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Plastic collapse in presence of non-linear kinematic hardening by the bipotential and the sequential limit analysis approaches

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ABSTRACT

The limit analysis approach as a direct method limit loads assessment of plastic deformable bodies is followed up to deal with plastic collapse of non-linear kinematic strain hardening materials. For this goal, the kinematic and static approaches of modern limit analysis are adopted. The non-linear kinematic hardening law is a non-associative plastic flow rule, but, it can be described by the bipotential concept. Based on this, an extension of the limit analysis approach is proposed. Limit loads, back stresses and other variables are assessed at each step by the sequential limit analysis method. Large plastic deformation could be taken into consideration by updating geometry after each sequence.

1. Introduction

In plasticity areas, incremental elastoplastic methods, which are deduced from continuous mechanics, and linked with plastic flow rules, lead to the determination of stresses and strains over a loading history up to the achievement of a flow mechanism. This latter is a limit state, well-known as a plastic collapse, which can be obtained directly by limit analysis methods without using cumbersome intermediate computations required by incremental methods. The limit analysis procedures have found wide applications in metal forming processes optimization, geotechnics and structural engineering optimal design. However, one of the main upsetting restrictive hypothesizes on which limit analysis approach is based is the perfect plastic behavior of the constitutive material. In order to include more accurate behaviors than perfectly plastic ones, there is no doubt that an attempt of limit analysis approach extending is of value. Moreover, we cannot deny that during plastic deformation of materials, strain hardening is one of the most important behaviors that takes place, such as isotropic hardening and linear or non-linear kinematic strain hardening. On the other hand, considering or not the strain hardening, we can note that the majority of plastic flow rules are non-associative in the sense that strain rates or kinematic hardening variables are not normal to the yield criterion. On the basis of the bipotential concept (De Saxcé, 1992), a previous extension of the limit analysis approach to the case of non-associative flow rules was presented in De Saxcé and Bousshine (1998) while its numerical treatment using the finite element method for the non-associative Drucker-Prager's model was presented in Chaaba et al. (2010).

Using sequential limit analysis, some authors (see e.g. Horne and Merchant, 1965; Hwan, 1997; Leu, 2003; Leu, 2005; Leu and Chen, 2006; Corradi and Panzeri, 2004; Seitzberger and Rammerstorfer, 1999; Yang, 1993; Huh et al., 1999; Huh et al., 2001; Kim and Huh, 2006; Leu, 2007) extended the conventional limit analysis concept to isotropic strain hardening materials instead of perfectly plastic ones, and small deformations to large ones. The idea of sequential limit analysis considers that the plastic collapse evolution is a sequence of conventional limit analyses with a yield stress updated at each one. By adopting a similar reasoning, as it has been done for the isotropic hardening case, this paper is dedicated to deal with limit analysis approaches in the presence of kinematic hardening, and in particular, the non-linear kinematic hardening law. This latter was proposed first by Armstrong and Frederic (Armstrong and Frederick, 1966) then developed more by Lemaître and Chaboche (1990) and Marquis (1979).

It is well known that in the framework of plasticity, the non-linear kinematic hardening belongs to the set of non-associative plastic laws that can be described by means of the bipotential concept (De Saxcé, 1992). For the plastic behavior with non-linear kinematic hardening, the bipotential is a theoretical powerful tool that helps us to build up a framework for both the plastic collapse and load factors definitions, in the sense of the limit analysis; whereas, the sequential limit analysis offers a computational procedure. In the framework of the sequential limit analysis, according to the von Mises criterion with isotropic hardening and by using the principle of virtual powers, the limit load factor at any sequence is defined as the total plastic dissipation over the volume divided by the power of the external reference loads. In order to extend this

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idea of sequential limit analysis to the case of kinematic hardening, and in particular to the non-linear case with the law of Armstrong-Frederick, we adopt the same reasoning as that used for the case of the isotropic hardening. Instead of the yield surface expansion associated to the isotropic hardening, we consider the yield surface translation through back stresses. Before closing this introduction, it is worthy to note that on the basis of the bipotential concept, the shakedown theorems were extended to the Armstrong-Frederick's non-linear kinematic hardening model as a non-associative law (Bouby et al., 2006; Bouby, 2006).

