Mechanical Quadrature Methods and Their Splitting Expansion Methods of The First Kind Boundary Integral Equations on Open Contours

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Abstract

This paper presents mechanical quadrature methods (MQM) for solving first-kind boundary integral equations (BIE) on open contours, which possesses high accuracy $O(h_0^3)$ and low computing complexities, where $h_0 = \max_{1\le m\le d} h_m$ and h_m $(m = 1, ..., d)$ is the mesh witdth of a curved edge Γ_m of open contours Γ. The paper shows that errors posses multivariate asymptotic expansions with h_m^3 $(m = 1, ..., d)$ for open contours. By using the splitting extrapolations the higher precision approximations and a posteriori estimates are obtained. Moreover, by the stability analysis, we conclude that mechanical quadrature methods provide not only high accuracy algorithms $O(h_0^3)$, but also excellent stability. Numerical examples are provided to support our theoretical analysiss.

key word: first-kind boundary integral equation, mechanical quadrature method, splitting extrapolation, a posteriori estimate, open contour, stability analysis.

1 Introduction

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By the layer potential theory, Dirichlet's problems of plane Laplace equations

$$\begin{cases} \Delta u = 0, \text{ in } \Omega, \\ u = f, \text{on } \Gamma, \end{cases} \tag{1.1}$$

are converted into the first kind boundary equation (BIE)

$$-\frac{1}{2\pi}\int_{\Gamma} v(x)\ln|x-y|ds_x = f(y),\ y\in\Gamma, \tag{1.2}$$

where $\Gamma = \cup_{m=1}^{d}\Gamma_m$ is the open contour with edges Γ_m, and $\Omega = R^2\backslash\Gamma$, and $|x-y|$ is the Euclidean distance. In (1.2) the unknown function $v(x) = \frac{\partial u(x)}{\partial n^-} - \frac{\partial u(x)}{\partial n^+}$, where n is a unit outward normal derivative at a point $x\in\Gamma$. From the known results[1,2,23,27,28,35], as the logarithmic capacity (transfinite diameter) $C_\Gamma \neq 1$, there exists a unique solution in (1.2). As soon as $v(x)$ is solved from (1.2), $u(y)$ ($y\in\Omega$) can be calculated by

$$u(y) = -\frac{1}{2\pi}\int_{\Gamma} v(x)\ln|x-y|ds_x,\ y\in\Omega. \tag{1.3}$$

Based on [3, 27,28 35], the solution $v(x)$ of (1.2) is usually singular at the endpoints and corner points. These singularities degrade the rates of convergence when numerical methods such as Galerkin and collocation methods are applied[1,3,6,7,12,27,28,31,35], and so one introduces modifications in order to restore the optimal rate of convergence. One possible modification is the augmented method for which the approximating spaces are augmented by appropriate singular functions which mimic the behaviour of the exact solution at the endpoints and corner points of the open contour Γ. This modification applied to the Galerkin method with piecewise polynomial test and trial functions has been analyzed by Stephan and Wendland in [31]. Another modification is to grade the mesh in a suitable way near the endpoints and corner points. Its application to the Galerkin method with piecewise constant test and trial functions has been analyzed by Yan and Sloan [36], and its application to the collocation method with piecewise linear trial functions has been done by Costabel, Ervin, & Stephan [8]. In [35] Yan also provided the method of cosine change of variable for the numerical solution of (1.2) with Galerkin and collocation methods, and obtained the superconvergence. However, quadrature methods are generally considered to be more practical since in their numerical implementation the computation of the matrix elements is less costly than in the corresponding collocation and Galerkin methods. Although some of the more practical quadrature methods have been considered by Kress and Sloan [19], by Saranen and Sloan [30] and by Saranen [29], one never provide mechanical quadrature methods (MQMs) for (1.2) with the open contour Γ.

In the paper, MQMs are constructed for solving BIE of the first kind with the open contour Γ, and the convergence theories are given. Firstly, we make use of the Sidi's quadrature rules[25] to calculate weakly singular integrals. Secondly, by calculating directly we get the eigenvalue expression of discrete matrices in the special case and estimate their super-bound and lower-bound. Finally, by using perturbation theory and Anselon's collective compact theory[4], we not only obtain that it is reasonable to construct MQMs, but also show that the condition number of discrete matrices is only $O(h^{-1})$. For MQMs the most of work can not only be saved, without calculating any singular integrals, but also the accuracies are very high $O(h^3)$. Especially, the singularity solutions at concave points and endpoints heavily dampen the approximate accuracy. The accuracy of Galerkin methods[27,28] is only $O(h^{1+\varepsilon})$ $(0 < \varepsilon < 1)$ and the accuracy of collocation methods[33] is even lower. In contrast, the accuracy of MQMs in the paper is as high as $O(h^3)$. In addition, collocation methods[33] are greatly restricted in practice, since the interior angle θ can only be in $\theta \in (29.85^0, 330.15^0)$.

It is a very important study field in numerical mathematical how to enhance further the approximate accuracy. Extrapolation algorithms and splitting extrapolation algorithms (SEM) are very effective methods to improve approximate accuracy. SEM[13,20,21] based on multivariate asymptotic expansions of errors are a very effective parallel algorithm, which possesses a high order of accuracy and almost optimal computational complexity. Since Lin and Lü published the first paper[20] in 1983, SEMs have been applied to many problems, e.g., the multidimensional numerical integrations[21], finite differential methods and finite element methods[21]. Using Galerkin methods, Rüde and Zhou[24] established multi-parameter extrapolation methods for BIE system of the second kind on polygonal domains. Assuming that Ω was a bounded, simply connected region with a smooth boundary Γ and the inverse matrix of discrete equation existed and was uniformly bounded, Xu and Zhao[32] established an extrapolation method for solving BIE from the boundary value problem of the third kind. Graham, Qun, and Rui- feng[13] established extrapolation of Nyström solutions of boundary integral equations of the second kind on non-smooth domains. Huang and Lü[15,16] constructed the MQM and their extrapolations for solving BIE of Steklov eigenvalue problems and MQM and SEM for solving BIE of linear elasticity Dirichlet problems on polygons. By MQMs, this paper shows that multivariate asymptotic expansions with h_i^3 $(i = 1, ..., d)$ for open contours. Thus, once discrete equations with some coarse meshed partitions are solved in parallel, the approximate accuracy can be greatly improved by the SEMs; moreover, a posteriori asymptotic error estimate as self-adaptive algorithms is derived.

This paper is organized as follows: In Section 2, the singularity of the integral kernels and solutions are eliminated for the first kind BIE. In Section 3, for the open contours, MQMs are constructed, and approximation convergences are proved. In Section 4, the multivariate asymptotic expansions with h_i^3 $(i = 1, ..., d)$ of errors are shown, and SEMs are established. In Section 5, the stability analysis is made. In Section 6, some numerical examples are reported and numerical results show further that the methods are worthy of recommending.

2 The singularity analysis of the integral kernels and solutions

Let $\Gamma = \cup_{m=1}^{d}\Gamma_m$ $(d > 1)$ be open contours with $C_\Gamma \neq 1$, and Γ_m $(m = 1, ..., d)$ be a piecewise smooth curve. Define boundary integral operators on Γ_m,

$$(K_{qm}v_m)(y) = -\frac{1}{2\pi}\int_{\Gamma_m} v_m(x)\log|y-x|ds_x, y \in \Gamma_q \ (m, q = 1, ..., d), \quad (2.1)$$

Thus Eq (1.2) can be converted into a matrix operator equation

$$Kv = F, \quad (2.2)$$

where $K = [K_{qm}]_{q,m=1}^{d}$, $v = (v_1(x), ..., v_d(x))^T$, $F = (f_1(y), ..., f_d(y))^T$. Assume that Γ_m can be described by the parameter mapping $x_m(s) = (x_{m1}(s), x_{m2}(s))$: $[0, T_m] \to \Gamma_m$ with $|x_m^{'}(s)| = [|x_{m1}^{'}(s)|^2 + |x_{m2}^{'}(s)|^2]^{1/2} > 0$, where T_m is the arc length of Γ_m. Using the $\sin^p-$transformation[26]

$$s = T_m\varphi_p(t) : [0, 1] \to [0, T_m], \ p \in N, \quad (2.3)$$

with $\varphi_p(t) = \vartheta_p(t)/\vartheta_p(1)$ and $\vartheta_p(t) = \int_0^t (\sin \pi t)^p dt$, the operators (2.1) will be converted into integral operators on [0,1]. Define

$$(A_{qq}w_q)(t) = \int_0^1 a_{qq}(t, \tau)w_q(\tau)d\tau, \ t \in [0, 1], \quad (2.4)$$

and

$$(B_{qm}w_m)(t) = \int_0^1 b_{qm}(t, \tau)w_m(\tau)d\tau, t \in [0, 1], \quad (2.5)$$

where $a_{qq}(t, \tau) = -\frac{1}{2\pi}\ln|2e^{-1/2}\sin \pi(t-\tau)|$, $w_m(t) = v_m(x_m(T_m\varphi_p(t)))|x_m'(T_m \varphi_p(t))|T_m\varphi_p'(t)$ and

$$b_{qm}(t, \tau) = \begin{cases} -\frac{1}{2\pi}\ln|\frac{x_q(t)-x_q(\tau)}{2e^{-1/2}\sin \pi(t-\tau)}|, \text{ for } q = m, \\ -\frac{1}{2\pi}\ln|x_q(t) - x_m(\tau)|, \text{ for } q \neq m, \end{cases}$$

and $x_m(t) = (x_{m1}(T_m\varphi_p(t)), x_{m2}(T_m\varphi_p(t)))$ $(m = 1, ..., d)$ and $|x_q(t)-x_m(\tau)| = [(x_{q1}(t) - x_{m1}(\tau))^2 + (x_{q2}(t) - x_{m2}(\tau))^2]^{1/2}$. Thus Eq (2.2) becomes

$$(A + B)W = G, \quad (2.6)$$

where $A =\text{diag}(A_{11}, ..., A_{qq})$, $B = [B_{qm}]_{q,m=1}^{d}$ and $W = (w_1, ..., w_d)^T$, $G = (g_1, ..., g_d)^T$ with $g_m(t) = f_m(x_m(t))$.

