A Boundary Element Algorithm for the Dirichlet Eigenvalue Problem of the Laplace Operator

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Abstract A novel boundary element method for the solution of the interior Dirichlet eigenvalue problem for the Laplace operator is presented and analyzed. Hereby, the linear eigenvalue problem for the partial differential operator is transformed into a nonlinear eigenvalue problem for an associated boundary integral operator. This nonlinear eigenvalue problem is solved by using a Newton scheme. We discuss the convergence and the boundary element discretization of this algorithm, and give some numerical results.

1 Introduction

We consider the interior Dirichlet eigenvalue problem of the Laplace operator,

$$-\Delta u(x) = \lambda u(x) \quad \text{for } x \in \Omega, \quad u(x) = 0 \quad \text{for } x \in \Gamma = \partial\Omega, \tag{1}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain. This eigenvalue problem can be transformed into the equivalent nonlinear eigenvalue problem, see [11],

$$\frac{1}{4\pi}\int_\Gamma \frac{\cos(\kappa|x-y|)}{|x-y|} t(y)\,ds_y = 0 \quad \text{for x } \in \Gamma \tag{2}$$

where the unknowns $\kappa = \sqrt{\lambda}$ and $t(y) = n_y \cdot \nabla u(y)$ for $y \in \Gamma$, the corresponding normal derivative, have to be found. Associated eigenfunctions of (1) can then be represented by

$$u(x) = \frac{1}{4\pi}\int_\Gamma \frac{\cos(\kappa|x-y|)}{|x-y|} t(y)\,ds_y \quad \text{for x } \in \Omega.$$

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In [3, 4] similar boundary element approaches for the eigenvalue problem (1) are suggested. The boundary element approximations in those works lead to polynomial approximations of the nonlinear eigenvalue problem (2).

In this work we consider an iterative solution approach for the nonlinear eigenvalue problem (2) which is an analogon of the inverse iteration for linear and for nonlinear matrix eigenvalue problems, see, e.g., [6, 8, 12]. In fact, we will apply a Newton scheme to solve the nonlinear equation (2) where in addition we introduce an appropriate scaling condition. However, our theoretical approach is restricted to simple eigenvalues only. Further a Galerkin boundary element method to solve the nonlinear eigenvalue problem is formulated and the results of the numerical analysis are presented. Numerical examples given in Section 3 confirm not only the theoretical results, the experiments indicate that our approach also works for multiple eigenvalues. For a detailed numerical analysis we refer to [11].

2 Boundary Element Methods

The nonlinear eigenvalue problem (2) can be written as

$$(V_\kappa t)(x) = \frac{1}{4\pi}\int_\Gamma \frac{\cos(\kappa|x-y|)}{|x-y|} t(y)\,ds_y = 0 \quad \text{for } x \in \Gamma \tag{3}$$

where for fixed κ the operator $V_\kappa : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is linear and bounded, see, e.g., [5]. To normalize the eigensolutions $t \in H^{-1/2}(\Gamma)$ of (3) we introduce a scaling condition by using an equivalent norm in $H^{-1/2}(\Gamma)$,

$$\|t\|_V^2 = \langle Vt, t\rangle_\Gamma = \frac{1}{4\pi}\int_\Gamma t(x)\int_\Gamma \frac{1}{|x-y|} t(y)\,ds_y\,ds_x = 1, \tag{4}$$

where $V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is the single layer potential of the Laplace operator. Now we have to find solutions $(t,\kappa) \in H^{-1/2}(\Gamma)\times\mathbb{R}$ of the nonlinear eigenvalue problem

$$F_1(t,\kappa) = (V_\kappa t)(x) = 0 \quad \text{for } x \in \Gamma, \quad F_2(t,\kappa) = \langle Vt, t\rangle_\Gamma - 1 = 0. \tag{5}$$

Hence we define the function $F : H^{-1/2}(\Gamma)\times\mathbb{R} \to H^{1/2}(\Gamma)\times\mathbb{R}$ as

$$F(t,\kappa) = \begin{pmatrix} F_1(t,\kappa) \\ F_2(t,\kappa) \end{pmatrix} = \begin{pmatrix} \dfrac{1}{4\pi}\displaystyle\int_\Gamma \frac{\cos(\kappa|x-y|)}{|x-y|} t(y)\,ds_y \\ \langle Vt, t\rangle_\Gamma - 1 \end{pmatrix}.$$

