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True and Spurious Eigensolutions of Two-dimensional Acoustic Cavities with the Mixed-type Boundary Conditions using BEMs

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ABSTRACT

In this paper, four different methods, the complex-valued BEM, the real-part BEM, the imaginary-part BEM and the multiple reciprocity method (MRM), are employed to solve the eigenproblem with the mixed-type boundary conditions and the FEM is implemented for comparisons. No spurious eignvalues are found by using the complex-valued BEM, while there are spurious eignvalues by using the real-part BEM, the imaginary-part BEM and the MRM. It is shown that the occurrence of spurious eignvalues depend on the adopted method (direct method: singular or hypersingular; indirect method: single layer or double layer) no matter what the given types of boundary conditions for the problem are.

Key words: mixed-type boundary conditions, boundary element method, singular equation, hypersingular equation

INTRODUCTION

Eigenproblems are often encountered not only in vibration but also in acoustics. By using the complex-valued BEM, the eigenvalues and eigenmodes can be determined [1] without any difficulty. Nevertheless, complex-valued computation is time consuming and not simple. A simplified method using only the real-part or imaginary-part kernel was presented by De Mey [2]. This novel method, using only the real-part BEM [3], was found to be equivalent to the multiple reciprocity method (MRM) if the zeroth-order fundamental solution is properly chosen [4]. The relations among the MRM, the real-part BEM and the complex-valued BEM were discussed in the keynote lecture by Chen [4]. In the reference of [5], Chen and Wong found that the real-part BEM and MRM result in spurious eigensolutions for one-dimensional examples. Kang *et al.* [7] employed the Nondimensional Dynamic Influence Function method (NDIF) to solve the eigenproblem and also suffered the problem of spurious eigensolutions. Chen *et al.* [8] commented that the NDIF method is a lumped case of the imaginary-part BEM. Kang and Lee [9] filtered out the spurious eigenvalues by using the net approach. The reason why spurious eigenvalues occur in the real-part BEM, the imaginary-part BEM or the MRM is the loss of the constraints, which was investigated by Yeih *et al.* [6]. The spurious eigensolutions arise from an improper approximation of the null space of operator [10]. Numerical experiments using the real-part kernels were performed for two-dimensional cases [11]. It is obvious that one advantage of using only the real-part kernels is that real-valued computation is employed instead of complex-valued computation in the complex-valued BEM. Another gain is that lengthy derivation as required for the MRM can be avoided. The main drawback of the real-part formulation has been found to be the occurrence of spurious eigenvalues [5, 11]. To deal with this problem, the framework of the real-part dual BEM [12] and dual MRM [13] were constructed to filter out spurious eigenvalues for the Dirichlet and Neumann probelm. Based on the circulant properties and degenerate kernels, the reason why the spurious eigensolutions occur can be easily understood analytically and numerically by demonstrating a circular case for the Dirichlet and Neumann problems [14]. The SVD updating documents in conjunction with the Fredholm's alternative theorem were employed to filter out spurious eigenvalues by assembling the dual equations [13, 12, 15]. The SVD updating terms were utilized to extract out the true eigenvalues for two-dimensional cavities, respectively. The same techniques will be chosen to sort out the true and spurious eigenvalues for the 2-D acoustics with the mixed-type boundary conditions in this paper.

The eigenproblems with either the Dirichlet or the Neumann boundary condition have been discussed by analytical and numerical methods, but only a few papers focused on the mixed-type boundary conditions to the authors' best knowledge. The real-part BEM, the imaginary-part BEM and the MRM, result in spurious eigenvalues for the Dirichlet and the Neumann eigenprobelms if the singular (UT method) or hypersingular (LM method) integral equation is used alone, while the complex-valued BEM is free of spurious eigenvalues. It is interesting to extend the above conclusion to the problem with mixed-type boundary conditions. A circular cavity will be discussed by using four different BEMs and will be compared with that of FEM.

THEORETICAL ANALYSIS

The governing equation for the eigenproblem is the Helmholtz equation as follows:

$$\nabla^2 u(\mathbf{x}) + k^2 u(\mathbf{x}) = 0, \quad \mathbf{x} \in D. \tag{1}$$

where D is the domain of interest, $\mathbf{x}$ is the domain point, k is the wave number and $u(\mathbf{x})$ is the acoustic potential, respectively. On the basis of the dual formulation, the null-field integral formulation for the Helmholtz equation can be written as

$$0 = \int_B T(\mathbf{s},\mathbf{x})u(\mathbf{s})dB(\mathbf{s}) - \int_B U(\mathbf{s},\mathbf{x})t(\mathbf{s})dB(\mathbf{s}), \quad \mathbf{x} \in D^e \tag{2}$$

