TORSION OF CRACKED COMPONENTS USING RADIAL BASIS FUNCTIONS

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Abstract

In this work, and in the context of collocation-based radial basis functions approach, a combination of regular and singular functions is considered for the analysis of cracked cross-sections subjected to torsional moments.

1 Introduction

Radial basis functions are essential ingredients of the techniques generally known as "meshless methods". In a way or another all of the meshless (or mesh reduction) techniques attempt to solve boundary-value problems without resorting to the definition of a "mesh" (at least not in the form meshes are usually defined in the context of finite or boundary element methods) thus requiring some sort of radial function to measure the influence of a given location on another part of the domain.
Techniques based on the partition of unity and the moving least squares concepts (such as the Element Free Galerkin or the hp-clouds methods) are, undoubtedly, the most popular meshless methods. A brief review of the various proposals that have been made (since the early work of Lucy on smoothed particle hydrodynamics) may include the works of Liszka on generalized finite-differences, that of Nayroles *et al.* on the diffuse element method), Belytschko and co-authors on the element-free Galerkin method ; Duarte and Oden on the h-p clouds method, Babuska and Melenk on

the partition of unity method, Liu and co-authors on the reproducing kernel method and that of De and Bathe on the finite sphere method. Other approaches include the works of Mukherjee and Mukherjee on the boundary node method and that of Atluri and Zhu on local forms of boundary integral equations. (for the complete references the reader is referred to [1].)
The use of radial basis functions (RBF) followed by collocation, a technique first proposed by Kansa [3], [4], after the work of Hardy [2] on multivariate approximation, is now becoming an established approach and various applications to problems of structures and fluids have been made in recent years.
Examples of application of the above technique (as well as the symmetric approach proposed by Fasshauer [5]) may be found, for example, in the works of the author (and co-authors), Leitão [1], [6], [7]. The range of problems analysed include: Kirchhoff plate bending, plate stretching, non-linear damage models for reinforced concrete, one-dimensional stability and one-dimensional free vibration problems.
Kansa's method (or asymmetric collocation) starts by building an approximation to the field of interest (normally displacement components) from the superposition of radial basis functions (globally or compactly supported) conveniently placed at points in the domain (and, or, at the boundary).
The unknowns (which are the coefficients of each RBF) are obtained from the (approximate) enforcement of the boundary conditions as well as the governing equations by means of collocation. Usually, this approximation only considers regular radial basis functions (such as the globally supported multiquadrics or the compactly supported Wendland [8] functions).
In order to model adequately the singular behaviour exhibited around cracks (or other sources of singularities) it is also necessary to superimpose singular functions as well. In fact standard (that is, regular) RBF approximations tend to lead to poor solutions around cracks or corners. This motivated Platte and Driscoll [9] to consider an augmented approximation by combining regular and singular functions for eigenmodes computation.
In this work, and in the context of collocation-based radial basis functions approach, the same idea is used (the combination of regular and singular functions) for the analysis of cracked cross-sections subjected to torsional moments. A different approach for treating singularities may be found in Wong *et al* [10].
In the following sections a brief description of the collocation approach is made. Then the problem of torsion is formulated and applied to the analysis of torsion of uncracked and cracked circular cross-sections.

2 Radial basis functions

Radial basis functions (RBFs) are all that exhibit radial symmetry, that is, may be seen to depend only (apart from some known parameters) on the distance $r = \|x - x_j\|$ between the center of the function and a generic point

x. These functions may be generically represented in the form $\phi(r)$.
These functions may be named globally supported (Multiquadrics (MQ), for example, $\sqrt{(x-x_j)^2+c_j^2}$, where c_j is a shape parameter) or compactly supported (Wendland[8], for example, $(1-r)_+^n p(r)$ where $p(r)$ is a polynomial and $(1-r)_+^n$ is 0 for r greater than the support) depending on their supports, that is, whether they are defined on the whole domain or only on part of it. Although no shape parameters are used in compact support RBF's it is important to appropriately define the size of the support (the distance r needs to be normalized with respect to the support radius).

