ON THE DUAL INTEGRAL REPRESENTATION OF BOUNDARY VALUE PROBLEM IN LAPLACE EQUATION

J.T. Chen & H.K. Hong
Department of Civil Engineering, National Taiwan University Taipei, Taiwan

1.Introduction

The hypersingular integral equation was first encountered by Hadamard [1] in treating the cylindrical wave equation by the spherical means of descent. In the meantime, Mangler derived the same mathematical form in solving the thin airfoil problem.[5] The improper integral was then defined by Tuck [2] as the ' Hadamard principal value'. Such a nonintegrable integral naturally arises in the dual boundary integral formulation especially with a degenerate boundary, e.g. crack problems in elasticity,[3,4,8,9] heat flow through baffle,[6] Darcy flow around cutoff wall,[7,13] or aerodynamic problem of thin airfoil.[10] In aerodynamic literature, it has been termed 'Mangler's principal value'.[5,10] A general application of the hypersingular integral equation in mechanics is discussed in Ref.11. Combining the conventional integral equation, e.g.Green's Identity or the Somigliana Identity, with the hypersingular integral equation, one obtains the dual integral equations, due to the pair of continuous and discontinuous properties of the potential as the field point x approaches the boundary.[12] In the above point of view, the definition of dual integral equations is quite different from the conventional one by Buecker.[14] These two equations are independent with respect to the undetermined coefficient of the complementary solution and in total have four kernel functions, which make a unified theory possible, encompassing different schemes, various derivations and interpretations. This type of singularity is one order stronger than the Cauchy principal value. The paradox of the nonintegrable kernel is introduced due to the wrong change of integral and trace operator from the point of view of the dual integral formulation. In order to ensure the finite value, Leibnitz' rule should be considered as the derivative of C.P.V. is taken so that the boundary terms $2/\varepsilon$ can not be neglected. Therefore, the finite value can be confirmed. Based on the theory of dual integral equations, the dual boundary element method can be implemented.[15] The purpose of this brief paper is to establish the general dual integral formulation and to examine the properties of the four kernel functions.

2.General dual integral representation of boundary value problem

Consider the classical Laplace problem as follows,

Governing equation:

$$\nabla^2\phi = 0, \;\; x \, in \, D \tag{1}$$

Boundary conditions:

$$\phi(x) = f(x), \;\; x \, on \, B_1 \tag{2}$$

$$\frac{\partial\phi(x)}{\partial n_x} = g(x), \;\; x \, on \, B_2 \tag{3}$$

The first equation of dual domain integral equations can be derived from the Green's third identity:

$$\begin{aligned} 2\pi\phi(x) &= \int_B T(s,x)\phi(s)dB(s) \\ &\quad - \int_B U(s,x)\frac{\partial\phi(s)}{\partial n_s}dB(s) \end{aligned} \tag{4}$$

After taking the normal derivative of eqn(4), the second equation of dual domain integral equations can be derived,

$$\begin{aligned} 2\pi\frac{\partial\phi(x)}{\partial n_x} &= \int_B M(s,x)\phi(s)dB(s) \\ &\quad - \int_B L(s,x)\frac{\partial\phi(s)}{\partial n_s}dB(s) \end{aligned} \tag{5}$$

where

$$T(s,x) = \frac{\partial U(s,x)}{\partial n_s} \tag{6}$$

$$L(s,x) = \frac{\partial U(s,x)}{\partial n_x} \tag{7}$$

$$M(s,x) = \frac{\partial^2 U(s,x)}{\partial n_x \partial n_s} \tag{8}$$

By tracing the field point x to the boundary, the dual boundary integral equations can be obtained.

$$\begin{aligned} \alpha\phi(x) &= C.P.V. \int_B T(s,x)\phi(s)dB(s) \\ &\quad - R.P.V. \int_B U(s,x)\frac{\partial\phi(s)}{\partial n_s}dB(s) \end{aligned} \tag{9}$$

$$\begin{aligned} \alpha\frac{\partial\phi(x)}{\partial n_x} &= H.P.V. \int_B M(s,x)\phi(s)dB(s) \\ &\quad - C.P.V. \int_B L(s,x)\frac{\partial\phi(s)}{\partial n_s}dB(s) \end{aligned} \tag{10}$$

where

R.P.V. : Riemann principal value
C.P.V. : Cauchy principal value
H.P.V. : Hadamard(Mangler) principal value
α depends on the solid angle, $\alpha = \pi$ if smooth boundary in 2-D.