$$0 = \int_B M(\mathbf{s},\mathbf{x})u(\mathbf{s})dB(\mathbf{s}) - \int_B L(\mathbf{s},\mathbf{x})t(\mathbf{s})dB(\mathbf{s}), \quad \mathbf{x} \in D^e \tag{3}$$

where D^e is the complementary domain of D, $\mathbf{x}$ is a field point, $\mathbf{s}$ is a source point, $u(\mathbf{s})$ and $t(\mathbf{s})$ is the potential and its normal derivative on the boundary respectively, $U(\mathbf{s},\mathbf{x})$ is the kermel funtion, which is difined in the second row of Table 1 for the different methods. Other kernel functions are defined as $T(\mathbf{s},\mathbf{x}) = \frac{\partial U(\mathbf{s},\mathbf{x})}{\partial n_{\mathbf{s}}}$, $L(\mathbf{s},\mathbf{x}) = \frac{\partial U(\mathbf{s},\mathbf{x})}{\partial n_{\mathbf{x}}}$

and $M(\mathbf{s},\mathbf{x}) = \frac{\partial^2 U(\mathbf{s},\mathbf{x})}{\partial n_{\mathbf{s}} n_{\mathbf{x}}}$. After moving the field point, $\mathbf{x} \in D^e$, to the boundary B, we can obtain the singular integral equations [4].

By discretizing the boundary B into boundary elements in Eq.(2) and Eq.(3), we obtain the dual algebraic system as follows [16]:

$$[U]\{t\} = [T]\{u\}, \tag{4}$$

$$[L]\{t\} = [M]\{u\}. \tag{5}$$

For the problem with mixed-type boundary conditions, Eq.(4) can be decomposed into

$$\begin{bmatrix} U_L & \vdots & U_R \end{bmatrix}_{N\times N} \begin{Bmatrix} t \\ \bar{t} \end{Bmatrix}_{N\times 1} = \begin{bmatrix} T_L & \vdots & T_R \end{bmatrix}_{N\times N} \begin{Bmatrix} \bar{u} \\ u \end{Bmatrix}_{N\times 1}, \tag{6}$$

where u and t are the unknown boundary potential and flux , $\bar{u}$ and $\bar{t}$ are the known boundary potential and flux, the subscripts L and R denote the left and right submatrices, respectively, and the symbol N is the number of boundary elements. By collecting the known and unknown sets, we rearrage the influences matrices into

$$\begin{bmatrix} U_L & \vdots & -T_R \end{bmatrix}_{N\times N} \begin{Bmatrix} t \\ u \end{Bmatrix}_{N\times 1} = \begin{bmatrix} T_L & \vdots & -U_R \end{bmatrix}_{N\times N} \begin{Bmatrix} \bar{u} \\ \bar{t} \end{Bmatrix}_{N\times 1}. \tag{7}$$

Therefore, Eq.(7) can be simplified to

$$[A]_{N\times N}\{p\}_{N\times 1} = [B]_{N\times N}\{q\}_{N\times 1}, \tag{8}$$

where

$$[A] = \begin{bmatrix} U_L & \vdots & -T_R \end{bmatrix}, \quad [B] = \begin{bmatrix} T_L & \vdots & -U_R \end{bmatrix}, \tag{9}$$

$$\{p\} = \begin{Bmatrix} t \\ u \end{Bmatrix} \text{ and } \{q\} = \begin{Bmatrix} \bar{t} \\ \bar{u} \end{Bmatrix}. \tag{10}$$

Similarly, Eq.(5) yields

$$[C]_{N\times N}\{p\}_{N\times 1} = [D]_{N\times N}\{q\}_{N\times 1}. \tag{11}$$

For the eigenproblem with homogeneous boundary condition of $\{q\} = \{0\}$, Eq.(8) and Eq.(11) reduce to

$$[A]_{N\times N}\{p\}_{N\times 1} = \{0\}, \tag{12}$$

and

$$[C]_{N\times N}\{p\}_{N\times 1} = \{0\}, \tag{13}$$

where $[A]$, $[B]$, $[C]$ and $[D]$ are obtained from the four influence matrices after considering the mixed-type boundary conditions.