Then, to obtain eigensolutions of the scaled eigenvalue problem (5) we have to find solutions $(t,\kappa) \in H^{-1/2}(\Gamma)\times\mathbb{R}$ of the nonlinear equation

$$F(t,\kappa) = 0 \tag{6}$$

which is to be solved by applying Newton's method. For the Fréchet derivative of $F(t,\kappa)$ we obtain

$$F'(t,\kappa) = \begin{pmatrix} V_\kappa & -A_\kappa t \\ 2\langle Vt,\cdot\rangle_\Gamma & 0 \end{pmatrix} : H^{-1/2}(\Gamma)\times\mathbb{R} \to H^{1/2}(\Gamma)\times\mathbb{R} \tag{7}$$

where

$$(A_\kappa t)(x) = \frac{1}{4\pi}\int_\Gamma \sin(\kappa|x-y|)t(y)ds_y \quad \text{for } x\in\Gamma.$$

When applying a Newton scheme to find solutions $(t_*,\kappa_*)\in H^{-1/2}(\Gamma)\times\mathbb{R}$ of the nonlinear equation (6) the new iterates $(t_{n+1},\kappa_{n+1})\in H^{-1/2}(\Gamma)\times\mathbb{R}$ can be determined by the linear operator equation

$$F'(t_n,\kappa_n)\begin{pmatrix} t_{n+1}-t_n \\ \kappa_{n+1}-\kappa_n \end{pmatrix} + F(t_n,\kappa_n) = 0. \tag{8}$$

In the following theorem we give sufficient conditions that equation (8) is unique solvable and that Newton's method locally converges to an eigensolution (t_*,κ_*).

Theorem 1. *[11] Let (t_*,κ_*) be a solution of $F(t,\kappa)=0$. Assume*

(A1) *κ_* is a simple eigenvalue of $V_\kappa t = 0$,*
(A2) *$A_{\kappa_*}t_* \notin \mathscr{R}(V_{\kappa_*})$.*

Then $F'(t_*,\kappa_*)$ is invertible and Newton's method converges for all initial values in a sufficient small neighborhood $U_\rho(t_*,\kappa_*)$ to (t_*,κ_*), where $\rho>0$ with

$$\|t_*-t_n\|^2_{H^{-1/2}(\Gamma)} + |\kappa_*-\kappa_n|^2 \le \rho^2. \tag{9}$$

Remark 1. For multiple eigenvalues κ_* the Fréchet derivative $F'(t_*,\kappa_*)$ is not invertible, because $F'(t_*,\kappa_*)$ is not injective. Nevertheless Newton's method may also converge [1, 2]. The convergence rate may then be smaller and the convergence domain is not a small neighborhood of the solution but rather a restricted region which avoids the set on which F' is singular. In our case numerical examples show that Newton's method converges also for multiple eigenvalues, see the numerical example in Section 3.

The linearized equation (8) is equivalent to a saddle point problem to find $(t_{n+1},\kappa_{n+1})\in H^{-1/2}(\Gamma)\times\mathbb{R}$ such that

$$\begin{aligned} \langle V_{\kappa_n}t_{n+1},w\rangle_\Gamma - \kappa_{n+1}\langle A_{\kappa_n}t_n,w\rangle_\Gamma &= -\kappa_n\langle A_{\kappa_n}t_n,w\rangle_\Gamma \\ 2\langle Vt_n,t_{n+1}\rangle_\Gamma &= \langle Vt_n,t_n\rangle_\Gamma + 1 \end{aligned} \tag{10}$$

is satisfied for all $w\in H^{-1/2}(\Gamma)$. For a Galerkin discretization of (10) we first define trial spaces $S_h^0(\Gamma)$ of piecewise constant basis functions ψ_k which are defined with respect to a globally quasi–uniform boundary element mesh of mesh size h. Then

the Galerkin discretization of (10) reads to find $(t_{n+1,h},\kappa_{n+1,h})\in S_h^0(\Gamma)\times\mathbb{R}$ such that

$$\begin{aligned}\langle V_{\kappa_n}t_{n+1,h},w_h\rangle_\Gamma-\kappa_{n+1,h}\langle A_{\kappa_n}t_n,w_h\rangle_\Gamma&=-\kappa_n\langle A_{\kappa_n}t_n,w_h\rangle_\Gamma\\ 2\langle Vt_n,t_{n+1,h}\rangle_\Gamma&=\langle Vt_n,t_n\rangle_\Gamma+1.\end{aligned}\tag{11}$$

is satisfied for all $w_h\in S_h^0(\Gamma)$. In the following theorem the solvability of the linear system (11) is discussed and an error estimate for the approximate solution $(t_{n+1,h},\kappa_{n+1,h})$ is given.