Table 1: The explicit form of kernel functions

Kernel function	$U(s,x)$	$T(s,x)$	$L(s,x)$	$M(s,x)$
Order of singularity	weak	strong	strong	hypersingular
Symmetry	$U(x,s)$	$L(x,s)$	$T(x,s)$	$M(x,s)$
Two dimensional case	$ln(r)$	$-y_i n_i/r^2$	$y_i\bar{n}_i/r^2$	$2y_iy_jn_i\bar{n}_j/r^4$ $-n_i\bar{n}_i/r^2$
Three dimensional case	$-1/r$	$-y_i n_i/r^3$	$y_i\bar{n}_i/r^3$	$3y_iy_jn_i\bar{n}_j/r^5$ $-n_i\bar{n}_i/r^3$
Remark	$r^2=y_iy_i$	$n_i=n_i(s)$	$\bar{n}_i=n_i(x)$	$y_i=x_i-s_i$

It must be noted that eqn(10) can be derived just by applying the normal derivative operator with respect to eqn(9). Differentiation of the Cauchy principal value should be carried out carefully by using the Leibnitz rule. The commutative property provides us with two alternatives to calculate the Hadamard principal value. For the degenerate boundary $B = C^+ + C^-$, eqns(4),(5),(9) and (10) can be reformed as,

For $x \in$ domain point,

$$2\pi\phi(x) = \int_{C+} T(s,x)\Delta\phi(s)dB(s) - \int_{C+} U(s,x)\Sigma\frac{\partial\phi(s)}{\partial n_s}dB(s) \qquad (11)$$

$$2\pi\frac{\partial\phi(x)}{\partial n_x} = \int_{C+} M(s,x)\Delta\phi(s)dB(s) - \int_{C+} L(s,x)\Sigma\frac{\partial\phi(s)}{\partial n_s}dB(s) \qquad (12)$$

For $x \in C^+$,

$$\alpha\Sigma\phi(x) = C.P.V.\int_{C+} T(s,x)\Delta\phi(s)dB(s) - R.P.V.\int_{C+} U(s,x)\Sigma\frac{\partial\phi(s)}{\partial n_s}dB(s) \qquad (13)$$

$$\alpha\Delta\frac{\partial\phi(x)}{\partial n_x} = H.P.V.\int_{C+} M(s,x)\Delta\phi(s)dB(s) - C.P.V.\int_{C+} L(s,x)\Sigma\frac{\partial\phi(s)}{\partial n_s}dB(s) \qquad (14)$$

where

$$\Sigma\phi(s) = \phi(s^+) + \phi(s^-) \qquad (15)$$

$$\Delta\phi(s) = \phi(s^+) - \phi(s^-) \qquad (16)$$

$$\Sigma\frac{\partial\phi}{\partial n}(s) = \frac{\partial\phi}{\partial n}(s^+) + \frac{\partial\phi}{\partial n}(s^-) \qquad (17)$$

$$\Delta\frac{\partial\phi}{\partial n}(s) = \frac{\partial\phi}{\partial n}(s^+) - \frac{\partial\phi}{\partial n}(s^-) \qquad (18)$$

Equations(15)-(18) reveal that as the unknowns on the degenerate boundary doubled, the additional hypersingular integral equation (14) is correspondingly necessary, i.e. the dual boundary integral equations provides the sufficient constraint relations of boundary data.

