{x}|\xi)$ is a solution of the equation $L[\Phi]=I(\boldsymbol{x}|\xi)$, where the right hand side is the truncated series

$$I(\boldsymbol{x}\,|\xi) = \sum_{\boldsymbol{n}=1}^{M} r_{\boldsymbol{n}} \psi_{\boldsymbol{n}}(\xi)\psi_{\boldsymbol{n}}(\boldsymbol{x}) \equiv \sum_{\boldsymbol{n}=1}^{M} c_{\boldsymbol{n}}(\xi)\psi_{\boldsymbol{n}}(\boldsymbol{x}) \tag{2}$$

over an orthogonal basis system $\psi_{\boldsymbol{n}}(\boldsymbol{x})$. Here $\boldsymbol{n}=(n_1,n_2)$ or (n_1, n_2, n_3) denotes a multi-index. In the general case $\psi_{\boldsymbol{n}}(\boldsymbol{x})$ is a solution of some *Sturm-Liouville* problem. The two basis systems

$$\psi_n(x)=\sin[\lambda_n(x+1)],\ \varphi_n(x)=\cos[\lambda_n(x+1)],$$
$$\lambda_n=0.5\pi(n-0.5) \tag{3}$$

and their products in many dimensional cases are used throughout the work. The regularizing coefficients $r_{\boldsymbol{n}}$ depend on the particular choice of the system $\psi_{\boldsymbol{n}}(\boldsymbol{x})$. For the trigonometric functions (3) in the one dimensional case, they are

$$r_n(M, \chi)=[\sigma_n(M)]^\chi,\ \sigma_n(M) = \frac{\sin[\nu(n, M)]}{\nu(n, M)},$$
$$\nu(n, M) = \frac{n\pi}{M+1} \tag{4}$$

where $\sigma_n(M)$ are the Lanczos sigma-factors which are used to overcome the so-called Gibb's phenomenon in the Fourier expansion of non smooth functions, χ is a parameter of regularization. The functions $I(\boldsymbol{x}|\xi)$ which essentially differ from zero only inside some neighborhood of the source point ξ are analogous in

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some sense to the Dirac delta functions $\delta(\boldsymbol{x}-\boldsymbol{\xi})$ in the MFS procedure.

In this work we consider the application of this technique to the time dependent incompressible Stokes problem. The paper is organized as follows. In Section II, we describe the main algorithm for the Stokes problem in the velocity-pressure formulation

$$\partial_t \boldsymbol{v}=-\nabla p+\Delta \boldsymbol{v},\ div\boldsymbol{v}=0,\ \boldsymbol{v}(\boldsymbol{x},0)=\boldsymbol{v}_0(\boldsymbol{x}),$$

$$\boldsymbol{v}|_{\partial\Omega}=\boldsymbol{v}_1(\boldsymbol{x},t) \tag{5}$$

inside a 2D arbitrarily bounded domain, Ω, with the boundary $\partial\Omega$. Here $\boldsymbol{v}=(u, v)$ is the velocity vector, p is the pressure, $\boldsymbol{v}_0(\boldsymbol{x})$, $\boldsymbol{v}_1(\boldsymbol{x}, t)$ are prescribed functions. The stationary case was considered in Pontrelli *et al.*, 1997. In Section III, we consider ω–ψ formulation of the problem. In Section IV, we extend this technique to 2D Navier-Stokes system at a low Reynolds number.

$$\partial_t \boldsymbol{v}+(\boldsymbol{v},\nabla)\boldsymbol{v}=-\nabla p+\frac{1}{\mathrm{Re}}\Delta\boldsymbol{v},\ div\boldsymbol{v}=0 \tag{6}$$

Numerical results are presented in Section V.