Because the operator A_{mm} $(m = 1, ..., d)$ is an isometry operator[1,35] from $H^s[0, 1]$ to $H^{s+1}[0, 1]$ for any real number s, A is also an isometry operator from $(H^s[0, 1])^d$ to $(H^{s+1}[0, 1])^d$. Hence Eq (2.6) is equivalent to

$$(E + A^{-1}B)W = A^{-1}G = \tilde{G}. \quad (2.7)$$

Since $\varphi_p(t) \in C^\infty[0,1]$, increases[26] on [0,1], and satisfies $\varphi_p(0) = 0$ and $\varphi_p(1) = 1$, the solutions of (2.6) are equivalent to those of (2.2).

Now we study the solution singularity for (2.2). We first suppose that the corner points are at $Q_1, ..., Q_d$, and at each corner point Q_m, the number $\chi_m \in (-1, 1)$ is defined by requiring $(1-\chi_m)\pi$ to be one of the angles $\angle Q_{m-1}Q_mQ_{m+1}$, where Q_1 and Q_d are the endpoints of the open contour Γ, i.e., $Q_1 \neq Q_d$. At the endpoints Q_1 and Q_d we define $\chi_1 = \chi_d = -1$, corresponding to an angle of 2π. Based on the potential theory, it is known that near the corner Q_m the solution $v_m(x) = \frac{\partial u_m(x)}{\partial n^-} - \frac{\partial u_m(x)}{\partial n^+}$ can generally be expected to have a singularity of the form $|s - s_m|^{\beta_m}$, where $\beta_m = -|\chi_m|/(1 + |\chi_m|) \geq -\frac{1}{2}$ and s with $s = s_m$ at Q_m is arc parameter. If Γ is a polygon[27,28,35], however, the singularity may be weaker than this. Since the singularity in v_m may be traced to the singularities in the potential $u(y)$, it turns out that if $u(y)$ is nonsingular in the exterior region then the singularity in v_m becomes $|s - s_m|^{\chi_m/(1-\chi_m)}$, and if it is nonsingular in the interior the singularity in v_m becomes $|s-s_m|^{-\chi_m/(1+\chi_m)}$.

Lemma 2.1. **(1)** Let a function $v_m(s) = s^\alpha g_m(s)$ $(0 > \alpha \geq -1/2)$, where $g_m(s)$ is differentiable on $[0, 1]$ a sufficient number of times and $g_m(0) \neq 0$. Then the function $w_m(t)$ takes the form

$$w_m(t) = c_1 g_m(0) t^{(p+1)\alpha+p}(1 + O(t^2)) \text{ as } t \to 0^+. \tag{2.8}$$

(2) Let a function $v_m(s) = (1 - s)^\beta \tilde{g}_m(s)$ $(0 > \beta \geq -1/2)$, where $\tilde{g}_m(s)$ is differentiable on $[0, 1]$ a sufficient number of times and $\tilde{g}_m(1) \neq 0$. Then the function $w_m(t)$ takes the form

$$w_m(t) = c_2 \tilde{g}_m(1)(1 - t)^{(p+1)\alpha+p}(1 + O((1 - t)^2))) \text{ as } t \to 1^-, \tag{2.9}$$

where c_1 and c_2 are constants.

Proof. (1) Since from the Taylor's rule we have

$$v_m(s) = \Sigma_{j=0}^{l} \frac{g_m^{(j)}(0)}{j!} s^{j+\alpha} + O(s^{l+\alpha+1}) \text{ as } s \to 0^+ \tag{2.10}$$

and

$$\varphi_p'(t) \sim \Sigma_{j=0}^{\infty} \delta_j t^{p+2j} \text{ as } t \to 0^+, \text{ and } \delta_0 > 0, \tag{2.11}$$

inserting (2.11) into (2.10), we obtain (2.8). Similarly, we can give the proof of (2).□

Although $v_m(x)$ at an angular point Q_m $(m = 2, ..., d-1)$ and the endpoints Q_1 and Q_d has the singularity, $w_m(t)$ $(m = 1, ..., d,\ t \in [0, 1])$ is a smooth function under (2.3). Below we study the singularities of the integral kernels in (2.6).

Lemma 2.2. (1) $a_{qq}(t, \tau)$ is a logarithmically singular function on $[0, 1]^2$. (2) For $|q-m| \neq 1$ (i.e., $\Gamma_q = \Gamma_m$ or $\Gamma_q \cap \Gamma_m = \emptyset$), $b_{qm}(t, \tau)$ is smooth functions on $[0, 1]^2$. (3) For $|q - m| = 1$ (i.e., $\Gamma_q \cap \Gamma_m = Q \in \{Q_m, m = 2, ..., d - 1\}$), $b_{qm}(t, \tau)$ is singular functions[] at the point (0,1) or (1,0), and $\tilde{b}_{qm}(t, \tau)$ $(= \sin^2(\pi t) b_{qm}(t, \tau))$ and $\frac{\partial^n}{\partial t^n} \tilde{b}_{qm}(t, \tau)$ $(n = 1, 2)$ are smooth functions on $[0, 1]^2$.

Proof. From the definition of $a_{qq}(t,\tau)$ and $b_{qm}(t,\tau)$, (1) and (2) are obvious[16,35]. Let $x_q(t) = x_q(T_q\varphi_p(t)) \in \Gamma_q$ and $x_m(\tau) = x_m(T_m\varphi_p(\tau)) \in \Gamma_m$. Without loss of generality, assume that $x_q(0) = (0,0) = x_m(0) = \Gamma_q \cap \Gamma_m$ is a vertex of Ω with the interior angle $\beta_q \in (0, 2\pi)$. Using the cosine theorem we have

$$\log|x_q(t) - x_m(\tau)| = \frac{1}{2}\log[(|x_q(t)| - |x_m(\tau)|)^2 + 2|x_q(t)||x_m(\tau)|\sin^2(\beta_q/2)], \tag{2.12}$$

where $|x_q(t)| = |x_q(t) - x_q(0)|$ and $|x_m(\tau)| = |x_m(\tau) - x_m(0)|$. It easily see that $|x_q(t) - x_m(\tau)| = 0$ only as $|x_q(t)| = |x_m(\tau)| = 0$ from (2.12). Hence, $b_{qm}(t,\tau)$ exists logarithmic singularity only at angular points $(0,1)$ or $(1,0)$. Also let the origin of coordinates $(0,0) = \Gamma_q \cap \Gamma_m$ be a vertex with interior angle θ_q.

Case I. For $\theta_q \in (0,\pi)\cup(\pi,2\pi)$, from (2.5) we make use of the cosine theorem and get

$$\begin{aligned}\tilde{b}_{qm}(t,\tau) &= -1/(4\pi)\sin^2(\pi t)\ln[a_0^2(t) + a_1^2(\tau) - 2a_0(t)a_1(\tau)\cos\theta_q] \\ &= -1/(4\pi)\sin^2(\pi t)\ln(a_0^2(t) + a_1^2(\tau)) \\ &\quad -1/(4\pi)\sin^2(\pi t)\ln[1 - 2a_0(t)a_1(\tau)\cos\theta_q/(a_0^2(t) + a_1^2(\tau))]\end{aligned}$$

as a new kernel of integral operator, where $a_0(t) = |x_q(T_q\varphi_p(t))|$, $a_1(\tau) = |x_m(T_m\varphi_p(\tau))|$. Obviously if $\frac{\partial^i}{\partial\tau^i}b(t,\tau)$ $(i = 0,1,2)$ is smooth, then the results of (3) holds. Without loss of generality, we assume $a_0(0) = a_1(0) = 0$. Since

$$|2a_0(t)a_1(\tau)\cos\theta_q/(a_0^2(t) + a_1^2(\tau))| \le |\cos\theta_q| < 1,$$

if we can prove that $b(t,\tau) = \sin^2(\pi t)\ln(a_0^2(t) + a_1^2(\tau))$ is a bounded function on $[0,1]$, then $\tilde{b}_{qm}(t,\tau)$ is continuous. In fact, from $\varphi_p^{(j)}(t)|_{t=0,1} = 0$, $j = 1,...,p$, we easily get $a_i^{(j)}(t)|_{t=0,t=1} = 0$, $i = 0,1$, $j = 1,...,p$. Thus we only require to prove that for an arbitrary real number $\varepsilon > 0$, $b(t,\tau)$ is bounded on $[\varepsilon/2,\varepsilon]^2$. For $(t,\, \tau) \in [\varepsilon/2,\varepsilon]^2$, it always holds that

$$|b(t,\tau)| = O(\varepsilon^2|\ln\varepsilon|) \to 0, \text{for } \varepsilon \to 0,$$

which means that $b(t,\tau)$ is bounded. Secondly, we can prove that $\frac{\partial}{\partial\tau}b(t,\tau)$ and $\frac{\partial^2}{\partial\tau^2}b(t,\tau)$ are continuous functions on $[0,1]^2$. For $(t,\tau) \in [\varepsilon/2,\varepsilon]$,we obtain

$$|\frac{\partial}{\partial\tau}b(t,\tau)| \le |\sin^2(\pi t)\frac{2a_1(\tau)|x_q'(\tau)|\varphi_p'(\tau)}{(a_0^2(t) + a_1^2(\tau))}| = O(\varepsilon^2)O(\varepsilon^{2p})/O(\varepsilon^{2p}) = O(\varepsilon^2).$$

Similarly, we have

$$|\frac{\partial^2}{\partial\tau^2}b(t,\tau)| = O(\varepsilon).$$

Therefor, $\frac{\partial^i}{\partial\tau^i}b(t,\tau), i = 0,1,2$ is a continuous function on $[0,1]^2$.