Detection of spurious eigensolutions by using SVD updating documents and Fredholm's alternative theorem

The linear algebraic equation $[K]\{u\} = \{b\}$ has a unique solution if and only if the only solution to the homogeneous equation,

$$[K]\{u\} = \{0\}, \tag{14}$$

is $\{u\} \equiv \{0\}$. Alternatively, the homogeneous equation has at least one solution if the homogeneous adjoint equation,

$$[K]^T\{\phi\} = \{0\}, \tag{15}$$

has a nontrivial vector, $\{\phi\}$, where $[K]^T$ is the transpose matrix of $[K]$ and $\{b\}$ must satisfy the constraint $\{b\}^T\{\phi\} = 0$. By employing the singular formulation (UT method) of Eq.(8), we have

$$[A(k)]\,\{p\} = [B(k)]\,\{q\} = \{b\}. \tag{16}$$

According to the Fredholm's alternative theorem, Eq.(16) has at least one solution for $\{p\}$ if the homogeneous adjoint equation,

$$[A(k_s)]^T\,\{\phi\} = \{0\}, \tag{17}$$

has a nontrivial vector, $\{\phi^{UT}\}$, of spurious boundary eigensolution, in which the constraint, $\{b\}^T\{\phi^{UT}\} = 0$, must be satisfied, where k_s denotes the spurious eigenvalues. Substituting $\{b\} = [B(k)]\{q\}$ in Eq.(16) into $\{b\}^T\{\phi^{UT}\} = 0$, we obtain

$$\{q\}^T\,[B(k_s)]^T\,\{\phi^{UT}\} = 0. \tag{18}$$

Since $\{q\}$ is an arbitrary vector, we have

$$[B(k_s)]^T\,\{\phi^{UT}\} = \{0\}, \tag{19}$$

Combining Eq.(17) and Eq.(19) together, we have

$$\begin{bmatrix} [A(k_s)]^T \\ [B(k_s)]^T \end{bmatrix} \{\phi^{UT}\} = \{0\} \;\text{ or }\; \{\phi^{UT}\}^T \begin{bmatrix} [A(k_s)] & [B(k_s)] \end{bmatrix} = \{0\}. \tag{20}$$

Equation (20) indicates that the two matrices have the same spurious mode $\{\phi^{UT}\}$ corresponding to the same zero singular value for the spurious eigenvalue k_s. The former one in Eq.(20) is a form of updating term and the latter one is a form of updating document. In the hypersingular formulation (LM method), the spurious eigenvalue satisfies

$$\begin{bmatrix} [C(k_s)]^T \\ [D(k_s)]^T \end{bmatrix} \{\phi^{UT}\} = \{0\} \;\text{ or }\; \{\phi^{UT}\}^T \begin{bmatrix} [C(k_s)] & [D(k_s)] \end{bmatrix} = \{0\}. \tag{21}$$

Four approaches, the complex-valued BEM, the real-part BEM, the imaginary-part BEM and the MRM are utilized to construct the $[A]$, $[B]$, $[C]$ and $[D]$ matrices in Tables 2(a)-(d), respectively.

Extraction of true eigensolutions by using SVD updating terms

Since the true eigenvalue, k_t, must satisfy Eq.(12) and Eq.(13) at the same time, we assemble

$$\begin{bmatrix} [A(k_t)] \\ [C(k_t)] \end{bmatrix} \{\psi^{AC}\} = \{0\}, \tag{22}$$

where $\{\psi^{AC}\}$ is the true boundary eigensolution for the original problem.
Similarly, we have

$$\begin{bmatrix} [B(k_t)] \\ [D(k_t)] \end{bmatrix} \{\psi^{BD}\} = \{0\}, \tag{23}$$

where $\{\psi^{BD}\}$ is the true eigenvalue for the complementary problem.

RESULTS AND DISCUSSIONS

A circular cavity with a radius 1 m is considered. The boundary conditions and the number of boundary elements are shown in the Tables 2(a)-2(d). Four approaches, the complex-valued BEM, the real-part BEM, the imaginary-part BEM and the MRM were utilized to solve the eigenproblem. The Fredholm's alternative theorem and the SVD updating documents were applied to filter out the spurious eigenvalues in the numerical experiments. The true eigenvalues were sorted out by using the SVD updating terms. Then the minimum singular values versus wave numbers for detecting the true and spurious eigenvalues were plotted in Tables 2(a)-2(d). Although the complex-valued BEM is free of spurious eigenvalues, the real-part BEM, the imaginary-part BEM and the MRM by using the singular formulation (UT method) result in the spurious eigenvalues at the zeros of $Y_n(k)$, $J_n(k)$ and $\bar{Y}_n(k)$ functions, respectively. For the hypersingular formulation (LM method), the spurious eigenvalues locate at the zeros of $Y_n'(k)$, $J_n'(k)$ and $\bar{Y}_n'(k)$ functions for the real-part, the imaginary-part BEMs and the MRM, respectively. The spurious eigenvalues occur at the same positions for each method even though they are employed to deal with either the Dirichlet or the Neumann problem [14]. There is no doubt that the diagrams plotted for the original problem are the same with those for the complementary problem due to symmetry. Based on the dual integral equations, the former five eigenmodes of the acoustic cavities were plotted in Table 1. To verify the accuracy, the FEM results are also shown for comparisons. It indicates that spurious eigenvalues depend on the formulation while true eigenvalues depend on the problem. The results show that good agreement is made.