II. STOKES PROBLEM IN *v-p* REPRESENTATION

Finite differencing in time transforms (5) to a sequence of generalized Stokes problems. For example, when the Crank-Nicholson scheme is applied, then one gets:

$$\Delta\boldsymbol{v}^{j+1}-s\boldsymbol{v}^{j+1}=-\Delta\boldsymbol{v}^{j}-s\boldsymbol{v}^{j}+2\nabla p^{j+1/2},\ div\boldsymbol{v}^{j+1}=0,\ s=2/\Delta t \tag{7}$$

The three sets of the basis functions

$$\varphi^{(u)}_{\boldsymbol{n}}(\boldsymbol{x})=\psi_{n_1}(x_1)\varphi_{n_2}(x_2),\ \varphi^{(v)}_{\boldsymbol{n}}(\boldsymbol{x})=\varphi_{n_1}(x_1)\psi_{n_2}(x_2),$$

$$\varphi^{(p)}_{\boldsymbol{n}}(\boldsymbol{x})=\psi_{n_1}(x_1)\psi_{n_2}(x_2) \tag{8}$$

are used to approximate the u, v and p fields correspondingly:

$$u(\boldsymbol{x})=\sum_{\boldsymbol{n}=(1,1)}^{M}U_{\boldsymbol{n}}\varphi^{(u)}_{\boldsymbol{n}}(\boldsymbol{x}),\ v(\boldsymbol{x})=\sum_{\boldsymbol{n}=(1,1)}^{M}V_{\boldsymbol{n}}\varphi^{(v)}_{\boldsymbol{n}}(\boldsymbol{x}),$$

$$p(\boldsymbol{x})=\sum_{\boldsymbol{n}=(1,1)}^{M}P_{\boldsymbol{n}}\varphi^{(p)}_{\boldsymbol{n}}(\boldsymbol{x}) \tag{9}$$

The two kinds of the source functions are used:

$$I_u(\boldsymbol{x}\,|\,\boldsymbol{\xi})=\sum_{\boldsymbol{n}=(1,1)}^{M}c^{(u)}_{\boldsymbol{n}}(\boldsymbol{\xi})\varphi^{(u)}_{\boldsymbol{n}}(\boldsymbol{x}),$$

$$I_v(\boldsymbol{x}\,|\,\boldsymbol{\xi})=\sum_{\boldsymbol{n}=(1,1)}^{M}c^{(v)}_{\boldsymbol{n}}(\boldsymbol{\xi})\varphi^{(v)}_{\boldsymbol{n}}(\boldsymbol{x}) \tag{10}$$

According to the method presented we write (7) in the form:

$$(\Delta-s)u^{j+1}=-(\Delta+s)u^{j}+2\partial_{x_1}p^{j+1/2}+\sum_{k=1}^{K}q^{j+1}_{u,k}I_u(\boldsymbol{x}\,|\,\boldsymbol{\xi}_k) \tag{11}$$

$$(\Delta-s)v^{j+1}=-(\Delta+s)v^{j}+2\partial_{x_2}p^{j+1/2}+\sum_{k=1}^{K}q^{j+1}_{v,k}I_v(\boldsymbol{x}\,|\,\boldsymbol{\xi}_k) \tag{12}$$

These equations are considered inside $\Omega_0=[-1,+1]\times[-1,+1]\supset\Omega$. The source points $\boldsymbol{\xi}_k$ are placed outside the solution domain Ω. Substituting (9), (10) in (11), (12) one gets:

$$-(\lambda^2_{\boldsymbol{n}}+s)U^{j+1}_{\boldsymbol{n}}=(\lambda^2_{\boldsymbol{n}}-s)U^{j}_{\boldsymbol{n}}+2\lambda_{n_1}P^{j+1/2}_{\boldsymbol{n}}+\sum_{k=1}^{K}q^{j+1}_{u,k}c^{(u)}_{\boldsymbol{n},k} \tag{13}$$

$$-(\lambda^2_{\boldsymbol{n}}+s)V^{j+1}_{\boldsymbol{n}}=(\lambda^2_{\boldsymbol{n}}-s)V^{j}_{\boldsymbol{n}}+2\lambda_{n_2}P^{j+1/2}_{\boldsymbol{n}}+\sum_{k=1}^{K}q^{j+1}_{v,k}c^{(v)}_{\boldsymbol{n},k} \tag{14}$$