Case II. For $\theta_q = \pi$, we have

$$\tilde{b}_{qm}(t,\tau) = -\sin^2(\pi t)\ln(a_0(t) + a_1(\tau))/(2\pi).$$

Imitating the above proof, we can obtain that $\frac{\partial^i}{\partial\tau^i}\tilde{b}_{qm}(t,\tau)$ $(i = 0,1,2)$ is continuous on $[0,1]^2$.□

3 The existence and convergence of approximations by MQM

Let $h_m = 1/n_m$, $n_m \in N$ $(m = 1, ..., d)$ be mesh widths and $t_j = \tau_j = (j - 1/2)h_m$ $(j = 1, ..., n_m)$ be node. By the trapezoidal or midpoint rule[10] we construct the Nyström's approximate operator B^h_{qm} of the integral operator B_{qm}

$$(B^h_{qm}w_m)(t) = h_m \sum_{j=1}^{n_m} b_{qm}(t, \tau_j)w_m(\tau_j),\ t \in [0, 1],\ (q, m = 1, ..., d), \tag{3.1a}$$

which has the error estimate

$$(B_{qm}w_m)(t) - (B^h_{qm}w_m)(t) = O(h^{2l}), \text{for } |q - m| \neq 1 \tag{3.1b}$$

and

$$(B_{qm}w_m)(t) - (B^h_{qm}w_m)(t) = O(h^{\omega}), \text{for } |q - m| = 1 \tag{3.1c}$$

with

$$\omega = \begin{cases} \min((p + 1)(\alpha + 1), p + 1),\ p \text{ odd} \\ \min((p + 1)(\alpha + 1), 2p + 2),\ p \text{ even}, \end{cases} \tag{3.1d}$$

at worst[10,26]. For the weakly singular operators A_{mm}, by the quadrature formula[25], we can construct Fredholm approximate operator A^h_{qq},

$$(A^h_{qq}w_q)(t_i) = -\frac{1}{2\pi}h_q\{\sum_{j=1, t\neq\tau_j}^{n_m} \ln|2e^{-1/2}\sin\pi(t_i - \tau_j)|w_q(\tau_j)\}$$

$$-|\ln|2\pi e^{-1/2}h_q/(2\pi)|w_q(t_i)\}, i = 1, ..., n_q, \tag{3.2}$$

which has the error estimate[25]

$$(A^h_{qq}w_q)(t) - (A_{qq}w_q)(t) =$$

$$\frac{-2}{\pi}\sum_{\mu=1}^{2l-1}\frac{\zeta'(-2\mu)}{(2\mu)!}[w_q(t)]^{(2\mu)}h_q^{2\mu+1} + O(h_q^{2l}), t \in \{t_i\}. \tag{3.3}$$

Set $t = t_i$ $(i = 1, ..., n_q)$, and we obtain the approximate equations of (2.6)

$$(A_h + B_h)W_h = G_h, \tag{3.4}$$

where $W_h = (w^h_1(t_1), ..., w^h_1(t_{n_1}), ..., w^h_d(t_1), ..., w^h_d(t_{n_d}))^T$, $A_h = \text{diag}(A^h_{11}, ..., A^h_{dd})$, $A^h_{qq} = [a_{qq}(t_j, \tau_i)]^{n_q}_{j,i=1}$, $B_h = [B^h_{qm}]^d_{q,m=1}$, $B^h_{qm} = [b_{qm}(t_j, \tau_i)]^{n_q,n_m}_{j,i=1}$, $G_h = (g_1(t_1), ..., g_1(t_{n_1}), ..., g_d(t_1), ..., g_d(t_{n_d}))^T$, and

$$a_{qq}(t_j, \tau_i) = \begin{cases} -[h_q \ln|2e^{-1/2}\sin\pi(t_i - \tau_j)|]/(2\pi),\ \text{as } i \neq j, \\ -[h_q|\ln|2\pi e^{-1/2}h_q/(2\pi)|]/(2\pi), \text{as } i = j. \end{cases} \tag{3.5}$$

Obviously, Eq (3.4) is a linear equation system with n $(= n_1+\cdots+n_d)-$unknown numbers. Once W_h is solved by (3.4), $u(y)$, $y \in \Omega$ can be computed by

$$u^h(y) = \frac{-1}{2\pi}\sum_{m=1}^{d}\sum_{i=1}^{n_m} h_m[\ln|y - x_m(t_i)|]|x'_m(t_i)|w_m^h(t_i). \tag{3.6}$$

From (2.4) and (3.5), we have

$$A_{mm}^h = -h_m/\pi \text{circular}(\ln(e^{-1/2}h_m), \ln(2e^{-1/2}\sin(\pi h_m)), ..., \ln(2e^{-1/2}\sin((n_m-1)\pi h_m))).$$

Lemma 3.1. The eigenvalues λ_k $(k = 1, ..., n_m)$ of A_{mm}^h are positive, and there exists a positive constant c such that $\lambda_k > c$ for $n_m < 4$, or $\lambda_k > 1/(2\pi n_m)$ for $n_m \geq 4$.

Proof. Since A_{mm}^h is a symmetric circulant matrix[11], we have $\lambda_k = F(\varepsilon_k)$ with

$$F(z) = -h_m[\ln|he^{-1/2}|+\sum_{j=1}^{n_m-1} z^j \ln|2e^{-1/2}\sin(j\pi/n_m)|], \text{ and } \varepsilon_k = \exp(2\pi ki/n_m).$$

If $n_m < 4$, then $\lambda_k > c$ can be easily verified by direct calculations. If $n_m \geq 4$, then λ_k is estimated as follows:

Step 1. Consider $k = 0$. Let

$$\lambda'_0 = \ln|h_m e^{-1/2}| + \sum_{j=1}^{n_m-1}\ln|2e^{-1/2}\sin(j\pi/n_m)|$$

$$= -n_m/2 - \ln n_m + \ln|2^{n_m-1}\prod_{j=1}^{n_m-1}\sin(j\pi/n_m)|. \tag{3.7}$$

We shall discuss the following two cases:

Case (1). For $n_m = 2l - 1$, by the inequality

$$2x/\pi < \sin x < x, \text{ if } 0 < x < \pi/2, \tag{3.8}$$

we have

$$\frac{2^{2l-2}[(l-1)!]^2}{(2l-1)^{2l-2}} < \prod_{j=1}^{l-1}\sin^2\frac{j\pi}{2l-1} = \prod_{j=1}^{n_m-1}\sin\frac{j\pi}{n_m} < \frac{\pi^{2l-2}[(l-1)!]^2}{(2l-1)^{2l-2}}.$$

Using Stirling's rule[9] $n! = \sqrt{2\pi n}(n/e)^n\exp(\theta/(12n))$ $(0 < \theta < 1)$, we obtain

$$\frac{2^{2l-2}[(l-1)!]^2}{(2l-1)^{2l-2}} > 2\pi e^{2-2l}(l-1)(1-1/l)^{2l-2}e^{\theta/[6(l-1)]}$$

and

$$\frac{\pi^{2l-2}[(l-1)!]^2}{(2l-1)^{2l-2}} < 2\pi(\frac{\pi}{2e})^{2l-2}(l-1)e^{\theta/[6(l-1)]} = 1/B.$$

Also since $\ln B < \ln |2^{n_m-1} \prod_{j=1}^{n_m-1} \sin(j\pi/n_m)|^{-1}$, we get

$$\lambda_0 = -\lambda_0'/(\pi n_m) > [3/2 - \ln \pi - 1/(2l-1) - 1/[6(2l-1)(l-1)]/\pi,$$

which implies that $\lambda_0 > 17/(150\pi)$ for $l \geq 3$.

Case (2). For $n_m = 2l$, from $0 < (l-1)/(2l) < 1/2$ and (3.8), we derive

$$\frac{[(l-1)!]^2}{l^{2l-2}} < \prod_{j=1}^{l-1} \sin^2 \frac{j\pi}{2l} = \prod_{j=1}^{n_m-1} \sin \frac{j\pi}{n_m} < \frac{\pi^{2l-2}[(l-1)!]^2}{(2l)^{2l-2}}.$$

Using

$$\frac{\pi^{2l-2}[(l-1)!]^2}{(2l)^{2l-2}} = 2l\pi(\frac{\pi}{2})^{2l-2} e^{(-2l+\theta/6l)}$$

and the above inequality, we have

$$\lambda_0 = -\lambda_0'/(\pi n_m) = [1/2 + 1/n_m \ln n_m + 1/n_m \ln |2^{n_m-1} \prod_{j=1}^{n_m-1} \sin(j\pi/n_m)|^{-1}]/\pi$$

$$> [3/2 - \ln \pi]/\pi,$$

which implies that $\lambda_0 > c > 0$ as $n_m \geq 4$.