2. Non-linear kinematic hardening rule

During the process of plastic deformation of materials, hardening phenomena appear in terms of expansion, translation in the deviatoric plan, or both simultaneously of the yielding surface. The expansion of this surface is connected to the isotropic hardening whereas the translation is related to the kinematic hardening. In order to take into consideration the kinematic hardening phenomenon, a second order tensor denoted by X (known as back stresses), and by duality, kinematic internal variables denoted by α (as a second order tensor too) are introduced. The non-linear kinematic hardening model of Armstrong and Frederic is characterized by three material parameters such as, the plasticity threshold R identical to the yield stress, hardening modulus C, and a saturation value X_∞. Let $\dot{\kappa} = (\dot{\varepsilon}, -\dot{\alpha}, -\dot{p})$ be the generalized velocities, which is regrouping the plastic strain rate $\dot{\varepsilon}$, kinematic $\dot{\alpha}$ and isotropic $\dot{p}$ hardening variables rates. Dual variables $\pi = (\sigma, X, R)$ are respectively the stress tensor σ, the back-stress tensor X and the threshold R. Let K_π be the set of plastically admissible stresses defined according to the von Mises criterion:

$$K_\pi = \left\{(\sigma, X, R) \quad \text{such as} \quad \sigma_{eq}(\sigma - X) - R \le 0\right\} \tag{1}$$

where $\sigma_{eq}(\sigma - X)$ is the usual norm defined as follows:

$$\sigma_{eq}(\sigma - X) = \sqrt{\tfrac{3}{2}(\sigma - X)'' : (\sigma - X)''}, \tag{2}$$

and (...)" means the devitoric part. Let $\varepsilon_{eq}(\dot{\varepsilon})$ be the equivalent strain rate defined as $\varepsilon_{eq}(\dot{\varepsilon}) = \sqrt{\frac{2}{3}\dot{\varepsilon} : \dot{\varepsilon}}$, since the threshold $R > 0$, the isotropic hardening rule implies:

$$\varepsilon_{eq}(\dot{\varepsilon}) = \dot{p}. \tag{3}$$

The non-linear kinematic hardening rule, introduced by Armstrong and Frederick (1966), developed and extensively used by Lemaître and Chaboche (1990) and Marquis (1979), consists in introducing a non-linearity relation between the evolution of back stresses and plastic strain rates as follows:

$$\dot{\alpha} = \dot{\varepsilon} - \frac{3}{2}\frac{X}{X_\infty}\dot{p}, \tag{4}$$

where back-stresses X and kinematic hardening variables α tensors are linearly related as:

$$X = \tfrac{2}{3}C\alpha. \tag{5}$$

On the other hand, it was shown in De Saxcé (1992), Bouby et al. (2006) and Bouby (2006) that the non-linear kinematic hardening law of Armstrong and Frederick is non-associative, and it may be represented by means of the bipotential concept. According to its definition, the bipotential is a scalar function that satisfies the fundamental inequality, which generalizes the one of Legendre–Fenchel (Fenchel, 1949):

$$\text{for any fields}: \quad (\dot{\kappa}^*, \pi^*) \quad b(\dot{\kappa}^*, \pi^*) \ge \dot{\kappa}^*.\pi^* \tag{6}$$

where "." means a bilinear form defined from the spaces of generalized velocities and forces into $]-\infty, +\infty[$: $(\dot{\kappa}^*, \pi^*) \to \dot{\kappa}^*.\pi^*$. From the point of view of the bipotential concept, the plastic flow rule can be written as an implicit normality law (De Saxcé, 1992):