Here $\lambda^2_{\boldsymbol{n}}=\lambda^2_{n_1}+\lambda^2_{n_2}$, $c^{(u)}_{\boldsymbol{n},k}=c^{(u)}_{\boldsymbol{n}}(\boldsymbol{\xi}_k)$, $c^{(v)}_{\boldsymbol{n},k}=c^{(v)}_{\boldsymbol{n}}(\boldsymbol{\xi}_k)$. The condition $div\boldsymbol{v}^{n+1}=0$ gives:

$$\lambda_{n_1}U^{j+1}_{\boldsymbol{n}}+\lambda_{n_2}V^{j+1}_{\boldsymbol{n}}=0 \tag{15}$$

System (12), (13), (14) can be resolved for each harmonic $\boldsymbol{n}=(n_1, n_2)$ separately. Coming back to the physical values we get:

$$\boldsymbol{v}^{j+1}=\boldsymbol{v}^{j+1}_p+\sum_{k=1}^{K}\hat{\Phi}_k(x)\boldsymbol{q}^{j+1}_k \tag{16}$$

Here $\boldsymbol{v}^{j+1}_p$ is a particular integral and $\hat{\Phi}_k(\boldsymbol{x})$ is the 2×2 matrix which does not depend on time. The free parameters $\boldsymbol{q}^{n+1}=(q^{j+1}_{u,k}, q^{j+1}_{v,k})$ should be determined from the boundary conditions $\boldsymbol{v}^{n+1}|_{\partial\Omega}=\boldsymbol{v}^{n+1}_1$. We would like to pay attention to the following two points of the algorithm: 1) particular solution $\boldsymbol{v}^{n+1}_p$ at each time layer can be found in the same form of the truncated series in an analytic way without any numerical integration; 2) the solution satisfies the condition $div\boldsymbol{v}^{n+1}=0$ exactly at each time layer.

III. *ω-ψ* REPRESENTATION

Another approach is based on the ω-ψ representation:

$$\partial_t\omega=\Delta\omega,\ \Delta\psi=\omega,\ u=\partial_y\psi,\ v=-\partial_x\psi \tag{17}$$

Differencing (17) in time one gets:

$$\Delta\omega^{j+1} - s\omega^{j+1} = -\Delta\omega^{j} - s\omega^{j},\ \Delta\psi^{j+1} = \omega^{j+1} \quad (18)$$

Here we use the same 2D basis system for ω and ψ approximation:

$$\omega(\boldsymbol{x}) = \sum_{\boldsymbol{n}=(1,1)}^{M} W_{\boldsymbol{n}}\phi_{\boldsymbol{n}}(\boldsymbol{x}),\ \ \psi(\boldsymbol{x}) = \sum_{\boldsymbol{n}=(1,1)}^{M} \Psi_{\boldsymbol{n}}\phi_{\boldsymbol{n}}(\boldsymbol{x}) \quad (19)$$

where $\phi_n(\boldsymbol{x}) = \varphi_{n_1}(x_1)\varphi_{n_2}(x_2)$ and φ_n is given in (3).

Similar to the previous section we replace (18) with the following system (cf. (11), (12)):

$$\Delta\omega^{j+1} - s\omega^{j+1} = -\Delta\omega^{j} - s\omega^{j} + \sum_{k=1}^{K} q_{1,k}^{j+1} I(\boldsymbol{x}|\boldsymbol{\xi}_k) \quad (20)$$

$$\Delta\psi^{j+1} - \omega^{j+1} = \sum_{k=1}^{K} q_{2,k}^{j+1} I(\boldsymbol{x}|\boldsymbol{\xi}_k) \quad (21)$$