CONCLUDING REMARKS

The study has successfully detected the spurious and the true eigensolutions for acoustic cavities with the mixed-type boundary conditions by using the complex-valued BEM, the real-part BEM, the imaginary-part BEM and the MRM. By using the Fredholm's alternative theorem and SVD updating documents in conjunction with the dual formulations, the spurious eigenvalues were sorted out successfully. It is numerically verified that the spurious eignvalues depend on the representation (singular or hypersingular; single layer or double layer) no matter what the given types of boundary conditions for the problem are.

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邊界元素法於混合型邊界條件之二維聲場真假特徵解探討

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摘要

本文採用複數型邊界元素法、實數部邊界元素法、虛數部邊界元素法及多倒易法有限元素法求解混合型邊界條件聲場之自然聲頻與自然聲模並與 FEM 的結果來比較。以複數型邊界元素法解內域問題不會有假特徵值。採用實數部邊界元素法、虛數部邊界元素法或多倒易法解內域問題則會產生假特徵值。此結果與 Dirichlet 邊界條件或 Neumann 邊界條件之聲場問題所產生的假特徵值相同，再次說明了假特徵值的出現只與使用的方法（奇異方程或超強奇異方程、單層勢能或雙層勢能）有關，而與問題的邊界條件型式無關。文中算例均成功地予以數值驗證。