$$\dot{\kappa} \in \partial_\pi b(\dot{\kappa}, \pi), \quad \pi \in \partial_{\dot{\kappa}} b(\dot{\kappa}, \pi), \tag{7}$$

where the symbol ∂_π ($\partial_{\dot{\kappa}}$ respectively) is the subdifferential with respect to scalar variables π (respectively $\dot{\kappa}$). The quantity $b(\dot{\kappa}, \pi)$ associated to the non-linear kinematic hardening rule of Armstrong-Frederick is a scalar function that is given by the following expression (De Saxcé, 1992; Bouby et al., 2006; Bouby, 2006):

$$b(\dot{\kappa}, \pi) = \frac{\left(\sigma_{eq}(X)\right)^2}{X_\infty}\dot{p} \tag{8}$$

where the admissibility conditions (1), (3), and (5) are satisfied. Finally, a couple $(\dot{\kappa}, \pi)$ that reaches the equality in (6), it is qualified to be extremal in the sense that:

$$b(\dot{\kappa}, \pi) = \dot{\kappa}\pi = \sigma : \dot{\varepsilon} - X : \dot{\alpha} - R\dot{p}. \tag{9}$$

3. Limit analysis procedure

3.1. Plastic collapse and load factors in presence of strain hardening

Let Ω be a deformable body made up of a rigid plastic material obeying to the non-linear kinematic hardening rule. It is subjected to proportional body forces $\bar{f} = \mu\bar{f}^0$ and surface tractions $\bar{t} = \mu\bar{t}^0$ on the part S_1 of the boundary S where $\bar{t}^0$ and $\bar{f}^0$ are fixed and refer to the reference loading; μ is a non negative number called load factor. On the other hand, the body is subjected to zero velocity via the second part S_0 of the whole boundary S. Kinematically admissible (k.a.) velocity field and statically admissible (s.a.) stress field, respectively denoted by $\dot{u}^k$ and σ^s, have to separately fulfill compatibility conditions ($\dot{\varepsilon}(\dot{u}^k) = grad_s\dot{u}^k$ within Ω, and $\dot{u}^k = 0$ on S_0) and equilibrium equations ($div\sigma^s + \mu\bar{f}^0 = 0$ within Ω and $t(\sigma^s) = \bar{t} = \mu\,\bar{t}^0$ on the boundary S_1).

Making use of the sequential limit analysis concept, we are going to use a similar reasoning adopted previously in the case of the non-associated limit analysis with the Drucker–Prager's model (De Saxcé and Bousshine, 1998; Chaaba et al., 2010) to define the plastic limit state and the limit loading. Recalling that, using the rigid-perfectly plastic behavior, the sequential limit analysis concept consists in solving a succession of conventional limit analysis problems. If the isotropic hardening is taken into consideration, the yield stress will be updated according to the adopted hardening law. In the same line of reasoning, this section aims at applying the sequential limit analysis to the kinematic hardening case, especially to the non-linear one. The idea developed here consists in setting up a sequence of rigid perfectly plastic limit analysis problems by considering at each sequence a new yield surface such as $f(\sigma - X) = \sigma_{eq}(\sigma - X) - R \le 0$. On the basis of this point of view, at any sequence of the plastic collapse evolution, the non-linear kinematic hardening plastic behavior has a limit surface defined by its position that is located by the back stresses X and its magnitude R. Then, such a plastic limit state of the body may be reached by performing a classical elastoplastic analysis as it has been done for the case of rigid perfectly plastic behavior. This leads to define the plastic collapse and the limit loading in presence of kinematic hardening, at each sequence of the evolution, by a similar way as in the case of rigid perfectly plastic behavior, however, instead of the yield surface $f(\sigma) \le 0$, we adopt the translated one in space stresses such as $f(\sigma - X) \le 0$. At each step of the collapse evolution, let $\bar{t}^l$ and $\bar{f}^l$ be respectively this limit loading of surface tractions and body forces causing the so-called plastic collapse. This latter is kinematically characterized by the set of variables $(\dot{\varepsilon}, -\dot{\alpha}, -\dot{p})$ and statically characterized by their dual variables (σ, X, R), or in a more compact form by the field couple $(\dot{\kappa}, \pi)$. Note that for this limit state,