Step 2. To estimate $\lambda_k, k = 1, ..., n_m - 1$, we write

$$\lambda_k^{'} = \ln |e^{-1/2}/n_m| + \sum_{j=1}^{n_m-1} \cos(2kj\pi/n_m) \ln |2e^{-1/2} \sin(j\pi/n_m)|$$

$$= -\ln n_m + \sum_{j=1}^{n_m-1} \cos(2kj\pi/n_m) \ln |2 \sin(j\pi/n_m)|. \qquad (3.9)$$

Using the expansions of the $\psi-$special function[9]

$$\psi(k/n) = -\gamma - \ln n - \pi/2 \cot(k\pi/n) + \sum_{j=1}^{n} \cos(2kj\pi/n) \ln |2 \sin(j\pi/n)|$$

and

$$\psi(z) = -\gamma - 1/z + z \sum_{j=1}^{\infty} 1/[j(j+z)],$$

we obtain

$$\sum_{j=1}^{n_m-1} \cos(2kj\pi/n_m) \ln |2 \sin(j\pi/n_m)|$$

$$= \ln n_m + \pi/2 \cot(k\pi/n_m) - n_m/k + k/n_m \sum_{j=1}^{\infty} [j(j + k/n_m)]^{-1}$$

and

$$\lambda_k^{'} = \pi/2\cot(k\pi/n_m) - n_m/k + k/n_m \sum_{j=1}^{\infty}[j(j+k/n_m)]^{-1}, 1 \le k \le n_m - 1, \quad (3.10)$$

where γ is a Euler's constant. Substituting

$$\cot(k\pi/n_m) = n_m/(k\pi) - k\pi/(3n_m) - 1/45(k\pi/n_m)^3 - \cdots$$
$$-2^{2j}B_j/(2j)!(k\pi/n_m)^{2j-1} - \cdots$$

into (3.10), we have

$$\lambda_k^{'} = -n_m/(2k) - k\pi^2/(6n_m) - \cdots - 2^{2j+1}B_j/(2j)!(k\pi/n_m)^{2j-1}\pi$$
$$- \cdots + k/n_m \sum_{j=1}^{\infty}[j(j+k/n_m)]^{-1}$$

and

$$\lambda_k = \{1/(2k) + k\pi^2/(6n_m^2) + \cdots + 2^{2j+1}B_j/(2j)!(k\pi/n_m)^{2j-1}\pi/n_m$$
$$+ \cdots - k/n_m^2 \sum_{j=1}^{\infty}[j(j+k/n_m)]^{-1}\}/\pi,$$

where B_j is the Bernoulli number. Since

$$k\pi^2/(6n_m^2) - k/n_m^2 \sum_{j=1}^{\infty}[j(j+k/n_m)]^{-1} > k/n_m^2\{\sum_{j=1}^{\infty}[j^{-2} - (j(j+1/2))^{-1}]\} > 0,$$

we obtain

$$\lambda_k > 1/(2\pi k) + 1/90(k/n_m)^3/n_m + ... > 1/(2\pi k) > 1/(2\pi n_m).$$

Combining the results of Step 1 and Step 2, the proof of Lemma 2 is completed. □

From Lemma 3.1 we have the following corollary.

Corollary 3.2. (1) A_{mm}^h is invertible, the conditional number of A_{mm}^h is $O(n_m)$, and $||(A_{mm}^h)^{-1}|| = O(n_m)$ holds, where $\|\cdot\|$ denotes the spectral norm. (2) A_h is invertible, the conditional number of A_h is $O(n_0)$, and $||(A_h)^{-1}|| = O(n_0)$ holds, where $\|\cdot\|$ denotes the spectral norm and $n_0 = \min n_m$.

In order to discuss the existence and convergence of approximations, we first introduce the subspace and some special operators. Define the subspace[16]

$$C_0[0,1] = \{v(t) \in C[0,1] : v(t)/\sin^2(\pi t) \in C[0,1]\}$$

of the space $C[0,1]$ with the norm $||v||^* = \max_{0\le t\le 1}|v(t)/\sin^2(\pi t)|$. Let S^{h_m} =span$\{e_j(t), j = 1, \cdots, n_m\} \subset C_0[0,1]$ be a piecewise linear function subspace with base points

$\{t_i\}_{i=1}^{n_m}$, where $e_j(t)$ is the basis functions satisfying $e_j(t_i) = \delta_{ji}$. Also define a prolongation operator $I^{h_m} : \Re^{n_m} \to S^{h_m}$ satisfying

$$I^{h_m} v = \sum_{j=1}^{n_m} v_j e_j(t), \forall v = (v_1, ..., v_{n_m}) \in \Re^{n_m}, \tag{3.11}$$

and a restricted operator $R^{h_m} : C_0[0,1] \to \Re^{n_m}$ satisfying

$$R^{h_m} v = (v(t_1), ..., v(t_{n_m})) \in \Re^{n_m}, \forall v \in C_0[0,1]. \tag{3.12}$$

To prove the approximation convergence, we first introduce the following lemma.

Lemma 3.3. The operator sequence $\{I^{h_q}(A_{qq}^h)^{-1} R^{h_q} A_{qq} : C^3[0,2\pi) \to C[0,2\pi)\}$ is uniformly bounded and convergent to embedding operator I.

Proof. From the quadrature rule (3.2)-(3.3), let $\forall \phi \in C^3[0,2\pi)$ and ϕ_q^h be solutions of auxiliary equations $A_{qq}\phi = \rho$ and $A_{qq}^h \phi^h = R^{h_q}\rho$. we have

$$A_{qq}\phi(t_i) = \int_0^{2\pi} a_{qq}(t_i,t)\phi(t)dt = \sum_{i\neq j, j=1}^{n_q} h_q a_{qq}(t_i,t_j)\phi(t_j) +$$

$$(\frac{-h_q}{2\pi})\ln|\frac{h_q e^{-1/2}}{2\pi}|\phi(t_j) + \varepsilon_i,\ \ \varepsilon_i = O(h^3), i = 1, ..., n_q.$$

Let $e(t_j) = \phi^h(t_j) - \phi(t_j)$, where $\phi^h(t_j)$ and $\phi(t_j)$ are the solutions of the above auxiliary equations at $t = t_j$ respectively. From (3.2), we lead to

$$\sum_{i\neq j, j=1}^{n_q} h_q a_{qq}(t_i,t_j)e(t_j) + (\frac{-h_q}{2\pi})\ln|\frac{h_q e^{-1/2}}{2\pi}|e(t_i) = \sum_{i\neq j, j=1}^{n_q} h_q a_{qq}(t_i,t_j)\phi_h(t_j) +$$

$$(\frac{-h_q}{2\pi})\ln|\frac{h_q e^{-1/2}}{2\pi}|\phi_h(t_i) - [\sum_{i\neq j, j=1}^{n_q} h_q a_{qq}(t_i,t_j)\phi(t_j) + (\frac{-h_q}{2\pi})\ln|\frac{h_q e^{-1/2}}{2\pi}|\phi(t_i)] =$$

$$\rho(t_i) - [\sum_{i\neq j, j=1}^{n_q} h_q a_{qq}(t_i,t_j)\phi(t_j) + (\frac{-h_q}{2\pi})\ln|\frac{h_q e^{-1/2}}{2\pi}|\phi(t_i)] = \int_0^{2\pi} a_{qq}(t_i,t)\phi(t)dt -$$

$$[\sum_{i\neq j, j=1}^{n_q} h_q a_{qq}(t_i,t_j)\phi(t_j) + (\frac{-h_q}{2\pi})\ln|\frac{h_q e^{-1/2}}{2\pi}|\phi(t_i)] = \varepsilon_i = O(h_q^3),$$

that is,

$$A_{qq}^h e = \varepsilon,\ \ e^T = (e(t_1), ..., e(t_n)), \varepsilon^T = (\varepsilon_1, ..., \varepsilon_n)^T, \tag{3.13}$$

and

$$e = (A_{qq}^h)^{-1}\varepsilon.$$

From Lemma 3.1, we have

$$||e|| = ||(A_{qq}^h)^{-1}\varepsilon|| = ||R^{h_q}A_{qq}^{-1}\rho - (A_{qq}^h)^{-1}R^{h_q}\rho|| = ||R^{h_q}\phi - (A_{qq}^h)^{-1}R^{h_q}A_{qq}\phi|| = O(h_q^2). \tag{3.14}$$

Since $I^{h_q}R^{h_q} \to I$, the proof of Lemma 3.3 is completed.

Lemma 3.4.Let the open contour $\Gamma = \cup_{m=1}^{d}\Gamma_m$ satisfy $C_\Gamma \neq 1$, and $\Gamma_q = \Gamma_m$ or $\Gamma_q \cap \Gamma_m = \emptyset$. Then under parameter transformations (2.3) Nyström's approximations B_{qm}^h of integral operators B_{qm} have

$$I^{h_q}(A_{qq}^{h_q})^{-1}R^{h_q}B_{qm}^{h_m} \overset{c.c}{\to} (A_{qq})^{-1}B_{qm}, \text{ in } C[0,1] \to C[0,1] \tag{3.15}$$

hold by the trapezoidal or midpoint rule.

Proof. Under the above assumptions and the parameter transformations (2.3) the kernels $b_{qm}(t,\tau)$ of operators B_{qm} are continuous and their high order derivable are, too, continuous[4,5,35]. By

$$I^{h_q}(A_{qq}^{h_q})^{-1}R^{h_q}B_{qm}^{h_m} = (I^{h_q}(A_{qq}^{h_q})^{-1}R^{h_q}A_{qq})((A_{qq}^{-1})B_{qm}^{h_m}),$$

we have

$$||I^{h_q}(A_{qq}^{h_q})^{-1}R^{h_q}B_{qm}^{h_m}||_{0,0} \leq ||(I^{h_q}(A_{qq}^{h_q})^{-1}R^{h_q}A_{qq})||_{0,3}||(A_{qq}^{-1})B_{qm}^{h_m}||_{3,0}.$$

According to Lemma 3.3, there exists a constant c such that

$$||I^{h_q}(A_{qq}^{h_q})^{-1}R^{h_q}A_{qq}||_{0,3} \leq c. \tag{3.16}$$

Using the results of [4,5,18] and Lemma 3.3, we obtain that the smooth operator sequence $\{(A_{qq})^{-1}B_{qm}^{h_m} : C[0,1] \to C^3[0,1]\ \}$ must be collectively compact convergent to $A_{qq}^{-1}B_{qm}$, which gets the proof of (3.15).

Corollary 3.5[16]**.** For $\Gamma_q \cap \Gamma_m = Q \in \{Q_m, m = 2, ..., d-1\}$, let the interior angle $\theta_q \in [0,\ 2\pi)$ $(q = 1,\ ...,\ d)$ of open contours $\Gamma = \cup_{m=1}^{d}\Gamma_m$ $(d > 1)$ with $C_\Gamma \neq 1$. Then under the parameter transformation (2.3) Nyström's approximations $\tilde{B}_{qm}^{h_m}$ of integral operators $\tilde{B}_{qm}$ by the trapezoidal or midpoint rule have

$$I^{h_q}(A_{qq}^{h_q})^{-1}R^{h_m}(\tilde{B}_{qm}^{h_m}) \overset{c.c}{\to} (A_{qq})^{-1}\tilde{B}_{qm}, \text{in } C[0,1] \to C[0,1], \tag{3.17}$$

hold, where the kernel $\tilde{b}_{qm}(t,\tau)$ of integral operator $\tilde{B}_{qm}$ is $\sin^2(\pi t)b_{qm}(t,\tau)$.