關鍵字：混合型邊界條件、對偶邊界元素法、奇異方程、超強奇異方程

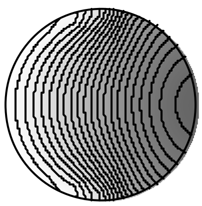
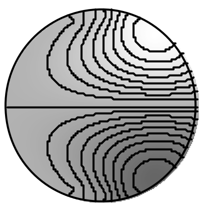
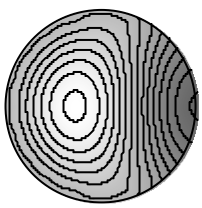
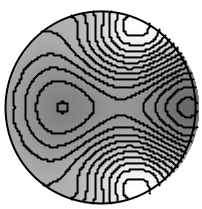
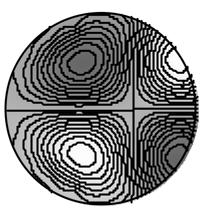
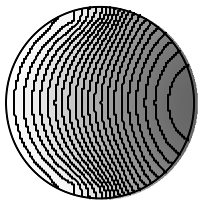
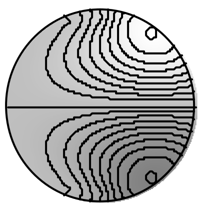
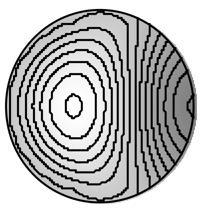
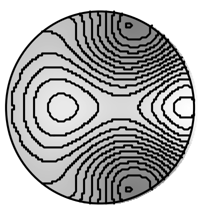
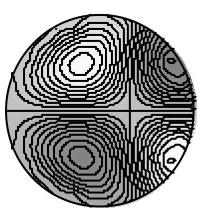
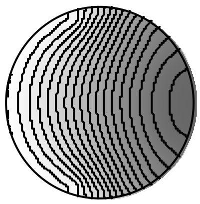
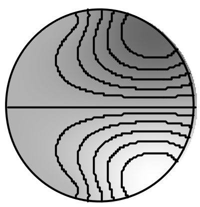
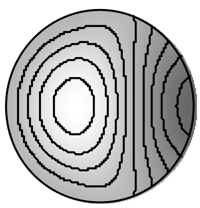
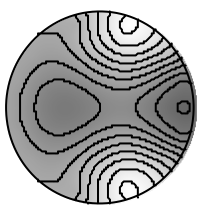
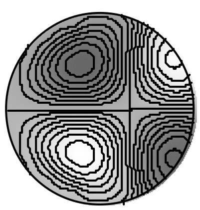
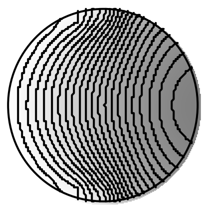
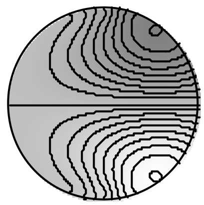
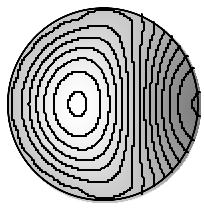
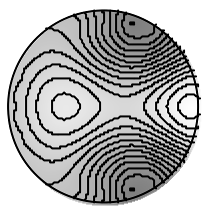
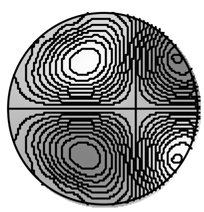
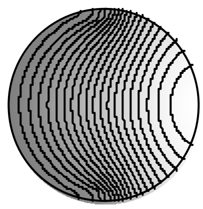
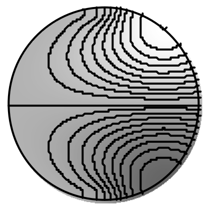
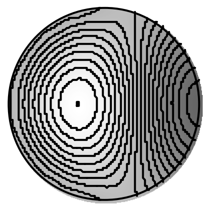
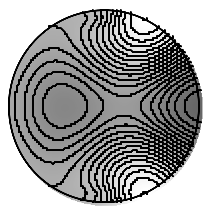
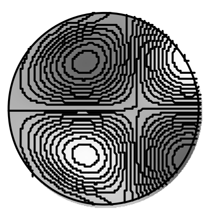
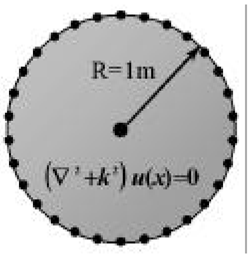
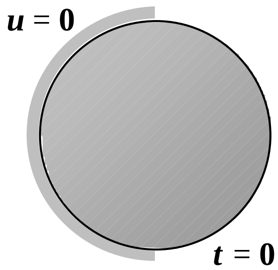
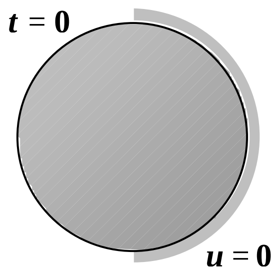
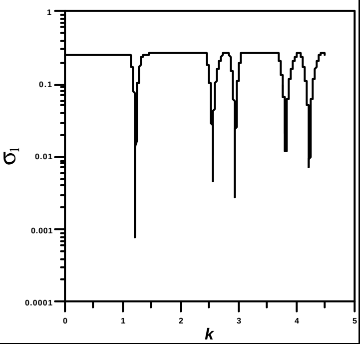
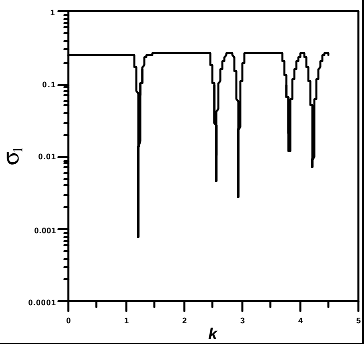
Complex–valued BEM	**Real-part BEM**	**Imaginary-part BEM**	**MRM**	**FEM**
	$U(s,x)=\frac{\pi Y_0(kr)}{2}$		$U(s,x)=\frac{\pi}{2}\bar{Y}_0(kr)$	-
 k_1 =1.233  k_2 =2.564  k_3 =2.955  k_4 =3.828  k_5 =4.238	 k_1=1.222  k_2=2.544  k_3=2.954  k_4=3.802  k_5=4.231	 k_1=1.235  k_2=2.585  k_3=3.019  k_4=3.873  k_5=4.352	 k_1 =1.222  k_2 =2.546  k_3 =2.956  k_4 =3.803  k_5 =4.206	 k_1 =1.254  k_2 =2.593  k_3 =2.934  k_4 =3.842  k_5 =4.194

Table 1. The comparison for the former five eigenvalues and eigenmodes by using different methods, the complex-valued BEM, the real-part BEM, the imaginary-part BEM, the MRM and the FEM, where $\bar{Y}_0(kr)=Y_0(kr)-\frac{2}{\pi}\left[\ln\frac{k}{2}+\gamma\right]J_0(kr)$, $\gamma=0.5772$.