Here $I(\boldsymbol{x}|\boldsymbol{\xi}_k)$ is the source expanded over the same basis system $\phi_{\boldsymbol{n}}(\boldsymbol{x})$. Substituting (19) one gets:

$$-(\lambda_{n_1}^2 + \lambda_{n_2}^2 + s)W_{\boldsymbol{n}}^{j+1} = (\lambda_{n_1}^2 + \lambda_{n_2}^2 - s)W_{\boldsymbol{n}}^{j} + \sum_{k=1}^{K} q_{1,k}^{j+1} c_{\boldsymbol{n}}(\boldsymbol{\xi}_k) \quad (22)$$

$$-(\lambda_{n_1}^2 + \lambda_{n_2}^2)\Psi_{\boldsymbol{n}}^{j+1} - W_{\boldsymbol{n}}^{j+1} = \sum_{k=1}^{K} q_{2,k}^{j+1} c_{\boldsymbol{n}}(\boldsymbol{\xi}_k) \quad (23)$$

The last part of the algorithm is the same as the one in the previous section. We solve Eqs. (22), (23) for each harmonic $\boldsymbol{n}$. Then, passing onto the physical values one gets an expression like (16) with the free parameters $q_{1,k}^{j+1}$, $q_{2,k}^{j+1}$. They are determined from the boundary conditions.

IV. NAVIER-STOKES PROBLEM AT LOW RE

Differencing (6) in time one gets:

$$\Delta\boldsymbol{v}^{j+1} - s\boldsymbol{v}^{j+1} = -\Delta\boldsymbol{v}^{j} - s\boldsymbol{v}^{j} + 2\mathrm{Re}\,\nabla p^{j+1/2} + 2\mathrm{Re}(\boldsymbol{v}^{j+1/2}, \nabla)\,\boldsymbol{v}^{j+1/2} \quad (24)$$

Here $s = 2\mathrm{Re}/\Delta t$ and the vector field $\boldsymbol{v}^{j+1}$ satisfies $div\boldsymbol{v}^{j+1} = 0$.

Let us assume that the velocity $\boldsymbol{v}^{j}$ is known. The nonlinear term is approximated using this field:

$$(\boldsymbol{v}^{j+1/2}, \nabla)\boldsymbol{v}^{j+1/2} \approx (\boldsymbol{v}^{j}, \nabla)\boldsymbol{v}^{j} = [s_u(\boldsymbol{x}), s_v(\boldsymbol{x})]^T \quad (25)$$

Similar to the method described in the previous section (24) is replaced by the following system (cf. (11), (12)):

$$(\Delta - s)u^{j+1} = -(\Delta + s)u^{j} + 2\mathrm{Re}(\partial_{x_1} p^{j+1/2} + s_u(\boldsymbol{x})) + \sum_{k=1}^{K} q_{u,k}^{j+1} I_u(\boldsymbol{x}|\boldsymbol{\xi}_k) \quad (26)$$

$$(\Delta - s)v^{j+1} = -(\Delta + s)v^{j} + 2\mathrm{Re}(\partial_{x_2} p^{j+1/2} + s_v(\boldsymbol{x})) + \sum_{k=1}^{K} q_{v,k}^{j+1} I_v(\boldsymbol{x}|\boldsymbol{\xi}_k) \quad (27)$$

The orthonormal basis systems $\varphi_{\boldsymbol{n}}^{(u)}$ and $\varphi_{\boldsymbol{n}}^{(v)}$ (see (8)) are used to approximate the functions $s_u(\boldsymbol{x})$ and $s_v(\boldsymbol{x})$. For example,

$$s_u(\boldsymbol{x}) \approx \sum_{\boldsymbol{n}=(1,1)}^{M} S_{u,\boldsymbol{n}}\varphi_{\boldsymbol{n}}^{(u)}(\boldsymbol{x}),$$

$$S_{u,\boldsymbol{n}} = \int_{\Omega_0} s_u(\boldsymbol{x})\varphi_{\boldsymbol{n}}^{(u)}(\boldsymbol{x})d\boldsymbol{x} = \sum_{k_1,k_2=1}^{n_q} A_{k_1}A_{k_2} s_u(x_{k_1}, x_{k_2})\varphi_{\boldsymbol{n}}^{(u)}(x_{k_1}, x_{k_2})$$