3.On the four kernel functions

There are four kernel functions in the dual integral equations. The properties of the four kernel functions are summarized in Table 1 and 2. The potential of the four kernel functions induced by a constant singularity source distributed at $s=0\sim1$ is shown in Fig.1. The behavior of the single layer, double layer, derivative of single layer and derivative of the double layer are displayed in this figure.

4.Conclusion

The general formulation of dual integral equations of the boundary value problem for the Laplace equation with a degenerate boundary has been derived in this paper. The properties of the four kernel functions in the dual integral equations are discussed and their potentials are also displayed.

Table 2: The properties of single and double layer potentials

Kernel function $K(s,x)$	$U(s,x)$	$T(s,x)$	$L(s,x)$	$M(s,x)$	$L^t(s,x)$	$M^t(s,x)$
Density function $\mu(s)$	$\partial\phi/\partial n$	ϕ	$\partial\phi/\partial n$	ϕ	$\partial\phi/\partial n$	ϕ
Potential type $\int K(s,x)\mu(s)ds$	single layer	double layer	normal derivative of single layer potential	normal derivative of double layer potential	tangent derivative of single layer potential	tangent derivative of double layer potential
Continuity across boundary	conti-nuous	disconti-nuous	disconti-nuous	pseudo conti-nuous	conti-nuous	disconti-nuous
Jump value	no jump	$2\pi\phi$	$-2\pi\partial\phi/\partial n$	no jump	no jump	$2\pi\partial\phi/\partial t$