3 Asymmetric collocation approach

In a very brief manner, interpolation with RBFs may take the form:

$$s(x) = \sum_{j=1}^{N} \alpha_j \phi(\|x - x_j\|) + \sum_{k=1}^{\hat{N}} \beta_k p_k(x) \tag{1}$$

where $f(x_i)$ is known for a series of points x_i and $p_k(x)$ is one of the $\hat{N}$ terms of a given basis of polynomials, see Buhmann [12].
This approximation is solved for the α_j unknowns from the system of N linear equations of the type:

$$s(x_i) = f(x_i) = \sum_{j=1}^{N} \alpha_j \phi(\|x_i - x_j\|) + \sum_{k=1}^{\hat{N}} \beta_k p_k(x_i) \tag{2}$$

subject to the conditions (for the sake of uniqueness) $\sum_{j=1}^{N} \alpha_j p_k(x_j) = 0$
By using the same reasoning it is possible to extend the interpolation problem to that of finding the approximate solution of partial differential equations. This is made by applying the corresponding differential operators to the radial basis functions and then to use collocation at an appropriate set of boundary and domain points.
In short, the non-symmetrical collocation is the application of the domain and boundary differential operators LI and LB, respectively, to a set of $N-M$ domain collocation points and M boundary collocation points.
From this, a system of linear equations of the following type may be obtained:

$$LIu_h(x_i) = \sum_{j=1}^{N} \alpha_j LI\phi(\|x_i - \varepsilon_j\|) + \sum_{k=1}^{\hat{N}} \beta_k LIp_k(x_i) \tag{3}$$

$$LBu_h(x_i) = \sum_{j=1}^{N} \alpha_j LB\phi(\|x_i - \varepsilon_j\|) + \sum_{k=1}^{\hat{N}} \beta_k LBp_k(x_i) \tag{4}$$

subject to the conditions $\sum_{j=1}^{N} \alpha_j p_k(x_j) = 0$ where the α_j and β_k unknowns are determined from the satisfaction of the domain and boundary constraints at the collocation points.

3.1 Augmenting the approximation by means of singular functions

The basic idea of combining regular RBF and singular solutions is to treat these singular terms in the way polynomials are added to the expansion, that is:

$$s(x_i) = f(x_i) = \sum_{j=1}^{N} \alpha_j \phi(\|x_i - x_j\|) + \sum_{k=1}^{\hat{N}} \beta_k \psi_k(x_i) \tag{5}$$

subject to the conditions $\sum_{j=1}^{N} \alpha_j \psi_k(x_j) = 0$
The singular functions to be used are, obviously, problem dependent. The case of torsion is described next.

4 The torsion problem

Consider a prismatic bar (along coordinate z) subjected to torsional end moments. If the cross section (in the x, y plane) is radially symmetric then the cross sections will remain plane and rotate (with a constant rate) without distortion during twist (Coulomb hypotheses).
For the non radially symmetric case the cross section no longer remains plane, ie, warping (extension of the fibers along the direction where the moment is applied) occurs. Saint-Venant's hypothesis consists in assuming warping to be constant.
By using the above simplifying hypotheses the displacement components for the torsion problem may be expressed as [13]:

$$u(x, y, z) = -\theta yz, \; v(x, y, z) = \theta xz \text{ and } w(x, y, z) = \theta\psi(x, y) \tag{6}$$

where θ is the twist angle per unit length and $\psi(x, y)$ is the warping function. The strain components are:

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = -y\theta + \theta\frac{\partial \psi}{\partial x} \text{ and } \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = x\theta + \theta\frac{\partial \psi}{\partial y} \tag{7}$$

and the corresponding stresses are:

$$\sigma_{xz} = -Gy\theta + G\theta\frac{\partial \psi}{\partial x} \text{ and } \sigma_{yz} = Gx\theta + G\theta\frac{\partial \psi}{\partial y} \tag{8}$$

where G is the shear modulus.