where A_k and x_k are weights and nodes of a one-dimensional quadrature. In particular, 32 points Gauss quadrature is used. The velocity components and the pressure are sought in the form of expansion (9). Similar to (13), (14) one gets:

$$-(\lambda_{\boldsymbol{n}}^2 + s)U_{\boldsymbol{n}}^{j+1} = (\lambda_{\boldsymbol{n}}^2 - s)U_{\boldsymbol{n}}^{j} + 2\mathrm{Re}(-\lambda_{n_1}P_{\boldsymbol{n}}^{j+1/2} + S_{u,\boldsymbol{n}}) + \sum_{k=1}^{K} q_{u,k}^{j+1} c_{\boldsymbol{n},k}^{(u)} \quad (28)$$

$$-(\lambda_{\boldsymbol{n}}^2 + s)V_{\boldsymbol{n}}^{j+1} = (\lambda_{\boldsymbol{n}}^2 - s)V_{\boldsymbol{n}}^{j} + 2\mathrm{Re}(-\lambda_{n_2}P_{\boldsymbol{n}}^{j+1/2} + S_{v,\boldsymbol{n}}) + \sum_{k=1}^{K} q_{v,k}^{j+1} c_{\boldsymbol{n},k}^{(v)} \quad (29)$$

Multiplying (28) and (29) on λ_{n_1} and λ_{n_2} and adding the results one gets the coefficients of the pressure field expansion:

$$P_{\boldsymbol{n}}^{j+1/2} = \frac{\lambda_{n_1}S_{u,\boldsymbol{n}} + \lambda_{n_2}S_{v,\boldsymbol{n}}}{\lambda_{\boldsymbol{n}}^2} + \frac{1}{2\mathrm{Re}}\sum_{k=1}^{K}\left[q_{u,k}^{j+1}\frac{\lambda_{n_1}c_{u,\boldsymbol{n}}}{\lambda_{\boldsymbol{n}}^2} + q_{v,k}^{j+1}\frac{\lambda_{n_2}c_{v,\boldsymbol{n}}}{\lambda_{\boldsymbol{n}}^2}\right] \quad (30)$$

Here we use:

$$\lambda_{n_1}U_{\boldsymbol{n}}^{j+1} + \lambda_{n_2}V_{\boldsymbol{n}}^{j+1} = \lambda_{n_1}U_{\boldsymbol{n}}^{j} + \lambda_{n_2}V_{\boldsymbol{n}}^{j} = 0\,.$$

Substituting (30) in (28) and (29) in a similar way one gets:

$$U_{\boldsymbol{n}}^{j+1} = \frac{s - \lambda_{\boldsymbol{n}}^2}{s + \lambda_{\boldsymbol{n}}^2}U_{\boldsymbol{n}}^{j} + \frac{2\mathrm{Re}[\lambda_{n_1}(\lambda_{n_1}S_{u,\boldsymbol{n}} + \lambda_{n_2}S_{v,\boldsymbol{n}}) - \lambda_{\boldsymbol{n}}^2 S_{u,\boldsymbol{n}}]}{\lambda_{\boldsymbol{n}}^2(s + \lambda_{\boldsymbol{n}}^2)} + \sum_{k=1}^{K}\left[q_{u,k}^{j+1}c_{u,\boldsymbol{n}}\left(\frac{\lambda_{n_1}^2}{\lambda_{\boldsymbol{n}}^2(s + \lambda_{\boldsymbol{n}}^2)} - \frac{1}{s + \lambda_{\boldsymbol{n}}^2}\right) + q_{v,k}^{j+1}c_{v,\boldsymbol{n}}\frac{\lambda_{n_1}\lambda_{n_2}}{\lambda_{\boldsymbol{n}}^2(s + \lambda_{\boldsymbol{n}}^2)}\right] \quad (31)$$