$\dot{u}$ and $\dot{\varepsilon}$ are related by strain-displacement equations; $\dot{\kappa}$ and π are related via the constitutive relations including the non-linear kinematic hardening laws. In other words, the couple $(\dot{\kappa}, \pi)$ is extremal for the bipotential b. And so, the plastic dissipation density $\sigma : \dot{\varepsilon}$ equals the sum of individual contributions of $b(\dot{\kappa}, \pi)$, $X : \dot{\alpha}$ and $R.\dot{p}$ (i.e. $\sigma : \dot{\varepsilon} = b(\dot{\kappa}, \pi) + X : \dot{\alpha} + R.\dot{p}$). The principle of virtual powers for this so-called plastic collapse can be stated as:

$$\int_{\Omega} (b(\dot{\kappa}, \pi) + X : \dot{\alpha} + R\dot{p})\, d\Omega = \int_{S_1} \bar{t}^l \dot{u} dS + \int_{\Omega} \bar{f}^l \dot{u} d\Omega. \tag{10}$$

Limit load factor: Let μ^l be the load factor such that $\bar{t}^l = \mu^l \bar{t}^0$ and $\bar{f}^l = \mu^l \bar{f}^0$; the positive number μ^l, can be defined unambiguously through the previous equation as follows:

$$\int_{\Omega} (b(\dot{\kappa}, \pi) + X : \dot{\alpha} + R\dot{p})\, d\Omega = \mu^l \left(\int_{S_1} \bar{t}^0.\dot{u} dS + \int_{\Omega} \bar{f}^0.\dot{u} d\Omega \right). \tag{11}$$

μ^l is called limit load factor. Thanks to the fact that $\dot{u}$ is a plastically admissible velocity field in the sense that $\int_{S_1} \bar{t}^0.\dot{u} dS + \int_{\Omega} \bar{f}^0.\dot{u} d\Omega > 0$:

$$\mu^l = \frac{\int_{\Omega} (b(\dot{\kappa}, \pi) + X : \dot{\alpha} + R\dot{p})\, d\Omega}{\int_{S_1} \bar{t}^0.\dot{u} dS + \int_{\Omega} \bar{f}^0.\dot{u} d\Omega}. \tag{12}$$

Kinematic load factor: analogically, associated to a velocity field $\dot{u}^k$ characterizing an admissible mechanism for what the condition $\int_{S_1} \bar{t}^0.\dot{u}^k dS + \int_{\Omega} \bar{f}^0.\dot{u}^k d\Omega > 0$ is assumed to be fulfilled, the kinematic load factor μ^k is defined by the following relation:

$$\mu^k = \frac{\int_{\Omega} \left(b(\dot{\kappa}^k, \pi) + X : \dot{\alpha}^k + R\dot{p}^k \right) d\Omega}{\int_{S_1} \bar{t}^0.\dot{u}^k dS + \int_{\Omega} \bar{f}^0.\dot{u}^k d\Omega}. \tag{13}$$

Thanks to the homogeneous feature of the bipotential function, homogeneity of order one, with respect to a velocity field $\dot{u}^k$ in the sense that, a mechanism velocity $\dot{u}^k$ can be multiplied by any $\lambda > 0$ without changing the definition, the following normalization condition could be considered as:

$$\int_{S_1} \bar{t}^0.\dot{u}^k dS + \int_{\Omega} \bar{f}^0.\dot{u}^k d\Omega = 1. \tag{14}$$