At this stage, and to simplify the boundary conditions, the stress function $\Psi(x, y)$ may be introduced thus leading to the following definitions for the stress components:

$$\sigma_{xz} = \frac{\partial \Psi}{\partial y} \text{ and } \sigma_{yz} = -\frac{\partial \Psi}{\partial x} \tag{9}$$

Using the equilibrium equation

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} = 0 \tag{10}$$

and after eliminating ψ the following Poisson problem is obtained,

$$\nabla^2 \Psi = -2G\theta, \text{ in } \Omega, \text{ subjected to } \Psi = 0 \text{ on all the boundary } \Gamma, \tag{11}$$

Now, it may be convenient to reformulate the problem such that a global solution is obtained from the superposition of the solution of the homogeneous problem u_h and a particular solution u_p,
So the actual problem that is going to be solved in this work is the following:

$$\nabla^2 u_h = 0, \text{ in } \Omega, \text{ subjected to } u_h = -u_p \text{ on all the boundary } \Gamma. \tag{12}$$

5 The case of a cracked circular cross section

Consider now the case of torsion of a cracked circular cross section. The crack tip is located at point (0,0) and the crack length is equal to the radius of the circular cross section. The sides of the crack are denoted by γ^+ and γ^- and Γ is the remaining boundary.
The torsion problem can now be stated as follows:

$$\nabla^2 u_h = 0, \text{ in } \Omega, \text{ subjected to } u_h = -u_p \text{ on all the boundary } \partial\Omega = \Gamma + \gamma^+ + \gamma^-. \tag{13}$$

where $u_p(x, y) = (x^2 + y^2)/2$ is a particular solution.
The analytic solution (the potential ϕ in polar coordinates r, φ centered at the crack tip for the circular cross section with a crack length a to diameter d ratio of $a/d = 0.5$ is presented by Chen *et al* [14] as follows:

$$\phi(r, \varphi) = 32\frac{a^2}{\pi} \sum_{n=0}^{\infty} \frac{(\frac{r}{a})^{(2n+1)/2} - (\frac{r}{a})^2}{(2n+1)[16 - (2n+1)^2]} \sin \frac{(2n+1)\varphi}{2} \tag{14}$$

$$\tag{15}$$

6 RBF analysis of the cracked circular cross section

The RBF analysis requires the definition of, first, the type and support of the RBF. In this work both global support and compact support RBF's are used. Multiquadric (MQ) was the global support RBF chosen. The chosen compact support RBF is one of the Wendland's (WD) family [8], $(1-r)^5(8r^2+5r+1)$

From previous tests (on other elliptic problems [1] and [6]) one might expect that factors, such as the number and location of the centers and of the collocation points, the type of RBF, the shape parameter (MQ) or the support radius (WD), could have an effect on the accuracy of the RBF approximations.

That was not the case for the torsion problem on uncracked cross sections. The reason is that this is a simple problem and any reasonably chosen distribution of RBF's will be capable of solving the problem.

Now, the picture is completely different in the presence of singularities such as the case of the cracked cross section.

In fact, even with very large number of centers and/or collocation points and after testing a large range of shape parameters or support radii it was not possible to satisfactorily approximate the singular behaviour by using only the standard MQ or WD RBFs, that is, without enriching the basis.

This is a conclusion which other authors, namely Platte and Driscoll [9] and Wong *et al* [10], had experienced for other types of singularities.

6.1 Enriching the approximation by adding singular fields

In order to model adequately the singular behaviour exhibited around cracks (or other sources of singularities) it is then necessary to superimpose (on the regular field) the effect of singular functions.

In the context of Trefftz methods this is standard procedure and has been used by various authors, namely by Leitão [16] for fracture mechanics problems.

In what concerns RBFs this superposition has been proposed by Platte and Driscoll [9] for eigenmodes computation on plates with reentrant corners (where singularities occur).