$$V_{\boldsymbol{n}}^{j+1} = \frac{s-\lambda_{\boldsymbol{n}}^2}{s+\lambda_{\boldsymbol{n}}^2}V_{\boldsymbol{n}}^j + \frac{2\mathrm{Re}[\lambda_{n_2}(\lambda_{n_1}S_{u,\boldsymbol{n}}+\lambda_{n_2}S_{v,\boldsymbol{n}})-\lambda_{\boldsymbol{n}}^2S_{v,\boldsymbol{n}}]}{\lambda_{\boldsymbol{n}}^2(s+\lambda_{\boldsymbol{n}}^2)} + \sum_{k=1}^{K}[q_{u,k}^{j+1}c_{u,\boldsymbol{n}}\frac{\lambda_{n_1}\lambda_{n_2}}{\lambda_{\boldsymbol{n}}^2(s+\lambda_{\boldsymbol{n}}^2)} + q_{v,k}^{j+1}c_{v,\boldsymbol{n}}(\frac{\lambda_{n_2}^2}{\lambda_{\boldsymbol{n}}^2(s+\lambda_{\boldsymbol{n}}^2)}-\frac{1}{s+\lambda_{\boldsymbol{n}}^2})] \quad (32)$$

Coming back to the physical values (31), (32) can be written in the form (cf. (16)):

$$\begin{pmatrix} u^{j+1} \\ v^{j+1}\end{pmatrix} = \begin{pmatrix} u_p^{j+1} \\ v_p^{j+1}\end{pmatrix} + \sum_{k=1}^{K}\begin{bmatrix}\Phi_{1,1} & \Phi_{1,2} \\ \Phi_{2,1} & \Phi_{2,2}\end{bmatrix}\begin{pmatrix} q_{u,k}^{j+1} \\ q_{v,k}^{j+1}\end{pmatrix} = \boldsymbol{v}_p^{j+1} + \sum_{k=1}^{K}\hat{\Phi}_k(\boldsymbol{x})q_k^{j+1} \quad (33)$$

Here the particular integral $\boldsymbol{v}_p^{j+1}=(u_p^{j+1}, v_p^{j+1})$ depends on the time layer. The coefficients of the expansion over $\varphi_{\boldsymbol{n}}^{(u)}(\boldsymbol{x})$ and $\varphi_{\boldsymbol{n}}^{(v)}(\boldsymbol{x})$ are

$$U_{p,\boldsymbol{n}}^{j+1} = \frac{s-\lambda_{\boldsymbol{n}}^2}{s+\lambda_{\boldsymbol{n}}^2}U_{\boldsymbol{n}}^j + \frac{2\mathrm{Re}[\lambda_{n_1}(\lambda_{n_1}S_{u,\boldsymbol{n}}+\lambda_{n_2}S_{v,\boldsymbol{n}})-\lambda_{\boldsymbol{n}}^2S_{u,\boldsymbol{n}}]}{\lambda_{\boldsymbol{n}}^2(s+\lambda_{\boldsymbol{n}}^2)} \quad (34)$$

and

$$V_{p,\boldsymbol{n}}^{j+1} = \frac{s-\lambda_{\boldsymbol{n}}^2}{s+\lambda_{\boldsymbol{n}}^2}V_{\boldsymbol{n}}^j + \frac{2\mathrm{Re}[\lambda_{n_2}(\lambda_{n_1}S_{u,\boldsymbol{n}}+\lambda_{n_2}S_{v,\boldsymbol{n}})-\lambda_{\boldsymbol{n}}^2S_{v,\boldsymbol{n}}]}{\lambda_{\boldsymbol{n}}^2(s+\lambda_{\boldsymbol{n}}^2)} \quad (35)$$

for u_p^{j+1} and v_p^{j+1} correspondingly.