Theorem 2. *[11] Let (t_*,κ_*) be a solution of $F(t,\kappa)=0$ and let the assumptions* (A1) *and* (A2) *be satisfied. Let $(t_n,\kappa_n)\in U_\rho(t_*,\kappa_*)$ be satisfied where ρ is appropriately chosen as discussed in Theorem* 1*. Then, for a sufficient small mesh size $h<h_0$, the Galerkin variational problem* (11) *has a unique solution $(t_{n+1,h},\kappa_{n+1,h})\in S_h^0(\Gamma)\times\mathbb{R}$ satisfying the error estimate*

$$\|t_{n+1}-t_{n+1,h}\|^2_{H^{-1/2}(\Gamma)}+|\kappa_{n+1}-\kappa_{n+1,h}|^2\le c\inf_{w_h\in S_h^0(\Gamma)}\|t_{n+1}-w_h\|^2_{H^{-1/2}(\Gamma)}.\tag{12}$$

In practical computations we have to replace in (11) $(t_n,\kappa_n)\in H^{-1/2}(\Gamma)\times\mathbb{R}$ by previously computed approximations $(\hat t_{n,h},\hat\kappa_{n,h})\in S_h^0(\Gamma)\times\mathbb{R}$. In particular we have to find $(\hat t_{n+1,h},\hat\kappa_{n+1,h})\in S_h^0(\Gamma)\times\mathbb{R}$ such that

$$\begin{aligned}\langle V_{\hat\kappa_{n,h}}\hat t_{n+1,h},w_h\rangle_\Gamma-\hat\kappa_{n+1,h}\langle A_{\hat\kappa_{n,h}}\hat t_{n,h},w_h\rangle_\Gamma&=-\hat\kappa_{n,h}\langle A_{\hat\kappa_{n,h}}\hat t_{n,h},w_h\rangle_\Gamma\\ 2\langle V\hat t_{n,h},\hat t_{n+1,h}\rangle_\Gamma&=\langle V\hat t_{n,h},\hat t_{n,h}\rangle_\Gamma+1\end{aligned}\tag{13}$$

is satisfied for all $w_h\in S_h^0(\Gamma)$. To analyze the perturbed variational problem (13) we also need to consider the continuous variational problem to find $(\hat t_{n+1},\hat\kappa_{n+1})\in H^{-1/2}(\Gamma)\times\mathbb{R}$ such that

$$\begin{aligned}\langle V_{\hat\kappa_{n,h}}\hat t_{n+1},w\rangle_\Gamma-\hat\kappa_{n+1}\langle A_{\hat\kappa_{n,h}}\hat t_{n,h},w\rangle_\Gamma&=-\hat\kappa_{n,h}\langle A_{\hat\kappa_{n,h}}\hat t_{n,h},w\rangle_\Gamma\\ 2\langle V\hat t_{n,h},\hat t_{n+1}\rangle_\Gamma&=\langle V\hat t_{n,h},\hat t_{n,h}\rangle_\Gamma+1\end{aligned}\tag{14}$$

is satisfied for all $w\in H^{-1/2}(\Gamma)$. Note that (13) is the Galerkin discretization of (14). In the next theorem we discuss the solvability of the linear system (14) and give an error estimate for the discrete Newton iterate $(\hat t_{n+1,h},\hat\kappa_{n+1,h})$ with respect to the continuous Newton iterate (t_{n+1},κ_{n+1}).