<table>
<tr><td colspan="3" rowspan="2"><u>Boundary element mesh</u>
N=30
</td><td colspan="2">SVD updating terms for the true eigensolutions are not required</td></tr>
<tr><td>Original problem
</td><td>Complementary problem
</td></tr>
<tr><td rowspan="2">SVD updating documents for the spurious eigensolutions are not required</td><td>Singular formulation (UT method)</td><td>BIE:
$pu = C.P.V.\int Tu\,dB - R.P.V.\int Ut\,dB$ [4]

BEM:
$[T]\{u\} = [U]\{t\}$</td><td>$[A]$
</td><td>$[B]$
</td></tr>
<tr><td>Hypersingular formulation (LM method)</td><td>BIE:
$pu = H.P.V.\int Mu\,dB - C.P.V.\int Lt\,dB$ [4]

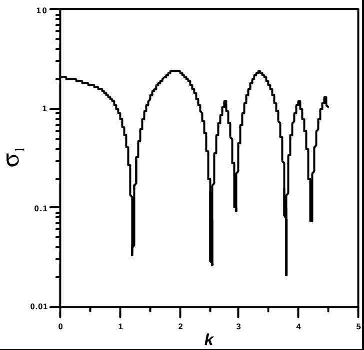
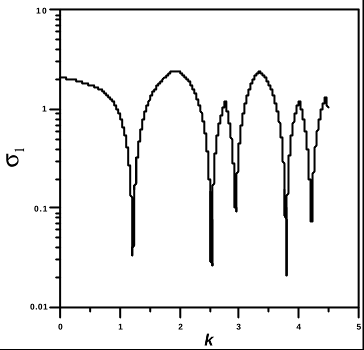
BEM:
$[M]\{u\} = [L]\{t\}$</td><td>$[C]$
</td><td>$[D]$
</td></tr>
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Table 2(a). Detection of true eigenvalues by using the complex-valued BEM.