The terms of $\hat{\Phi}_k(\boldsymbol{x})$ do not change in time. $\Phi_{1,1}$ and $\Phi_{1,2}$ are expanded over $\varphi_{\boldsymbol{n}}^{(u)}(\boldsymbol{x})$ system. The coefficients are:

$$\Phi_{1,1} \sim c_{u,\boldsymbol{n}}(\frac{\lambda_{n_1}^2}{\lambda_{\boldsymbol{n}}^2(s+\lambda_{\boldsymbol{n}}^2)}-\frac{1}{s+\lambda_{\boldsymbol{n}}^2}),$$

$$\Phi_{1,2} \sim c_{v,\boldsymbol{n}}\frac{\lambda_{n_1}\lambda_{n_2}}{\lambda_{\boldsymbol{n}}^2(s+\lambda_{\boldsymbol{n}}^2)}.$$

For the terms $\Phi_{2,1}$ and $\Phi_{2,2}$ the coefficients of expansion over $\varphi_{\boldsymbol{n}}^{(u)}(\boldsymbol{x})$ are:

$$\Phi_{2,1} \sim c_{u,\boldsymbol{n}}\frac{\lambda_{n_1}\lambda_{n_2}}{\lambda_{\boldsymbol{n}}^2(s+\lambda_{\boldsymbol{n}}^2)},$$

$$\Phi_{2,2} \sim c_{v,\boldsymbol{n}}(\frac{\lambda_{n_2}^2}{\lambda_{\boldsymbol{n}}^2(s+\lambda_{\boldsymbol{n}}^2)}-\frac{1}{s+\lambda_{\boldsymbol{n}}^2})$$

The free parameters $\boldsymbol{q}_k^{j+1}$ are determined from the boundary conditions.

There are two following possibilities here: 1) $u^{j+1}(\boldsymbol{x})$ and $v^{j+1}(\boldsymbol{x})$ can be considered as the final values of the velocity components at the j+1th time layer; 2) these functions can be considered as intermediate values $\tilde{u}^{j+1}(\boldsymbol{x})$, $\tilde{v}^{j+1}(\boldsymbol{x})$ and can be used for correction of the nonlinear term $(v^{j+1/2}, \nabla)v^{j+1/2}$:

$$u^{j+1/2}(\boldsymbol{x}) = \frac{1}{2}(\tilde{u}^{j+1}(\boldsymbol{x}) + u^j(\boldsymbol{x})),$$

$$v^{j+1/2}(\boldsymbol{x}) = \frac{1}{2}(\tilde{v}^{j+1}(\boldsymbol{x}) + v^j(\boldsymbol{x})) \quad (36)$$

Then, the final values, $u^{j+1}(\boldsymbol{x})$ and $v^{j+1}(\boldsymbol{x})$ are obtained by repeating the algorithm (26)-(33) with this corrected nonlinear term.

V. NUMERICAL EXAMPLE

1. Stokes Problem

As an example, consider the problem when an infinitely long cylinder with the radius a containing a viscous liquid begins to rotate with the angular velocity ε at the time moment t=0. The solution domain is a disk with the radius a. The initial condition is: $u(\boldsymbol{x}, 0)=v(\boldsymbol{x}, 0)=0$ i.e., the liquid is at rest for $t\le 0$. The boundary conditions are:

$$v(\boldsymbol{x}, t)\cos\theta - u(\boldsymbol{x}, t)\sin\theta = \varepsilon,$$

$$u(\boldsymbol{x}, t)\cos\theta + v(\boldsymbol{x}, t)\sin\theta = 0, \ |\boldsymbol{x}| = a$$

Here r, θ are the polar coordinates of the point $\boldsymbol{x}$ with the origin at the centre of the disk. This problem has an analytic solution (Sneddon, 1951): $u(\boldsymbol{x}, t)= -V(r, t)\sin\theta$, $v(\boldsymbol{x}, t)=V(r, t)\cos\theta$, where

$$V(r,t) = \varepsilon a[\frac{r}{a} + 2\sum_{k=1}^{\infty}\exp(-\frac{\mu_k^2}{a^2}t)\frac{J_1(r\mu_k/a)}{\lambda_k J_1'(\mu_k)}] \quad (37)$$

and μ_k is the k^{th} root of the quation $J_1(\mu)=0$.