Replacing $(A_{qq}^{h_q})^{-1}$ and $B_{qm}^{h_m}$ $(m, q = 1, ..., d)$ by $I^{h_q}(A_{qq}^{h_q})^{-1}R^{h_m}$ and $I^{h_q}B_{qm}^{h_m}R^{h_m}$, we construct an operator $\hat{L}_h : (C_0[0.1])^d \to \cup_{m=1}^{d}S^{h_m}$. Consider the following operator equation

$$(I + \hat{L}_h)\hat{W}_h = I^h\tilde{G}_h, \tag{3.18}$$

where $\tilde{G}_h = (A_h)^{-1}R^hG_h$ and $\tilde{G} = A^{-1}G$. Obviously, if $\hat{W}_h$ is a solution of (3.18), then $R^h\hat{W}_h$ is a solution of (3.4) ; reversely, if W_h is a solution of (3.4), then I^hW_h is a solution of (3.18); where $\hat{L}_h = (\hat{A}_h)^{-1}\hat{B}_h = I^h(A_h)^{-1}R^hB_h$, $I^h = diag(I_{h_1}, ..., I_{h_d})$, $R^h = diag(R_{h_1}, ..., R_{h_d})$. Below We shall prove that W_h converge to W.

Theorem 3.6. Let the open contour $\Gamma = \cup_{m=1}^{d}\Gamma_m$ satisfy $C_\Gamma \neq 1$ and Γ_m $(m = 1,\ ...,\ d)$ be smooth curve. Then the operator sequence$\{(\hat{A}_h)^{-1}\hat{B}_h\}$ is collectively compact convergent to $A^{-1}B$ in $V = (C_0[0.1])^d$,i.e.

$$(\hat{A}_h)^{-1}\hat{B}_h \overset{c.c}{\to} A^{-1}B. \tag{3.19}$$

Proof. Let $\Theta = \{z : ||z|| \leq 1, z \in V\}$ be a unit ball and $H = \{H^{(1)}, H^{(2)}, ...,\}$ be a mesh sequence, where $H^{(n)} = \{h_1^{(n)}, ..., h_d^{(n)}\}$ denote a multi-parameter step size with $\max_{1\leq m\leq d} h_m^{(n)} \to 0$ as $n_m \to \infty$. Take an arbitrary sequence $\{Z_h, h \in H\} \subset \Theta \subset V$ and $Z_h = \{z_{11}^h, ..., z_{1n_1}^h, ..., z_{d1}^h, ..., z_{dn_d}^h\}$ with

$$\max_{1\leq m\leq d} \max_{1\leq t\leq n_m} |z_{mi}(t)/\sin^2(\pi t)| \leq 1.$$

Under the above assumptions we assure that there exists a convergent subsequence in $\{(\hat{A}_h)^{-1}\hat{B}_h Z_h\}$. We consider the first complement

$$\sum_{m=1}^{d} I^{h_1}(A_{11}^{h_1})^{-1} R^{h_m} B_{1m}^{h_m} R^{h_q} z_{1h} \tag{3.20}$$

of $(\hat{A}_h)^{-1}\hat{B}_h Z_h$. For $\Gamma_q = \Gamma_m$ or $\Gamma_q \cap \Gamma_m = \emptyset$, by the lemma 3.4 we have

$$I^{h_1}(A_{11}^{h_1})^{-1} R^{h_m} B_{1m}^{h_m} \xrightarrow{c.c} A_{11}^{-1} B_{1m}, \text{ in } C[0,1] \to C[0,1].$$

For $\Gamma_q \cap \Gamma_m = Q \in \{Q_m, m = 2, ..., d-1\}$, we have

$$||I^{h_1}(A_{11}^{h_1})^{-1} R^{h_m} B_{1m}^{h_m} R^{h_m} z_{1m}^h||_{0,0} = ||I^{h_1}(A_{11}^{h_1})^{-1} R^{h_m} \tilde{B}_{1m}^{h_m} (R^{h_m} z_{1m}^h / \sin^2(\pi t))||_{0,0}$$

$$\leq ||I^{h_1}(A_{11}^{h_1})^{-1} R^{h_m} A_{11}||_{0,3} ||(A_{11}^1)^{-1} \tilde{B}_{1m}^{h_m}||_{3,0} ||z_{1m}^h||^*.$$

According to Lemma 3.4 and [4,5,18] there exists a convergent subsequence in $\{I^{h_1}(A_{11}^{h_1})^{-1} \, R^{h_m} B_{1m}^{h_m} R^{h_m} z_{1m}^h\}$. However $C_0[0,1] \subset C[0,1]$, based on the above two cases, using the results of [4,5,18], we can find a infinite subsequence $H^{(1)} \subset H$ such that (3.20) converge as $h \to 0$, $h \in H_1$. Imitating the above methods, we can find a infinite subsequence $H_d \subset H_1 \subset H$ such that $\{(\hat{A}_h)^{-1}\hat{B}_h Z_h, \, h \in H_d\}$ is a convergent sequence in $V = (C_0[0.1])^d$. Obviously it means

$$(\hat{A}_h)^{-1}\hat{B}_h \xrightarrow{p} A^{-1}B = L,$$

where the notation $\xrightarrow{P}$ denotes the pointwise convergence. Based on the base of [5, 7], this completes the proof of Theorem 3.7.

Corollary 3.7[4,18]**.** Let the open contour $\Gamma = \cup_{m=1}^d \Gamma_m$ satisfy $C_\Gamma \neq 1$ and Γ_m $(m = 1, ..., d)$ be smooth curve. Also let $h_0 = \max_{1\leq m\leq d} h_m$ be sufficiently small. Then there exits the unique solution W^h in (3.18) and its error estimate under the norm of V has (3.21) hold at node points.

$$||W^h - W|| \leq ||(I+L)^{-1}|| \frac{||(\hat{L}_h - L)\tilde{G}|| + ||(\hat{L}_h - L)\hat{L}_h W||}{1 - ||(I - \hat{L}_h)^{-1}(\hat{L}_h - L)\hat{L}_h||}. \tag{3.21}$$

4 Multiparameter asymptotic expansions of errors and splitting extrapolation algorithms

Now we prove the main result.

Theorem 4.1. Let the open contour $\Gamma=\cup_{m=1}^{d}\Gamma_m$ satisfy $C_\Gamma\neq 1$ and Γ_m ($m=1,\ ...,\ d$) be smooth curve. Also let $f_m\in C^4(\Gamma_m)\times C^4(\Gamma_m)$. Then when $p\geq 6$, there exists vector function $\phi=(\phi_{1,}...,\phi_d)^T\in(C_0[0,1])^d$ independent of $h=(h_1,...,h_d)^T$ such that the following multi-parameter asymptotic expansions

$$W-\hat{W}^h=h^3\phi+0(h_0^3), \tag{4.1}$$

hold at node points, where $h^3=(h_1^3,...,h_d^3)$, $h_0=\max_{1\leq m\leq d}h_m$.

Proof. When $p\geq 6$, based on (3.1)-(3.4), we have

$$(\hat{A}_h+\hat{B}_h)(W-\hat{W}^h)=I^hR^h(A+B)W-(\hat{A}_h+\hat{B}_h)W$$

$$=diag(h_1^3,...,h_d^3)I^hR^h\varpi+0(h_0^3),$$

where $\varpi^T=(\varpi_1,...,\varpi_d),\varpi_m=\zeta^{'}(-2)W''$, and

$$(I+\hat{L}_h)(W-\hat{W}^h)=diag(h_1^3,...,h_d^3)(\hat{A}_h)^{-1}I^hR^h\varpi+0(h_0^3).$$

We construct the following auxiliary equations

$$(I+L)\phi=A^{-1}\varpi; \tag{4.2}$$

and its approximate equations

$$(I+\hat{L}_h)\phi^h=(\hat{A}_h)^{-1}I^hR^h\varpi. \tag{4.3}$$

Substituting (4.3) into (4.2), we get

$$(I+\hat{L}_h)(W-\hat{W}^h-diag(h_1^3,...,h_d^3)\phi^h)=0(h_0^3).$$

Since $(I+\hat{L}_h)^{-1}$ is uniformly bounded from the theorem 3.6, we obtain

$$W-\hat{W}^h-diag(h_1^3,...,h_d^3)\phi^h=0(h_0^3). \tag{4.4}$$

Because ϕ^h is the approximate solution of (4.3); replacing ϕ^h of (4.3) by ϕ, we get the proof of Theorem 4.1.

Making use of the splitting extrapolation algorithms[13,16,20,24,32] according to the multi-parameter asymptotic expansions (4.3), we can get approximations of a higher order accuracy $0(h_0^3)$ by solving some coarse grid discrete equations in parallel. The process of the splitting extrapolation algorithms is as follows.

Step1. Take $h^{(0)}=(h_1,...,h_d)$ and $h^{(m)}=(h_1,...,h_m/2,...,h_d)$, and solve (3.4) according to $h^{(m)}$, $m=1,\ ...,\ d$ in parallel and obtain the solutions $W_{h^{(m)}}(t_i)$, $m=1,\ ...,\ d$.

Step2. Compute h^3-Richardson extrapolation on the coarse grid points

$$W^*(t_i)=\frac{8}{7}[\sum_{m=1}^{d}W_{h^{(m)}}(t_i)-(d-\frac{7}{8})W_{h^{(0)}}(t_i)]. \tag{4.5}$$

Then compute $u_i(y)$ ($y\in\Omega\backslash\Gamma$) according to (3.6).