Theorem 3. *[11] Let (t_*,κ_*) be a solution of $F(t,\kappa)=0$ and let the assumptions* (A1) *and* (A2) *be satisfied. Let $(\hat t_{n,h},\hat\kappa_{n,h})\in S_h^0(\Gamma)\times\mathbb{R}\cap U_\rho(t_*,\kappa_*)$ be satisfied where ρ is appropriately chosen as discussed in Theorem* 1*. Then, for a sufficient small mesh size $h<h_0$, the Galerkin variational problem* (13) *has a unique solution $(\hat t_{n+1,h},\hat\kappa_{n+1,h})\in S_h^0(\Gamma)\times\mathbb{R}$ satisfying the error estimate*

$$\|t_{n+1}-\hat{t}_{n+1,h}\|^2_{H^{-1/2}(\Gamma)}+|\kappa_{n+1}-\hat{\kappa}_{n+1,h}|^2 \\ \leq c\left[\|t_n-\hat{t}_{n,h}\|^2_{H^{-1/2}(\Gamma)}+|\kappa_n-\hat{\kappa}_{n,h}|^2+\inf_{w_h\in S^0_h(\Gamma)}\|t_{n+1}-w_h\|^2_{H^{-1/2}(\Gamma)}\right] \tag{15}$$

where the constant c depends on (t_*,κ_*)*, and on* ρ.

Considering the approximation property of $S^0_h(\Gamma)$ we get from (15) the following error estimate, see [11],

$$\|t_{n+1}-\hat{t}_{n+1,h}\|^2_{H^{-1/2}(\Gamma)}+|\kappa_{n+1}-\hat{\kappa}_{n+1,h}|^2 \leq c\left[\rho^4+h^3\right]$$

when assuming $t_*\in H^1_{\mathrm{pw}}(\Gamma)$. The constant c depends on (t_*,κ_*), n, and on ρ.

When using the Aubin–Nitsche trick, see for example [10], it is possible to derive error estimates in Sobolev spaces with lower Sobolev index. In particular we obtain the error estimate

$$\|t_{n+1}-\hat{t}_{n+1,h}\|^2_{H^{-2}(\Gamma)}+|\kappa_{n+1}-\hat{\kappa}_{n+1,h}|^2 \leq c\left[\rho^4+h^6\right]$$

when assuming $t_*\in H^1_{\mathrm{pw}}(\Gamma)$. Hence we can expect a cubic convergence rate for the eigenvalues,

$$|\kappa_{n+1}-\hat{\kappa}_{n+1,h}| \leq c\left[\rho^4+h^6\right]^{1/2}=\mathcal{O}(h^3).$$

3 Numerical Results

In this section we present some numerical results to investigate the behavior of the nonlinear boundary element approach as presented in this paper. As a model problem we consider the interior Dirichlet eigenvalue problem (1) where the domain $\Omega=(0,\frac{1}{2})^3$ is a cube. Hence the eigenvalues are given by

$$\lambda_k=4\pi^2\left[k_1^2+k_2^2+k_3^2\right]$$

and the associated eigenfunctions are

$$u_k(x)=(\sin 2\pi k_1x_1)(\sin 2\pi k_2x_2)(\sin 2\pi k_3x_3).$$

It turns out that the first eigenvalue ($k_1=k_2=k_3=1$)

$$\lambda_1=12\pi^2,\quad \kappa_1=2\sqrt{3}\pi$$

is simple, while the second eigenvalue ($k_1=2,k_2=k_3=1$)

$$\lambda_2=24\pi^2,\quad \kappa_2=2\sqrt{6}\pi$$

is multiple.

For the boundary element discretization the boundary $\Gamma = \partial\Omega$ was decomposed into N uniform triangular boundary elements. The numerical results to approximate the simple eigenvalue $\kappa_1 = \sqrt{\lambda_1}$ are given in Table 1.

Table 1 Approximation of $\kappa_1 = 2\sqrt{3}\pi \approx 10.8828$, simple eigenvalue

N	$\kappa_{1,N}$	$\lvert\kappa_1 - \kappa_{1,N}\rvert$	rate
384	10.8768	6.0e-03	-
1536	10.8821	7.0e-04	8.6
6144	10.8827	8.6e-05	8.1

Note that the convergence rate of approximately 8 corresponds to the cubic convergence as predicted in (2). Next we consider the case of a multiple eigenvalue, the results to approximate $\kappa_2 = \sqrt{\lambda_2}$ are given in Table 2.