Static load factor: In the static approach, we define the static load factor as it has been defined in the classical limit analysis for rigid perfectly plastic material through the static equilibrium conditions and the plasticity criterion. If σ^s is a statically admissible field, the static limit load factor μ^s is defined by the set of the following conditions:

$$\begin{aligned} &div\sigma^s + \mu^s \bar{f}^0 = 0 \ \text{within}\ \Omega \quad \text{and} \quad t(\sigma^s) = \sigma^s.n = \bar{t} = \mu^s \bar{t}^0 \ \text{on}\ S_1 \\ &f(\sigma^s - X) \le 0 \ \text{within}\ \Omega \end{aligned} \tag{15}$$

Two bound theorems underpin the background of the limit analysis approach and make up a framework of the effective application of this approach to practical engineering problems. These basic theorems, which provide lower and upper bounds of the true limit loads, are known as static theorem or lower bound theorem and kinematic or upper bound theorem. The remainder of this paragraph will be devoted to the presentation of these two bound theorems: the kinematic/upper and static/lower ones.

3.2. Kinematic approach in limit analysis sense

On the basis of the bipotential theory, our goal here consists in proposing a kinematic approach in the limit analysis sense to include the non-associated non-linear kinematic hardening flow rule. So, first, as the limit state $(\dot{\kappa}, \pi)$ satisfies the constitutive law (7), for any admissible field $\dot{\kappa}^k$, the definition (6) leads to $b(\dot{\kappa}^k, \pi) - b(\dot{\kappa}, \pi) \ge \pi : (\dot{\kappa}^k - \dot{\kappa})$ in Ω. Since the field σ is statically admissible (with the limit loading: $\bar{t}^l = \mu^l \bar{t}^0$ and $\bar{f}^l = \mu^l \bar{f}^0$), $(\dot{u}^k - \dot{u})$ is kinematically admissible, and by applying the principle of virtual powers, the following inequality holds:

$$\begin{aligned} &\int_{\Omega} \left(b(\dot{\kappa}^k, \pi) + X : \dot{\alpha}^k + R.\dot{p}^k \right) d\Omega - \int_{\Omega} (b(\dot{\kappa}, \pi) + X : \dot{\alpha} + R.\dot{p})\, d\Omega \\ &\ge \mu^l \left(\int_{S_1} \bar{t}^0 (\dot{u}^k - \dot{u}) dS + \int_{\Omega} \bar{f}^0.(\dot{u}^k - \dot{u}) d\Omega \right) \end{aligned} \tag{16}$$

Second, taking into account the two equations (12) and (13), which respectively define the limit load factor and the kinematic one, the following inequality holds:

$$\mu^k \left(\int_{S_1} \bar{t}^0 \dot{u}^k dS + \int_{\Omega} \bar{f}^0 \dot{u}^k d\Omega \right) \ge \mu^l \left(\int_{S_1} \bar{t}^0 \dot{u}^k dS + \int_{\Omega} \bar{f}^0 \dot{u}^k d\Omega \right) \tag{17}$$

Because the field velocity $\dot{u}^k$ is plastically admissible in the sense that $\int_{S_1} \bar{t}^0 \dot{u}^k dS + \int_{\Omega} \bar{f}^0 \dot{u}^k d\Omega > 0$, the integrals in the two sides of (17) are strictly positive. Thus the following proposition states:

Let μ^l be a limit multiplier associated to a limit state $(\dot{\kappa}, \pi)$, and μ^k is a multiplier associated to an admissible velocity field $\dot{\kappa}^k$: $\mu^k \ge \mu^l$.