Step3. According to (4.7), have

$$|W(t_i)-\frac{1}{d}\sum_{m=1}^{d}W_{h^{(m)}}(t_i)|\le|W(t_i)-\frac{8}{7}[\sum_{m=1}^{d}W_{h^{(m)}}(t_i)-(d-\frac{7}{8})W_{h^{(0)}}(t_i)]|+$$

$$(\frac{8d-7}{7})|\frac{1}{d}\sum_{m=1}^{d}W_{h^{(m)}}(t_i)-W_{h^{(0)}}(t_i)|\le(\frac{8d-7}{7})|\frac{1}{d}\sum_{m=1}^{d}W_{h^{(m)}}(t_i)-W_{h^{(0)}}(t_i)|+0(h_0^3). \tag{4.6}$$

Then compute the right side of (4.6) and obtain a posteriori error estimate.

5 The stability analysis

To study the stability of MQM, we first introduce the following known result.

Lemma 5.1[14]**.** (1) Define $|D|=[|d_{ij}|]$ for any matrix $D=[d_{ij}]$, then its spectral radius holds $\rho(D)\le\rho(|D|)$. (2) If $|D|\le C$, then $\rho(D)\le\rho(C)$. (3) If the diagonal matrix $D\ge 0$ and the matrix $C\ge 0$ with $\mathrm{Re}\lambda_i(D-C)>0$; the matrix $|M_1|\ge D$ and $|M_2|\le C$, then (a) M_1-M_2 is a nonsingular matrix; (b) $|(M_1-M_2)^{-1}|\le(D-C)^{-1}$; (3) $|\det(M_1-M_2)|\ge\det(D-C)$.

Lemma 5.2. Let Γ ($C_\Gamma\neq 1$) be an arbitrarily closed smooth curve. Assume that A_h and B_h are the discrete matrices defined by (3.2) and (3.1) respectively. Then the eigenvalues $|\lambda_i|$ $(i=0,1,...,n-1)$ of discrete matrix K_h=A_h+B_h satisfy

$$\check{c}\ge|\lambda_i|\ge\hat{c}h^{-1}, i=0,1,...,n-1, \tag{5.1}$$

where $\check{c}$ and $\hat{c}$ are two positive constants independent of h.

Proof. From (3.1) and (3.2), the diagonal entries of discrete matrices A_h and B_h are a_{ii}= $a_0=\frac{-h}{2\pi}\ln|\frac{e^{-1/2}h}{2\pi}|$ and $b_{ii}=-\frac{h}{2\pi}\ln|e^{1/2}x'(t_i)|$ respectively. Two cases are discussed.

Case I. When $1\ge e^{1/2}\check{\mu}\ge e^{1/2}|x^{'}(t)|\ge e^{1/2}\hat{\mu}>0$, we choose $\alpha=a_0+c_0$, where $c_0=-\frac{h}{2\pi}\ln(e^{1/2}\check{\mu})\ge 0$. Let D =diag(α, ..., α) and $C=[\frac{c_0}{n}]_{i,j=1}^{n}$. Obviously, the matrix $D-C$ is a circular matrix[9]. From the theory of circular matrix, the eigenvalues $\bar{\lambda}_k$ of matrix $D-C$ are given by

$$\bar{\lambda}_k=((a_0+c_0)-\frac{c_0}{n})-\sum_{j=1}^{n-1}\frac{c_0}{n}\varepsilon_k^j,\ k=0,...,n-1,$$

where $\varepsilon_k=\exp(2k\pi i/n)$ and $i=\sqrt{-1}$.Hence, by some manipulations we have

$$\bar{\lambda}_k=\begin{cases}\frac{1}{2n}+\frac{1}{n}\ln n\ge ch, \text{ as } k=0,\\ \frac{1}{2n}+\frac{1}{n}\ln n-\frac{1}{n}\ln(e^{1/2}\check{\mu})\ge ch, \text{ as } k=1,...,n-1,\end{cases}$$

where c is a positive constant number independent of n.

Case II. When $e^{1/2}\check{\mu}\ge e^{1/2}|x^{'}(t)|\ge e^{1/2}\hat{\mu}\ge 1$, we choose $\alpha=a_0-c_0$, where $c_0=\frac{h}{2\pi}\ln(e^{1/2}\check{\mu})\ge 0$. Let D =diag(α, ..., α) and $C=[\frac{c_0}{n}]_{i,j=1}^{n}$. Eigenvalues $\bar{\lambda}_k$

of matrix $D{-}C$ are given by

$$\bar{\lambda}_k = ((a_0 - c_0) - \frac{c_0}{n}) - \sum_{j=1}^{n-1} \frac{c_0}{n} \varepsilon_k^j$$

$$= \begin{cases} \frac{1}{2n} + \frac{1}{n} \ln n - 2c_0, \text{ as } k = 0, \\ \frac{1}{2n} + \frac{1}{n} \ln n - c_0, \text{ as } k = 1, ..., n-1, \end{cases}$$

where $\varepsilon_k = \exp(2k\pi i/n)$ and $i = \sqrt{-1}$. Since $\breve{\mu}$ is a bounded positive number, there always exists a positive integer number n_0 such that $n \geq n_0 \geq (e^{1/2}\breve{\mu})^2 \geq 1$. Hence, when $n \geq n_0$ we have $\bar{\lambda}_k \geq ch$, where c is a positive constant number independent of n.

Denote $M_1 = K_h{=}A_h{+}B_h$ and $M_2 = [0]_{i,j=1}^n$. Obviously, $|M_1| \geq D$, $|M_2| \leq \mathbf{C}$ and $K_h = M_1{-}M_2$. Since $\operatorname{Re} \bar{\lambda}_i(D{-}C) = ch$, from Lemma 5.1, $K_h = M_1{-}M_2$ is invertible and

$$0 < |\bar{\lambda}_i({K_h}^{-1})| \leq \rho({K_h}^{-1}) \leq \rho(|{K_h}^{-1}|)$$

$$\leq \rho((D{-}C)^{-1}) \leq ch^{-1},$$

i.e.,

$$|\bar{\lambda}_i(K_h)| \geq (\rho((D{-}C)^{-1}))^{-1} \geq ch, \ i = 0, ..., n-1.$$

This is the lower bound of $|\lambda_i|$ in (5.1).

Next we derive the upper bound of eigenvalues $|\lambda_i|$ of $K_h{=}A_h{+}B_h$. From [14] we have $\rho(K_h) = \rho(A_h{+}B_h) \leq \rho(A_h) + \rho(B_h)$. Moreover, from Lemma 3.1 and [5] we obtain $\rho(A_h) = c_1$, and from [5] we have $\rho(B_h) \leq c_2$, where c_1 and c_2 are two positive constants. Hence, $\rho(K_h) \leq c_1 + c_2$ and the upper bound of $|\lambda_i|$ in (5.1) follows. This completes the proof of Lemma 5.2.

Based on Lemma 5.2, we have the following theorem.

Theorem 5.3. Let the open contour $\Gamma = \cup_{m=1}^d \Gamma_m$ satisfy $C_\Gamma \neq 1$ and Γ_m ($m = 1, ..., d$) be smooth curve. Also assume that A_h and B_h are discrete matrices defined by (3.2) and (3.1) respectively. Then the bound of condition number is

$$\text{Cond.}(K_h) = O(\bar{h}^{-1}), \, \bar{h} = \min_{1 \leq m \leq d} h_m. \tag{5.2}$$

6 The numerical experiments

We carry out the numerical experiments for the problem (1.1) by MQM and $h^3{-}$Richardson extrapolation or SEM algorithms, and verify the errors and stability analyses made in the above sections.

Example 1[28,34,36]**.** Here the equation is

$$-\int_\Gamma \log |x - y| v(x) ds_x = 1 \ (y \in \Gamma), \tag{6.1}$$

with Γ a straight-line segment of length 2: specifically,

$$\Gamma = \{(x_1, 0) : -1 \le x_1 \le 1\}.$$

Since the right-hand side of (6.1) is 1, its solution[28,34,36] is

$$v(x) = (\pi \log 2)^{-1}(1 - x_1^2)^{-\frac{1}{2}} \ (-1 < x_1 < 1),$$

and

$$C_\Gamma = \exp[-(\int_\Gamma v(x) ds_x)^{-1}],$$

which has the exact value $C_\Gamma = 0.5$ for an interval of length 2. We compute the values of $u(0,1)$ and $u(1.2,0)$, where

$$u(y) = -\int_\Gamma \log|x-y| v(x) ds_x. \qquad (6.2)$$

$u(0,1)$ $(= -\frac{1}{\log 2}\log(\frac{1+\sqrt{2}}{2}))$ and $u(1.2,0)$ $(= -\frac{1}{\log 2}\log(\frac{1.2^2+\sqrt{1.2^2-1}}{2}))$ are the potential at two points on the plane, one on the peependicular bisector of Γ, and one on the axis. Since Γ has open ends, the exact solution $v(x)$ is expected to have singularities of the form $|x - x_0|^{-\frac{1}{2}}$ at the two ends.