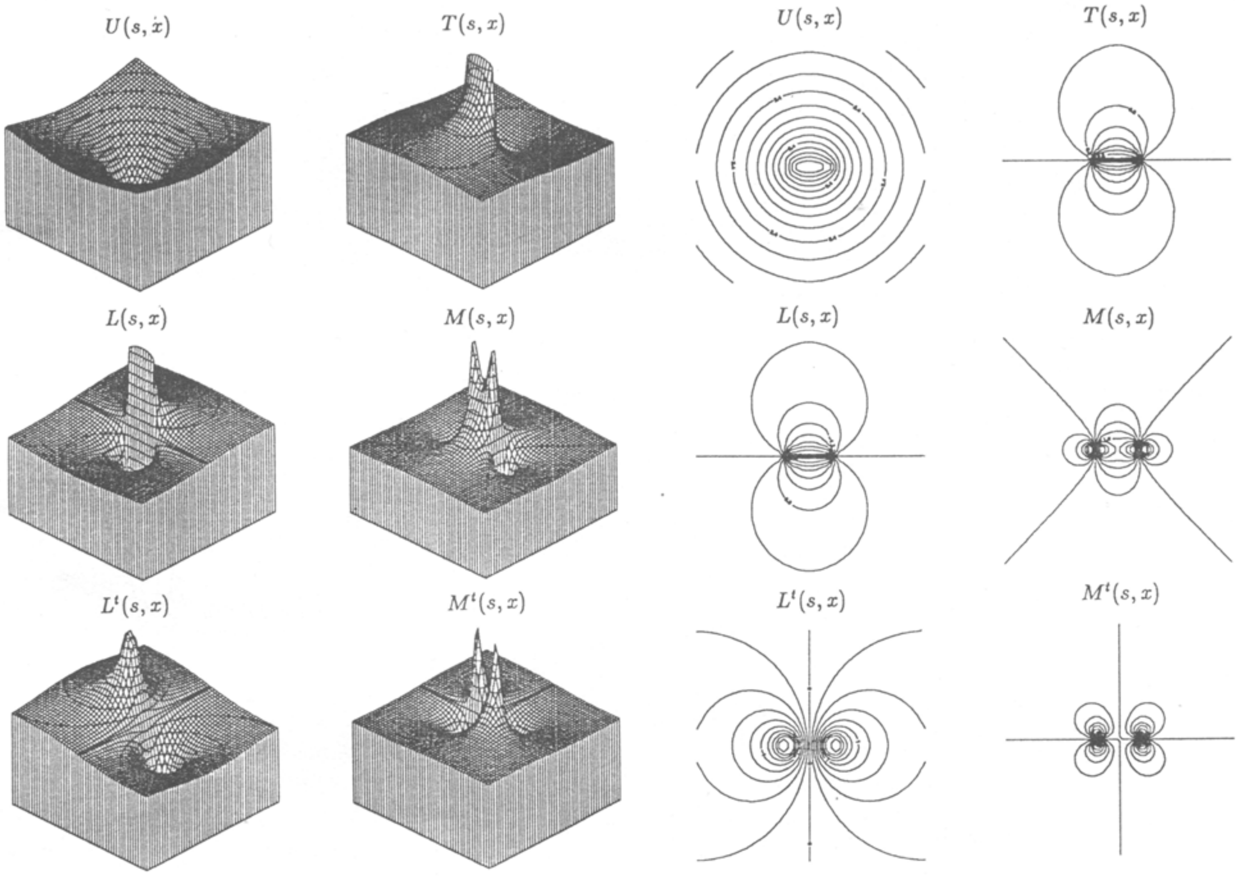

Figure 1: The potential distribution of the four kernel functions.

References

1.Hadamard, J. *Lectures on Cauchy's Problem in Linear Partial Differential Equations*, Dover, 1952.

2.Tuck, E.O. Application and solution of Cauchy singular integral equations, *The Application and Numerical Solution of Integral Equations*, ed. R.S. Anderson *et al.*, Sijthoff and Noordhoff, 1980.

3.Hong, H.K. & Chen, J.T. Derivations of integral equations in elasticity, J*ournal of ASCE*, June 1988, **114**(6), Em5, 1028-1044.

4.Chen, J.T. On Hadamard principal value and boundary integral formulation of fracture mechanics, Master Thesis, Institute of Applied Mechanics, National Taiwan University, Taiwan, 1986.

5.Mangler, K.W. Improper integrals in theoretical aerodynamics, RAE Report, No.2424, 1951.

6.Chen, J.T. & Hong, H.K. Application of integral equations with superstrong singularity to steady state heat conduction, *Thermochimica Acta*, 1988, **135**, 133-138.

7.Chen, J.T. & Hong H.K. Singularity in Darcy flow around a cutoff wall, *Advances in Boundary Elements*, Vol.2, Field and Flow Solution, ed. C.A. Brebbia & J.J. Conner, 1989, pp.15-27.

8.Hong, H.K. & Chen, J.T. Generality and special cases of dual integral equations of dual integral equations of elasticity, *J. CSME*, 1988, **9**(1), 1-19.

9.Chen, J.T. & Hong, H.K. On Hadamard principal value and its application to crack problems through BEM, The 11th National Conference on Theo. and Appl. Mech., Taiwan, 1987.

10.Cheng-Sheng Wang, Sing Chu & Jeng-Tzong Chen, Boundary element method for predicting store airloads during its carriage and separation procedures, *Computational Engineering with Boundary Elements*, Vol.1, Fluid and Potential Problems, ed. Grilli *et al.*, CMP, 1990.

11.Martin, P.A., Rizzo, F.J. & Gonsalves, I.R. On hypersingular integral equations for certain problems in mechanics, *Mech. Res. Commun.*, 1989, **16**(2), 65-71.

12.Chen, J.T. & Hong, H.K. *Boundary Element Method,* Second Edition, New World Press, Taipei, Taiwan, 1992, (in Chinese).

13.Gray, L.J. Boundary element method for regions with thin internal cavities, *Engineering Analysis with Boundary Elements*, 1989, **6**(4).

14.Buecker, H.F. Field singularities and related integral representations, *Mechanics of Fracture*, Vol.1, ed. G.C. Sih, 1973.

15.Portela, A., Aliabadi, M.H. & Rooke, D.P. The dual boundary element method : Effective implementation for crack problems, *Int. J. Num. Meth. Engng*, 1992, **33**, 1269-1287.