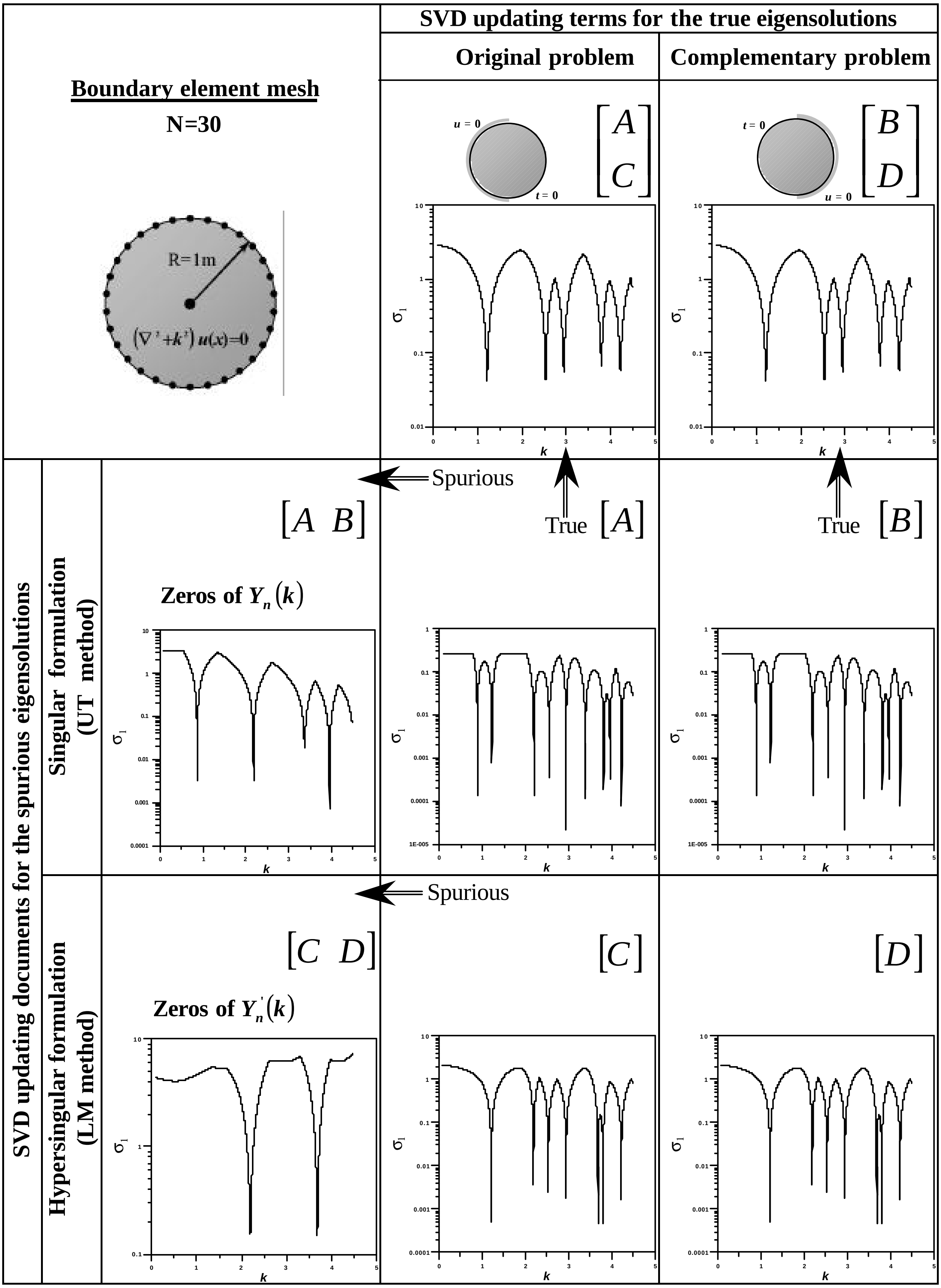

Table 2(b). Detection of true and spurious eigenvalues by using the SVD updating terms and updating documents for the real-part BEM.

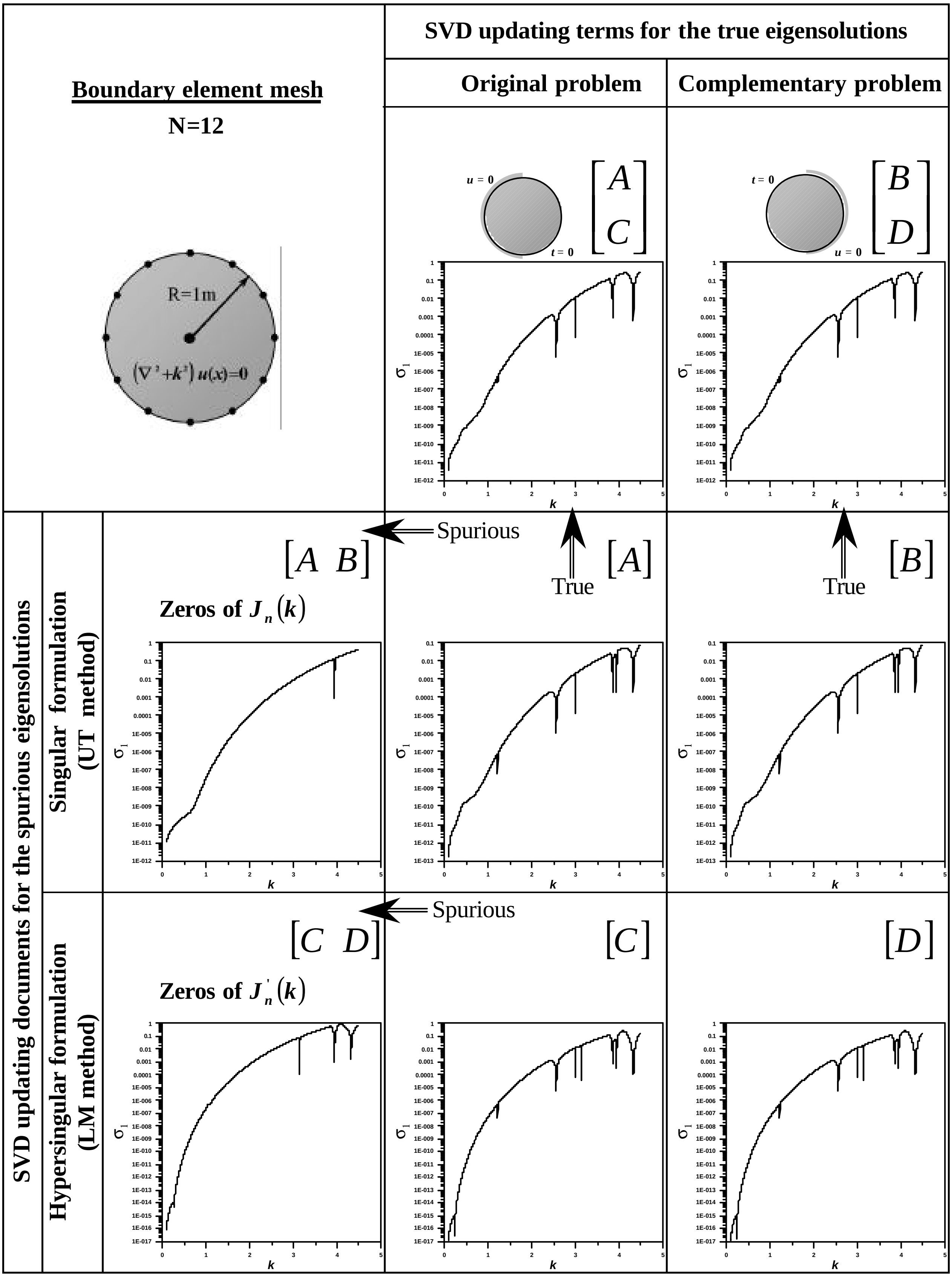

Table 2(c). Detection of true and spurious eigenvalues by using the SVD updating terms and updating documents for the imaginary-part BEM.