Table 1.1. Compute results using φ_2

n	$e_{\max}$	e_{1n}	e_{1n}^E	e_{2n}	e_{2n}^E	e_{Cn}	e_{Cn}^E
2^3	7.290E-2	9.807E-5	1.148E-5	3.868E-4	2.588E-5	3.331E-5	9.568E-6
2^4	5.138E-2	2.212E-6	1.837E-6	2.570E-5	3.000E-6	1.253E-5	1.127E-6
2^5	3.632E-2	1.884E-6	2.258E-7	5.884E-7	1.969E-7	2.553E-6	1.387E-7
2^6	2.569E-2	4.332E-7	2.081E-8	2.458E-7	4.540E-8	4.405E-7	1.727E-8
2^7	1.816E-2	7.876E-8	3.513E-9	7.046E-8	5.670E-9	7.018E-8	2.157E-9
2^8	1.284E-2	1.291E-8	4.390E-10	1.376E-8	7.087E-10	1.066E-8	2.695E-10
2^9	9.084E-3	1.999E-9	5.487E-11	2.341E-9	8.859E-11	1.568E-9	3.369E-11
2^{10}	6.423E-3	2.979E-10	6.859E-12	3.701E-10	1.107E-11	2.255E-10	4.212e-12
2^{11}	4.542E-3	4.324E-11		5.596e-11		3.187E-11	

Table 1.2. Compute results using φ_4

n	$e_{\max}$	e_1	e_{1n}^E	e_{2n}	e_{2n}^E	e_{Cn}	e_{Cn}^E
2^3	1.397E-2	1.816E-4	8.366E-6	3.448E-5	1.143E-5	4.772E-4	6.081E-6
2^4	5.320E-3	3.002E-5	3.047E-7	1.431E-5	4.852E-7	5.433E-5	1.921E-7
2^5	1.956E-3	3.486E-6	9.801E-9	2.214E-6	1.534E-8	6.623E-6	6.106E-9
2^6	7.001E-4	4.272E-7	3.124E-10	2.902E-7	4.886E-10	8.226E-7	1.943E-10
2^7	2.483E-4	5.313E-8	9.951E-12	3.670E-8	1.556E-11	1.026E-7	6.184E-12
2^8	8.791E-5	6.633E-9	3.169E-13	4.601E-9	4.958E-13	1.282E-8	1.966E-13
2^9	3.205E-5	8.289E-10	1.011E-14	5.756E-10	1.628E-14	1.603E-9	6.700E-15
2^{10}	1.813E-5	1.036E-10	5.868E-16	7.196E-11	9.536E-16	2.003e-10	3.447E-16
2^{11}	9.143E-6	1.295E-11		8.996e-12		2.504e-11	

Using the $\sin^p-$transformation φ_2 and φ_4, we compute the results in Table 1.1 and 1.2, respectively. We list the maximal errors $e^n_{\max} = \max_{1\le i\le n} |v - v^h|$ ($n = 2^i$, $i = 3, ..., 11$), where v and v^h are the solutions of (6.1) and the corresponding to discrete equation (3.4) respectively. the errors $e_{1n} = |u(0,1) - u^h(0,1)|$, $e_{2n} = |u(1.2,0) - u^h(1.2,0)|$ and $e_{Cn} = |C_\Gamma - C^h_\Gamma|$ ($n = 2^i$, $i = 3, ..., 11$), and the first extrapolation errors $e^E_{1n} = |8e_{1\frac{n}{2}} - e_{1n}|/7$, $e^E_{2n} = |8e_{2\frac{n}{2}} - e_{2n}|/7$ and $e^E_{Cn} = |8e^E_{C\frac{n}{2}} - e^E_{Cn}|$. Based on (3.1d), when $p = 2$ and 4, $\omega = \min(3/2, 6)$ and $\min(5/2, 10)$ respectively. Hence, there do not exist asymptotic expansions of solution errors, which imply that the extrapolation algorithm is not effective. In Tables 1.1 and 1.2 the numerical results of the second, forth, sixth and eighth column completely agree with our theory.

Table 1.3. Compute results using φ_6

n	$\|\lambda_{\min}\|$	$\|\lambda_{\max}\|$	Cond	Eff_C	$e_{\max}$	$e_{\min}$
2^3	0.179	2.161	12.02	1.108	9.201E-3	3.848E-3
2^4	0.087	2.183	25.01	6.383	1.163E-3	4.456E-5
2^5	0.043	2.188	50.43	12.83	1.455E-4	2.155E-5
2^6	0.021	2.189	101.0	25.70	1.817E-5	5.524E-7
2^7	0.018	2.190	202.2	51.42	2.272E-6	2.128E-8
2^8	0.005	2.190	404.4	102.8	2.841E-7	1.664E-10
2^9	0.002	2.190	808.9	205.7	3.551E-8	8.137E-11

In Table 1.3, we list the minimal eigenvalue $|\lambda_{\min}|$ and maximal eigenvalue $|\lambda_{\max}|$, and the condition numbers, Cond=$\frac{|\lambda_{\max}|}{|\lambda_{\min}|}$, and the effective condition numbers Eff_C=$\frac{ff}{uu\cdot|\lambda_{\min}|}$, as well as the minimal errors $e^n_{\min}$ =$\min_{1\le j\le n} |v - v^h|$, and the maximal errors $e^n_{\max} = \max_{1\le i\le n} |v - v^h|$ ($n = 2^i$, $i = 3, ..., 9$). In Table 1.3, we have

$$\frac{\mathrm{Cond}|_{n=2^{m+1}}}{\mathrm{Cond}|_{n=2^m}} \approx 2 \text{ and } \frac{e_{\max}|_{n=2^{m+1}}}{e_{\max}|_{n=2^m}} \approx 8 \ (m = 3, ..., 8),$$

which indicate (5.2) and (4.1) perfectly.

Table 1.4. Compute results using φ_6

n	2^3	2^4	2^5	2^6	2^7	2^8	2^9
e_{1n}	4.246E-4	2.293E-5	2.895E-6	3.612E-7	4.513E-8	5.641E-9	7.051E-10
e^E_{1n}	8.672E-5	3.228E-8	7.219E-10	2.213E-11	7.044E-13	2.237E-14	
e^{EE}_{1n}	2.764E-6	2.961E-10	4.432E-13	1.325E-14	3.690E-16		
e_{2n}	5.686E-4	6.422E-5	8.034E-6	1.004E-6	1.255E-7	1.568E-8	1.960E-9
e^E_{2n}	7.834E-6	6.871E-9	1.701E-10	9.790E-12	3.451E-13	1.111E-14	
e^{EE}_{2n}	1.813E-8	4.605E-10	4.618E-10	4.042E-14	3.361E-16		
e_{Cn}	6.588e-4	8.120E-5	1.012E-5	1.264E-6	1.580E-7	1.975E-8	2.468E-9
e^E_{Cn}	1.317E-6	3.302E-8	9.465E-10	2.941E-11	9.285E-13	2.890E-14	
e^{EE}_{Cn}	8.398E-9	8.812E-11	1.735E-13	9.741E-15	1.193E-16		

In Table 1.4, we list the errors $e_{1n} = |u(0,1) - u^h(0,1)|$, $e_{2n} = |u(1.2,0) - u^h(1.2,0)|$ and $e_{Cn} = |C_\Gamma - C_\Gamma^h|$ ($n = 2^i$, $i = 3, ..., 9$), and the first extrapolation errors $e_{1n}^E = |8e_{1\frac{n}{2}} - e_{1n}|/7$, $e_{2n}^E = |8e_{2\frac{n}{2}} - e_{2n}|/7$ and $e_{Cn}^E = |8e_{C\frac{n}{2}}^E - e_{Cn}^E|$, as well as the second extrapolation errors $e_{1n}^{EE} = |32e_{1\frac{n}{2}}^E - e_{1n}^E|/31$, $e_{2n}^{EE} = |32e_{2\frac{n}{2}}^E - e_{2n}^E|/31$ and $e_{Cn}^{EE} = |32e_{C\frac{n}{2}}^E - e_{Cn}^E|/31$.

Now, let us examine the numerical data In Table 1.4. we have numerically

$$\frac{e_{1n}|_{n=2^{m+1}}}{e_{1n}|_{n=2^m}} \approx 8, \frac{e_{2n}|_{n=2^{m+1}}}{e_{2n}|_{n=2^m}} \approx 8 \text{ and } \frac{e_{Cn}|_{n=2^{m+1}}}{e_{Cn}|_{n=2^m}} \approx 8 \ (m = 3, ..., 8)$$

to indicate (4.1) perfectly, which imply that the first extrapolation is very effective. However, when $p = 6$, from (3.1d), $\omega = \min(7/2, 14)$. Based on (4.1), there only exists the first extrapolation. For Table 1.4, although e_{1n}^{EE}, e_{2n}^{EE} and e_{Cn}^{EE} are improved, the effectiveness is not remarkably. If we us $\varphi_p(t)$ ($p \geq 10$), these can largely be improved.

Table 1.5. Compute results using φ_6

$y_1 \backslash e_n \backslash n$	2^3	2^4	2^5	2^6	2^7	2^8	2^9
$1+2*10^{-2}$	7.917E-4	*1.148E-4	1.419E-5	1.774E-6	2.216E-7	2.771E-8	3.463E-9
$1+2*10^{-3}$	4.317E-3	3.197E-4	*3.302E-5	4.124E-6	5.154E-7	6.442E-8	8.052E-9
$1+2*10^{-4}$	4.243E-3	2.457E-4	4.689E-5	*5.775E-6	7.217E-7	9.020E-8	1.127E-8
$1+2*10^{-5}$	1.560E-3	7.555E-4	5.690E-5	*6.350E-6	7.938E-7	9.921E-8	1.240E-8
$1+2*10^{-6}$	5.362E-3	1.930E-4	3.171E-5	6.247E-6	*7.734E-7	9.667E-8	1.208E-8
$1+2*10^{-7}$	7.066E-3	1.492E-4	8.266E-5	6.028E-6	*7.068E-7	8.837E-8	1.104E-8
$1+2*10^{-8}$	6.913E-3	4.367E-4	5.651E-5	3.181E-6	6.263E-7	*7.804E-8	9.754E-9
$1+2*10^{-9}$	6.764E-3	6.116E-4	1.604E-5	7.841E-6	5.515E-7	*6.765E-8	8.454E-9
$1+2*10^{-10}$	6.706E-3	5.991E-4	3.454E-5	3.763E-6	3.114E-7	5.796E-8	*7.250E-9
$1+2*10^{-11}$	6.687E-3	5.846E-4	5.330E-5	1.703E-6	6.970E-7	4.872E-8	*6.172E-9
$1+2*10^{-12}$	6.681E-3	5.789E-4	5.254E-5	2.640E-6	1.128E-7	3.131E-8	5.168E-9
$1+2*10^{-13}$	6.679E-3	5.770E-4	5.114E-5	4.660E-6	1.723E-7	5.951E-8	4.038E-9
$1+2*10^{-14}$	6.678E-3	5.763E-4	5.058E-5	4.644E-6	1.906E-7	1.803E-8	2.147E-9

Below we calculate the values of $u(y_1, y_2)$ at the neighbor of singular point $(1,0)$. From (6.2), for $|y_1| > 1$ and $y_2 = 0$, exact solution $u(y_1, y_2)$ is $u(y_1, 0)$ $= -\frac{1}{\log 2} \log \frac{y_1 + \sqrt{y_1^2 - 1}}{2}$. In Table 1.5, we list the errors $e_n = |u(y_1, 0) - u^h(y_1, 0)|$ ($n = 2^i$, $i = 3, ..., 9$). From Table 1.5, as $y_1 \to 1$, the approximate accuracy is still $O(h_0^3)$, which does not happen.