The kinematic approach of limit analysis is usually set up as a minimization problem; this is also the case of the present proposition. In fact, thanks to the above inequality and taking into consideration the normalization condition (14) and the load factors definitions (12) and (13), the limit state can be determined by the solving of the following non-linear mathematical program:

Inf $\int_{\Omega} (b(\dot{\kappa}, \pi) + X : \dot{\alpha} + R\dot{p})\, d\Omega$ subjected to:

$$\begin{cases} \varepsilon_{eq}(\dot{\varepsilon}) = \dot{p};\ \dot{\alpha} = \dot{\varepsilon} - \dfrac{3}{2} \dfrac{X}{X^{\infty}} \dot{p} \ \text{ and } \ X = \dfrac{2}{3} C\alpha \\ \int_{S_1} \bar{t}^0 \dot{u}^k dS + \int_{\Omega} \bar{f}^0 \dot{u}^k d\Omega = 1 \ \text{ and } \ \dot{u} = 0 \ \text{on}\ S_0 \end{cases} \tag{18}$$

The plasticity threshold R is considered to be identical to the yield stress σ_Y; it is easy to see that the case with rigid perfectly plastic behavior will be reduced to the problem of classical limit analysis, excluding any effect of hardening:

$$\begin{aligned} &Inf \int_{\Omega} \sigma_Y \dot{p} d\Omega \\ &\text{subjected to} : \left(\int_{S_1} \bar{t}^0 \dot{u}^k dS + \int_{\Omega} \bar{f}^0 \dot{u}^k d\Omega \right) = 1 \ \text{and}\ \dot{u} = 0 \ \text{on}\ S_0 \end{aligned} \tag{19}$$

3.3. Static approach in limit analysis sense

For the static approach, In order to include the non-associated non-linear kinematic hardening flow rule, and on the basis of the bipotential theory, our goal here consists in proposing a static formulation in the limit analysis sense. The limit state is always characterized by the couple $(\dot{\kappa}, \pi)$ which satisfies the constitutive law (7) with the bipotential (8). So, for any admissible field $\pi^s = (\sigma^s, X^s, R^s)$:

$$b(\dot{\kappa}, \pi^s) - b(\dot{\kappa}, \pi) \ge \dot{\kappa}.(\pi^s - \pi) \ \text{in}\ \Omega \tag{20}$$

By developing the right side of the inequality: $\dot{\kappa}.(\pi^s - \pi) = \dot{\varepsilon} : (\sigma^s - \sigma) - \dot{\alpha} : (X^s - X) - \dot{p}(R^s - R)$, and taking into account that $R^s = R$, the following inequality holds:

$$\left(b(\dot{\kappa}, \pi^s) + \dot{\alpha} : X^s\right) - (b(\dot{\kappa}, \pi) + \dot{\alpha} : X) \geq \dot{\varepsilon} : (\sigma^s - \sigma) \quad (21)$$

Because $\dot{u}$ is kinematically admissible, and $(\sigma^s - \sigma)$ is statically admissible, the principle of virtual powers enables us to replace the quantity $\int_\Omega \dot{\varepsilon} : (\sigma^s - \sigma) d\Omega$ by $\left(\mu^s - \mu^l\right)\left(\int_{S_1} \bar{t}^0.\dot{u} dS + \int_\Omega \bar{f}^0.\dot{u} d\Omega\right)$. By integrating (21) over the volume Ω, and considering the normalization condition (14), we get the following inequality:

$$\mu^l - \int_\Omega (b(\dot{\kappa}, \pi) + X : \dot{\alpha}) d\Omega \geq \mu^s - \int_\Omega \left(b(\dot{\kappa}, \pi^s) + X^s : \dot{\alpha}\right) d\Omega \quad (22)$$

On the basis of the last inequation, the static approach in the limit analysis sense can be stated as a maximization problem:

$$Sup\, \mu^s - \int_\Omega \left(b(\dot{\kappa}, \pi^s) + X^s : \dot{\alpha}\right) d\Omega; \quad Subjected\ to : \pi^s \in K_\pi, \quad \mu^s\ associated\ to\ \pi^s \quad and \quad \int_{S_1} \bar{t}^0.\dot{u}\, dS + \int_\Omega \bar{f}^0.\dot{u}\, d\Omega = 1 \quad (23)$$

It can be observed that in the case of strain hardening absence, the maximization problem (23) will be reduced to the well known one of classical limit analysis, operating with perfectly plastic behavior.