Basically, the following expansion is assumed, $u(x) = u_R(x) + u_S(x), \forall x \in \Omega$ where the actual solution is the sum of a regular (the combined effect of the RBFs) and a singular part (a suitably chosen function (or functions) centered at the singularity location):

$$u_R(x) \approx \sum_{j=1}^{N} \alpha_j \phi_j(x) \text{ and } u_S(x) \approx \sum_{k=1}^{\hat{N}} \beta_k \Phi_k^{sing}(x) \tag{16}$$

The suitably chosen functions are such that fully satisfy the boundary condition ($u(x) = 0$) on the crack faces $\gamma^+ + \gamma^-$), that is, functions of the form:

$$\psi_k(\rho,\varphi) = \rho^{p_k} \cos(p_k\varphi) \text{ and } \psi_k(\rho,\varphi) = \rho^{p_k} \cos(p_k\varphi), \tag{17}$$

in polar coordinates, centered at the crack tip (ie. $\rho = |x - c_{tip}|$ and φ is the angle defined by the point $\frac{x-c_{tip}}{\rho}$ in the unit circle).
The above functions constitute a T-complete series of functions. and therefore, a conveniently truncated series of those functions followed by boundary-only collocation (that is, the Trefftz collocation approach) leads to accurate results, [15] and [16].
In what concerns this work it is possible to consider $p_k = k - 0.5, k$ being an integer, and use exclusively the expansion $u_S(x) \approx \sum_{k=0}^{\infty} \beta_k\ \rho^{k+0.5} \cos((k+0.5)\theta)$.
Thus, the solution to this combined approximation may be obtained by:

$$LIu_h(x_i) = \sum_{j=1}^{N} \alpha_j LI\phi(\|x_i - \varepsilon_j\|) + \sum_{k=1}^{\hat{N}} \beta_k LI\Phi_k^{sing}(x_i) \tag{18}$$

$$LBu_h(x_i) = \sum_{j=1}^{N} \alpha_j LB\phi(\|x_i - \varepsilon_j\|) + \sum_{k=1}^{\hat{N}} \beta_k LB\Phi_k^{sing}(x_i) \tag{19}$$

subject to the conditions $\sum_{j=1}^{N} \alpha_j \Phi_k^{sing}(x_j) = 0$ where the α_j and β_k unknowns are determined from the satisfaction of the domain and boundary constraints at the collocation points.

6.2 Numerical tests

Results are presented for the following numerical tests on the circular cross section (among the many that were carried out): global support RBF (multiquadric) with and without adding singular terms; compact support RBF (Wendland) with and without adding singular terms.
Due to symmetry only half of the circular domain was modelled. All these tests used the same distribution of RBF centers and collocation points (see figure 1):a "grid" of 14×14 centers distributed concentrically at increasing distances from the center; a "grid" of 20×20 domain collocation points distributed concentrically at increasing distances from the center; 140 collocation points distributed uniformly along the circular cross section boundary; 50 collocation points distributed uniformly along the crack face.
The number of equations (600) outnumbered that of unknowns by a factor of, approximately, two and a half. A least squares solver was used (within the Matlab [17] environment) to solve the system of equations. Reasonably accurate results may be obtained with smaller systems of equations but the strategy here was to try to get as good a solution as possible, thus the high number of equations and unknowns.
These results are plotted against the exact solution (along the horizontal line containing the crack tip) in figures 2 and 3 (for the multiquadric and the Wendland types of RBF).

Figure 1: Distribution of centers and points.

Figure 2: Comparison of Ψ with x-coordinate (at $y = 0$) for enriched and standard RBF (Multiquadric).

Figure 3: Comparison of Ψ with x-coordinate (at $y = 0$) for enriched and standard RBF (Wendland).

It is evident that the results achieved without adding the singular terms, without enriching the basis, are not reasonable. The singular behaviour is not captured at all. It is possible that a domain decomposition technique would diminish those inaccuracies but it does not seem a viable way of addressing this type of problems.
On the other hand, by adding extra terms (of a series that fully represents the singular fields around the crack tip) good results are obtained: the effect of the singularity is now fully captured. It seems that the chosen compact support RBFs, figure 3, performs slightly better than the multiquadric shown in figure 2.

7 Conclusions

The main purpose of this work was that of contributing for the development of the radial basis functions asymmetric collocation approach by extending its range of applications to the analysis of singular problems, namely the torsion of cracked components.
Standard RBF approximations do not seem to appropriately describe the singular behaviour around cracks. The addition of extra (singular) terms is, as may be concluded from the above results, one way of addressing problems exhibiting singularities. It is expected that the coupling of this approach with domain decomposition techniques would improve the results. Further work on the subject is now being carried out.

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