Next, we compute the derivable values $\frac{\partial u(y_1, y_2)}{\partial y_1}$ at the neighbor of singular point $(1,0)$. From (6.2), at points $y = (y_1, 0)$ ($|y_1| > 1$), we derive

$$\frac{\partial u(y_1, 0)}{\partial y_1} = -\int_{-1}^{1} \frac{1}{y_1 - x_1} v(x_1) dx_1$$

$$= -\frac{1}{\pi \log 2} \int_{-1}^{1} \frac{1}{y_1 - x_1} \frac{1}{\sqrt{1-x_1^2}} dx_1 = -\frac{1}{\log 2} \frac{1}{\sqrt{y_1^2 - 1}}.$$

Table 1.6. Compute results using φ_6

n	$e\backslash y_1$	1+10^{-1}	1+10^{-2}	1+10^{-3}	1+10^{-4}	1+10^{-5}
2^5	e_5	1.950E-6	8.245E-5	1.953E-4	1.577E-4	4.685E-3
2^6	e_6	2.438E-7	1.029E-5	2.535E-5	3.220E-5	1.002E-5
	e_6^E	4.979E-11	1.258E-8	1.071E-6	5.933E-5	6.578E-4
2^7	e_7	3.046E-8	1.286E-6	3.167E-6	4.030E-6	1.096E-6
	e_7^E	2.397E-11	6.086E-10	1.691E-9	4.814E-9	1.792E-7
2^8	e_8	3.806E-9	1.607E-7	3.958E-7	5.036E-7	1.372E-7
	e_8^E	1.004E-12	2.164E-11	6.886E-11	1.414E-10	1.523E-10
2^9	e_9	4.758E-10	2.009E-8	4.948E-8	6.294E-8	1.716E-8
	e_9^E	3.484E-14	7.150E-13	2.391E-12	6.265E-12	2.085E-11

Table 1.7. Compute results using φ_6

n	$e\backslash y_1$	1+10^{-6}	1+10^{-7}	1+10^{-8}	1+10^{-9}	1+10^{-10}
2^5	e_5	3.677E-2	1.265E-1	1.381E-1	7.215E-1	1.103E+0
2^6	e_6	5.760E-4	2.827E-3	2.688E-2	9.648E-2	1.790E-1
	e_5^E	5.912E-3	1.485E-2	5.046E-2	2.133E-1	3.622E-1
2^7	e_7	2.625E-5	1.052E-4	5.247E-4	6.163E-4	1.597E-2
	e_6^E	5.229E-5	2.835E-4	4.440E-3	1.307E-2	4.382E-2
2^8	e_8	3.276E-6	1.374E-5	4.474E-5	1.278E-4	3.544E-4
	e_7^E	6.062E-9	6.718E-7	2.382E-5	5.807E-5	2.686E-3
2^9	e_9	4.095E-7	1.719E-6	5.600E-6	1.655E-5	4.687E-5
	e_8^E	8.834E-11	7.462E-10	8.965E-9	6.505E-7	2.936E-6

In Table 1.6 and 1.7, we list the errors $e_n = |\frac{\partial u(y_1,0)}{\partial y_1} - \frac{\partial u^h(y_1,0)}{\partial y_1}|/|\frac{\partial u(y_1,0)}{\partial y_1}|$ ($n = 2^m$, $m = 5, 6, 7, 8, 9$) and the extrapolation errors $e_n^E = |8e_{\frac{n}{2}} - e_n|/7$ at the neighbor of singular point $(1, 0)$.

Example 2[28]**.** Let Γ be an open contour of length 2, in the form of a right-angled wedge:

$$\Gamma = \{(x_1, 0) : 0 \le x_1 \le 1\} \cup \{(0, x_2) : 0 \le x_2 \le 1\}.$$

The integral equation is chosen as

$$-\int_{\Gamma} \ln |y - x| v(x) ds_x = 1, \text{ for } (y_1, y_2) \in \Gamma. \tag{6.3}$$

We compute the numerical solution of

$$u(y) = -\int_{\Gamma} \ln |y - x| v(x) ds_x$$

at $(0.5, 0.5)$, whose true value $u(0.5, 0.5)$ takes 0.621455343.

From [28], although the exact solution $v(x)$ is expected to have a $O(|x - x_0|^{-\frac{1}{3}})$ singularity at the right-angled corner, the dominant singularities in $v(x)$ occur at the two ends, with $O(|x - x_0|^{-\frac{1}{2}})$. Based on [15,16], using $\varphi_6(t)$ in the periodical transformation (5.3), we obtain the numerical results at $Q = (0.5, 0.5)$ by MQMs and list Cond. and Cond-eff in Tables 3.1. Let n_m $(m = 1, 2)$ be the number of uniform partition on $[0, 1]$ corresponding to the mth edge Γ_m of Γ. Based on (5.9), we can obtain the splitting extrapilation errors $e^E(Q) = |u_E(Q) - u(Q)|$, where

$$u_E(Q) = \frac{8}{7}[\sum_{m=1}^{d} u_{h^{(m)}}(Q) - (d - \frac{7}{8})u_{h^{(0)}}(Q)],\ d = 2,$$

is the splitting extrapilation values. The errors $|u^h(Q) - u(Q)|$ and the splitting extrapilation errors $e^E(Q)$ are also listed in Table 2.1, where $(n_1, n_2) = (8, 8)$ and $(16, 16)$.

Table 2.1. The errors, Cond. and Cond_eff for (6.3).

(n_1, n_2)	$\lvert u^h - u\rvert$	$\lvert\lambda_1\rvert$	$\lvert\lambda_n\rvert$	Cond.	Cond-eff
(4,4)	4.413E-2	0.104	4.287	40.865	12.042
(8,4)	2.166E-2	0.113	4.147	36.420	11.979
(4,8)	2.166E-2	0.113	4.147	36.420	12.049
e^E	7.229E-3				
(8,8)	1.738E-3	0.055	4.374	78.474	22.395
(16,8)	9.452E-4	0.056	4.312	76.007	23.820
(8,16)	9.452E-4	0.056	4.312	76.007	23.856
e^E	7.495E-5				
(16,16)	1.383E-4	0.028	4.378	154.828	44.046
(32,16)	7.805E-5	0.028	4.357	153.680	47.609
(16,32)	7.805E-5	0.028	4.357	153.680	47.618
e^E	5.184E-7				
(32,32)	1.725E-5	0.014	4.375	308.348	87.754
(64,32)	9.703E-6	0.014	4.369	308.234	95.208
(32,64)	9.703E-6	0.014	4.369	308.234	95.210
e^E	1.350E-9				

From Table 2.1 we have

$$\frac{|u^h - u|_{(4,4)}}{|u^h - u|_{(8,8)}} = 25.39, \frac{|u^h - u|_{(8,8)}}{|u^h - u|_{(16,16)}} = 12.56 \text{ and } \frac{|u^h - u|_{(16,16)}}{|u^h - u|_{(32,32)}} = 8.01, \quad (6.4)$$

Hence, the SEMs can provide more accurate solutions. Note that from Table 2.1 with the total number $n = \sum_{m=1}^{2} n_m = 32$ and 64, the error of SEMs is $5.184E$-7 and $1.350E - 9$, respectively. In contrast, when $n = 256$ the $u^h = 0.62125$ is given in [28] by Galerkin methods, where the approximating space S^h is the piecewise constant space. This fact displays the efficiency of MQMs and SEMs.

Since all corners of Γ are right-angled, it follows from results that in [27,28] $v(x)$ is expected to have singularities of the form $|x-x_0|^{-1/3}$ at the corners in the exterior. By Galerkin methods, the errors are $||v_1^h - v_1||^2 = O(h^{4/3})$ and $|u_1^h - u_1|^2 = O(h^{4/3})$. However, for the case in which the point x lies in the interior, $v(x)$ is much less singular, that is, the singularities are only of the form $|x-x_0|$. By Galerkin methods, the errors are $||v_2^h - v_2||^2 = O(h^3)$ and $|u_2^h - u_2| = O(h^{13/6})$.

To close this paper, let us make a few concluding remarks.

1. The above numerical results show that the MQMs not only possess high accuracy, but also h^3-Richaedson extrapolation or SEM and a posteriori error estimate are very effective. Since the discrete matrix of BIE is full, by using SEM the larger the scale of problems are, the more effective the methods are. These results further verify that it is reasonable to construct the MQMs in the paper and it is correct to give the convergence theory. Especially, the results in tables display that the approximate accuracies are very high. It further shows that the accuracy order can be largely enhanced by h^3-Richaedson extrapolation or SEM.

2. To the MQM there exist the following advantages: (1) each elements of discrete matrixes calculated are very simple and straightforward, not need calculate any singular integrals; (2) it is a high accuracy algorithm $O(h^3)$. However, the theoretic study of the MQM is more difficult than that of Galerkin and collocation methods, because its theory is no longer within the framework of the projection theory. In the present paper we only discuss Dirichlet's problems, and to mixed boundary problems or Neumann's problems we can also establish the corresponding to algorithms by using these results.

3. This paper explores the traditional stability analysis, the traditional Cond. $= O(h^{-1})$. The small bounds of condition number are significant to stability of numerical the first kind boundary integral equations for Laplace's equations. This paper is the first time to explore stability analysis for the open contour, and to derive (5.2), which grants the MQMs an excellent stability.

4. Numerical experiments are carried out for the arbitrary boundary Γ with $C_\Gamma \neq 1$ by MQM and SEM, and the computed results coincide with the new stability analysis perfectly.

5. The extrapolation and SEM techniques are applied to the first kind boundary integral equations, to improve the solution accuracy.

7 References

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