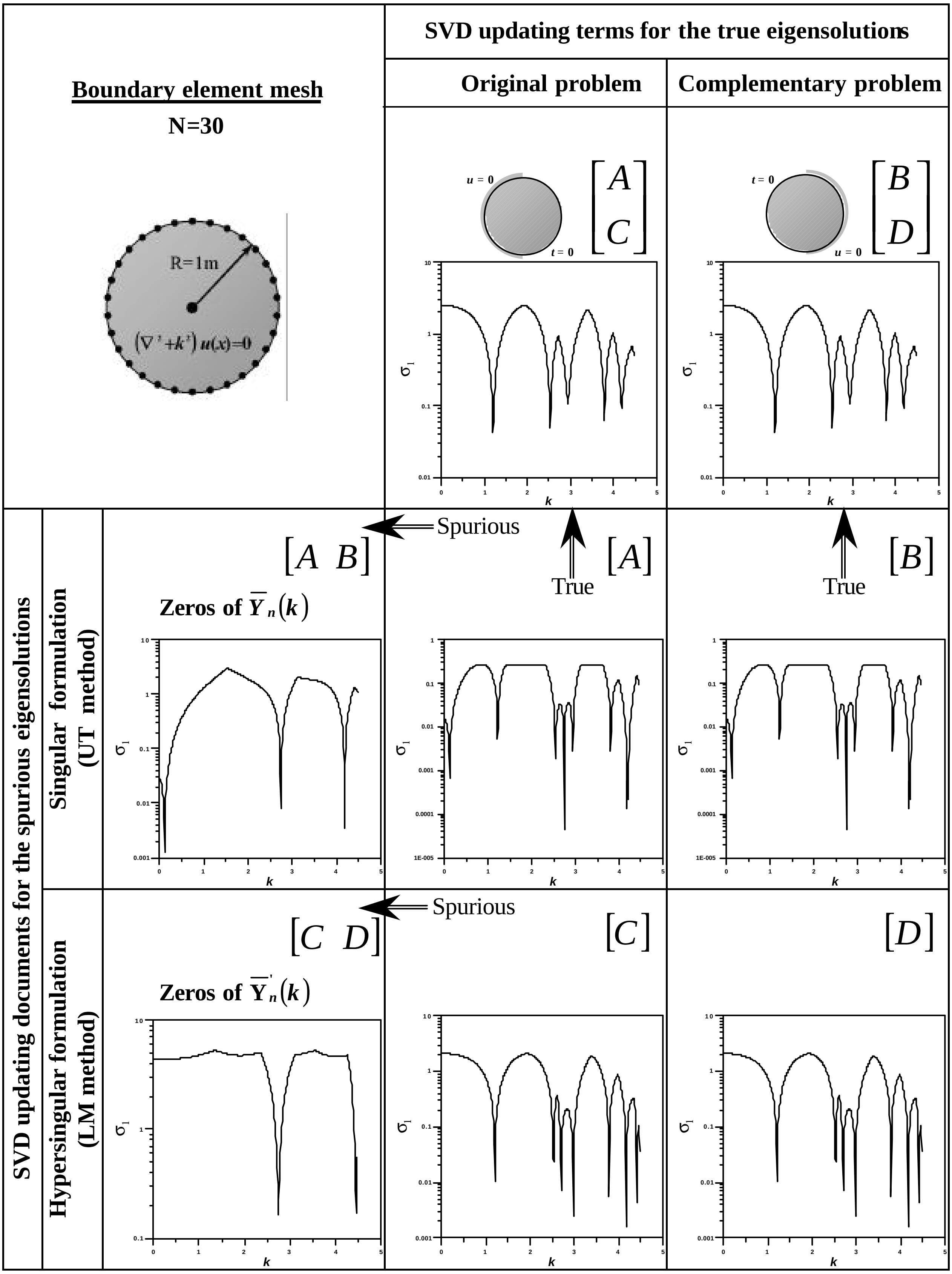

Table 2(d). Detection of true and spurious eigenvalues by using the SVD updating terms and updating documents for the MRM.