The source points are placed on the circle with the radius 0.95. The number of the source points, i.e. the number of free parameters, is K=25.

The computations show that stability of the solution process is managed by the parameter of regularization χ in (4). Namely, for each M, there exists a minimal χ_{div} such that the solution process diverges for $x<\chi_{div}$ and it converges for $\chi\ge\chi_{div}$. These values are χ_{div}=3, 4, 6, 7 for M=10,15,20, 25 correspondingly. In Table 1, we present the mean square root error

$$e_{sq}(t) = \sqrt{\frac{1}{2N}\sum_{i=1}^{N}[v_\theta(\boldsymbol{x}_i, t) - V(r_i, t)]^2 + [v_r(\boldsymbol{x}_i, t))]^2} \quad (38)$$

Table 1 The mean square root error in solution of the 2D Stokes problem

t	$\boldsymbol{v}-p$				$\omega-\psi$	
	M=10, χ=3		M=20, χ=6		M=10, χ=3	
	Δt_1	Δt_2	Δt_1	Δt_2	Δt_1	Δt_2
0.05	$6.1\cdot10^{-2}$	$1.2\cdot10^{-3}$	$6.2\cdot10^{-2}$	$1.5\cdot10^{-3}$	$6.2\cdot10^{-2}$	$1.3\cdot10^{-3}$
0.1	$1.5\cdot10^{-3}$	$3.3\cdot10^{-4}$	$1.6\cdot10^{-2}$	$6.1\cdot10^{-5}$	$1.6\cdot10^{-2}$	$1.0\cdot10^{-4}$
1	$3.7\cdot10^{-4}$	$3.8\cdot10^{-4}$	$4.4\cdot10^{-4}$	$5.1\cdot10^{-6}$	$2.2\cdot10^{-4}$	$1.3\cdot10^{-4}$

corresponding to a=0.5 and ε=1. The time steps are Δt_1=0.05 and Δt_2=0.005.

One can see that when t is small, a main error occurs due to the discontinuity at the first moment of motion. Its value decreases together with Δt and does not depend on the number of harmonics M. For a large t, the error $e_{sq}(t)$ depends on M. When we decrease the time step Δt in 10 steps from 0.05 to 0.005 this makes little change in the error for M=10. So, in this case the main error is the one due to a low number of harmonics in the approximate solution. At the same time, this decreasing reduces the error in almost 100 times for M=20. So, the error introduced by the Crank-Nicholson scheme is dominant for these values of M. Besides, one can see that the error in the $\boldsymbol{v}-p$ model is less than the one in $\omega-\psi$ approach with the same number of harmonics and degrees of freedom.

2. Navier-Stokes Equation

As an example of the method described in Section III consider the stationary Couette flow between two cylinders with the radiuses a_1 and a_2 ($>a_1$) . The inner cylinder is at rest and the outer one is rotated with the angular velocity ε. The stationary solution for the fluid flow between the cylinders is:

$$v_\theta(r,\theta)=\frac{\varepsilon a_2^2(r-a_1^2/r)}{a_2^2-a_1^2}\,,\quad v_r(r,\theta)=0 \tag{39}$$

The calculations were performed with the following parameters: ε=1, a_1=0.4, a_2=0.6. In Table 2, the mean square root error is placed. The source points ξ_k are placed on the two circles. On the circle with the radius 0.1 16 sources are placed and 24 sources are placed on the circle with the radius 0.95.

Table 2 The mean square root error in solution of the 2D Navier-Stokes problem for stationary Couette flow

Re	M=10	M=20	M=30
1	$5.3\cdot10^{-3}$	$2.6\cdot10^{-3}$	$2.0\cdot10^{-4}$
3	$5.2\cdot10^{-3}$	$2.6\cdot10^{-3}$	$2.1\cdot10^{-4}$
5	$1.1\cdot10^{-2}$	$2.6\cdot10^{-3}$	$3.3\cdot10^{-4}$

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