Table 2 Approximation of $\kappa_2 = 2\sqrt{6}\pi \approx 15.3906$, multiple eigenvalue

N	$\kappa_{2_1,N}$	$\lvert\kappa_2 - \kappa_{2_1,N}\rvert$	rate	$\kappa_{2_2,N}$	$\lvert\kappa_2 - \kappa_{2_2,N}\rvert$	$\kappa_{2_3,N}$	$\lvert\kappa_2 - \kappa_{2_3,N}\rvert$
384	15.3739	1.7e-02	-	14.7057	0.68	15.8867	0.50
1536	15.3887	1.9e-03	8.9	14.6902	0.70	15.8579	0.47
6144	15.3904	2.3e-04	8.3	14.6839	0.71	15.8499	0.46

As in other boundary element approaches for eigenvalue problems [3, 4, 13] the problem of the so–called spurious eigenvalues occurs close to multiple eigenvalues. In particular, several distinct discrete eigenvalues are obtained to approximate a multiple eigenvalue. This phenomenon also occurs for algebraic eigenvalue problems when an approximation of the matrix is used, see e.g. [9].

The spurious eigenvalues can be filtered out with an a posteriori error control by using the complex valued fundamental solution for an eigensolution (t,κ),

$$\frac{1}{4\pi}\int_\Gamma \frac{e^{i\kappa|x-y|}}{|x-y|}t(y)ds_y = (V_\kappa t)(x) + i\frac{1}{4\pi}\int_\Gamma \frac{\sin\kappa|x-y|}{|x-y|}t(y)ds_y = 0. \tag{16}$$

Then the norm of the residual

$$r(t_h,\kappa_h) = \frac{1}{4\pi}\int_\Gamma \frac{e^{i\kappa_h|x-y|}}{|x-y|}t_h(y)ds_y$$

for actual approximations of eigensolutions (t_h,κ_h) is significant smaller than for spurious eigensolutions, see Table 3.

When an analogous algorithm is used which is based on the complex valued fundamental solution (16) no spurious eigenvalues occur. But then complex arithmetics

Table 3 Residual for true and spurious eigenvalues

N	$\|r(t_{1,N}, \kappa_{1,N})\|$	$\|r(t_{2_1,N}, \kappa_{2_1,N})\|$	$\|r(t_{2_2,N}, \kappa_{2_2,N})\|$	$\|r(t_{2_3,N}, \kappa_{2_3,N})\|$
384	6.3e-05	1.9e-04	7.5e-03	2.3e-02
1536	7.7e-06	2.2e-05	3.8e-03	1.2e-02
6144	9.6e-07	2.5e-06	1.9e-03	6.1e-03

has to be used so that the computational complexity is twice expensive as for the real valued version. In Table 4 and 5 the approximations for κ_1 and κ_2 are given.

Table 4 Approximation of $\kappa_1 = 2\sqrt{3}\pi \approx 10.8828$, simple eigenvalue

N	$\kappa_{1,N}$	$\|\kappa_1 - \kappa_{1,N}\|$	rate
384	10.8768-1.0e-06i	6.0e-03	-
1536	10.8821-2.4e-07i	7.0e-04	8.6
6144	10.8827-6.0e-09i	8.6e-05	8.1

Table 5 Approximation of $\kappa_2 = 2\sqrt{6}\pi \approx 15.3906$, multiple eigenvalue

N	$\kappa_{2,N}$	$\|\kappa_2 - \kappa_{2,N}\|$	rate
384	15.3739-5.1e-06i	1.7e-02	-
1536	15.3887-9.4e-07i	1.9e-03	8.9
6144	15.3904-2.1e-08i	2.3e-04	8.3

Note that the real part of the approximations of the complex valued algorithm are the same as of the real valued version.

4 Conclusions

In this paper we have presented and analyzed a boundary element method for the solution of the interior Dirichlet eigenvalue problem for the Laplace operator. Hereby, the linear eigenvalue problem for the partial differential operator is transformed into a nonlinear eigenvalue problem for an associated boundary integral operator which is solved via a Newton iteration. The discretization by using a Galerkin boundary element method gives a cubic order of convergence of the approximated eigenvalues. When using fast boundary element methods [7] an almost optimal computational complexity can be obtained. For this, also efficient preconditioned iterative solution methods to solve the Galerkin equations (13) are mandatory. As already mentioned in Remark 1 a further analysis in the case of multiple eigenvalues is needed.

Finally we mention that the proposed approach can be used to solve the interior Neumann eigenvalue problem for the Laplace operator, and to solve related eigenvalue problems in linear elastostatics.

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