4. Solving methodology of the bound problems

The solving of the upper bound problem (18) and the lower bound one (23) will not be directly obtained, but, it will be done by making use of the sequential limit analysis approach as follows. For reasons of simplicity, we consider only the case of the upper bound problem (18). The first step consists in solving a classical upper bound limit analysis problem without taking into account any kind of strain hardening. In other words, we solve first the mathematical program (19) with the rigid perfectly plastic model. In the following sequences, both kinematic strain hardening variables and geometry for successive configurations are updated by using displacement velocity field that is obtained from the previous sequence. For the new configuration, back stresses tensor X and their dual variables α, are computed and/or updated by calculating, first, the strain rate $\dot{\varepsilon}$, from the displacement velocity of the previous sequence, then $\dot{\varepsilon}$ is multiplied by a small enough time increment Δt such as:

$$\Delta\alpha = \dot{\alpha}\, \Delta t \ \text{ and } \alpha = \alpha_0 + \Delta\alpha; \ \Delta X = \tfrac{2}{3} C \Delta\alpha \ \text{ and } X = X_0 + \Delta X \quad (24)$$

where α_0 and X_0 are the initial values of the two tensors α and X, which are set to equal zero for the first step. Viewed as parameters only, the obtained new values of back stresses are introduced in the mathematical programming (18). Once again, the solving of this minimization problem provides a new limit load and a new displacement velocity vector. These new results permit the computation of kinematical hardening variables α and X using Eq. (24) again. When large deformation is considered, the geometry will be updated by multiplying the velocity vector with the same pseudo-time increment Δt, as it was used for the updating of hardening variables in (24).

5. Concluding remarks

As a direct computational method by the use of kinematic and static approaches, the application of limit analysis, to non-associative kinematic hardening plastic materials, has been investigated. The bipotential concept enables us to model the non-associative character of the plastic hardening flow rule, and the sequential limit analysis helps us to make up sequences for strain hardening and large plastic deformations computation. Using the bipotential properties and the limit analysis approach, the problem of plastic collapse considering the non-linear kinematic hardening is represented by a non-linear minimization problem (kinematic approach) and a non-linear maximization problem (static approach). The solving of these two bound problems is first, based on the classical solution with the perfectly plastic model without taking into account any kind of strain hardening. Second, the evolution of the limit load and the hardening variables are iteratively computed as described in the section 4 above. Using finite element method, some algorithmic schemes that could deal with the numerical processing of this proposal are published before in Chaaba et al. (2010), Chaaba et al. (2003) and Chaaba and Bousshine (2009).

Before ending this conclusion, we make two remarks. First, the solving of the bounding problems (18) and (23), within the context of sequential limit analysis, is iterative, and the convergence issue of the algorithm may arise. A tentative answer to this question could be found in the applying of the proposed approach for various examples by testing the influence of the size of time increment on the convergence aspect. We have noticed no problem with the convergence of the iterative solution for all examples and applications we have studied, providing that the time increment be small enough in order to realize the small deformation hypothesis. Otherwise, a complete and comprehensive proof of the convergence matter is so far an open question. Second, a word regarding the difference between the isotropic and kinematic hardening in the sequential limit analysis context is in order. Actually, during the plastic deformation process, the consideration of the kinematic hardening implies the existence of back-stresses and dual kinematic variables that need to be updated at each step. However, in the case of the use of isotropic hardening, only the threshold yield stress changes with respect to the accumulated plastic strain. Furthermore, if the nonlinear kinematic hardening is considered with a yield stress that varies with respect to the accumulated plastic strain, the isotropic hardening will be as a particular case of the kinematic one.

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