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An integral–differential equation approach for the free vibration of a SDOF system with hysteretic damping

J.T. Chen, D.W. You

Department of Harbor and River Engineering, Taiwan Ocean University, Keelung, Taiwan, ROC

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Abstract

An integral–differential equation (IDE) in the time domain is proposed for the free vibration of a single-degree-of-freedom (SDOF) system with hysteretic damping which is different from the conventional complex stiffness model as employed in the frequency domain. The integral of the Hilbert transform is embedded in the IDE and is calculated in the Cauchy principal value sense by using a numerical folding technique. Numerical experiments show that the free vibration obtained by the frequency domain approach satisfies the IDE in the time domain. A successive iteration algorithm is employed to solve the IDE subject to forced vibration, and a convergent solution for the hysteresis loop is constructed, which matches the solution found by using the frequency domain approach. Both models, the time domain and frequency domain approaches, present the noncasual effect since they are equivalent in the mathematical sense.

1. Introduction

The damping characteristic is often utilized to suppress the vibration level using various dissipation mechanisms. In this decade, two books on the topic of damping have been published [1, 2]. A great deal of effort has been focused on the frequency domain approach, especially for the hysteretic damping model. Recently, Chen et al. [3] have successfully solved the single-degree-of-freedom (SDOF) hysteretic damping model in the time domain by using the concept of phase plane. However, Crandall [4] criticized the model in Ref. [3] for not being fully equivalent to the hysteretic damping model in the frequency domain. The time-domain governing equation with the Hilbert transform for hysteretic damping was derived by Chen [3], and Inaudi and Kelly [5] independently. Inaudi and Kelly solved the time-domain equation by using the iteration technique with fictitious viscous damping and solved the transient solution for the pulse forcing function. In this paper, we employ a direct iteration technique to solve the time-domain governing equation for harmonic loading. Also, the hysteresis loop is constructed by using the time-domain approach. To deal with the numerical integral of the Hilbert transform, a folding technique for the Cauchy principal value is employed [6]. It is demonstrated that the free vibration obtained by the frequency domain approach using a real integral and fast Fourier transform (FFT) satisfies the governing equation in the time domain.

2. Method of solution

2.1. Frequency domain approach

The governing equation of a SDOF system for hysteretic damping model has been formulated as [7]:

$$m\ddot{x} + \frac{h}{\bar{\omega}}\dot{x} + kx = \bar{p}(\bar{\omega})e^{i\bar{\omega}t} \tag{1}$$

where m, h and k represent the mass, hysteretic damping coefficient and stiffness, respectively. $\bar{p}$ and $\bar{\omega}$ are the amplitude of the harmonic loading and the exciting frequency, respectively. To make the transfer functions conjugate for $-\bar{\omega}$ and $\bar{\omega}$, the governing equation has been modified to be

$$m\ddot{x} + \frac{h}{|\bar{\omega}|}\dot{x} + kx = \bar{p}(\bar{\omega})e^{i\bar{\omega}t} \tag{2}$$

Although good for harmonic motion, Eq. (2) is invalid for free vibration since, when the forcing term, $\bar{p}e^{-i\bar{\omega}t}$, is set to vanish, the presence of $|\bar{\omega}|$ in the denominator of Eq. (2) is ambiguous. Therefore, only the steady-state solution can be obtained. In other words, the hysteretic damping is focused on the frequency domain model [8, 9] as follows:

$$-m\bar{\omega}^2\bar{x} + k(1 + sgn(\bar{\omega})i\eta)\bar{x} = \bar{p} \tag{3}$$

where η is the loss factor, $k\eta$ is equal to h, and $sgn(\bar{\omega})$ is $+1$ when $\bar{\omega} > 0$ and -1 when $\bar{\omega} < 0$. To solve for the

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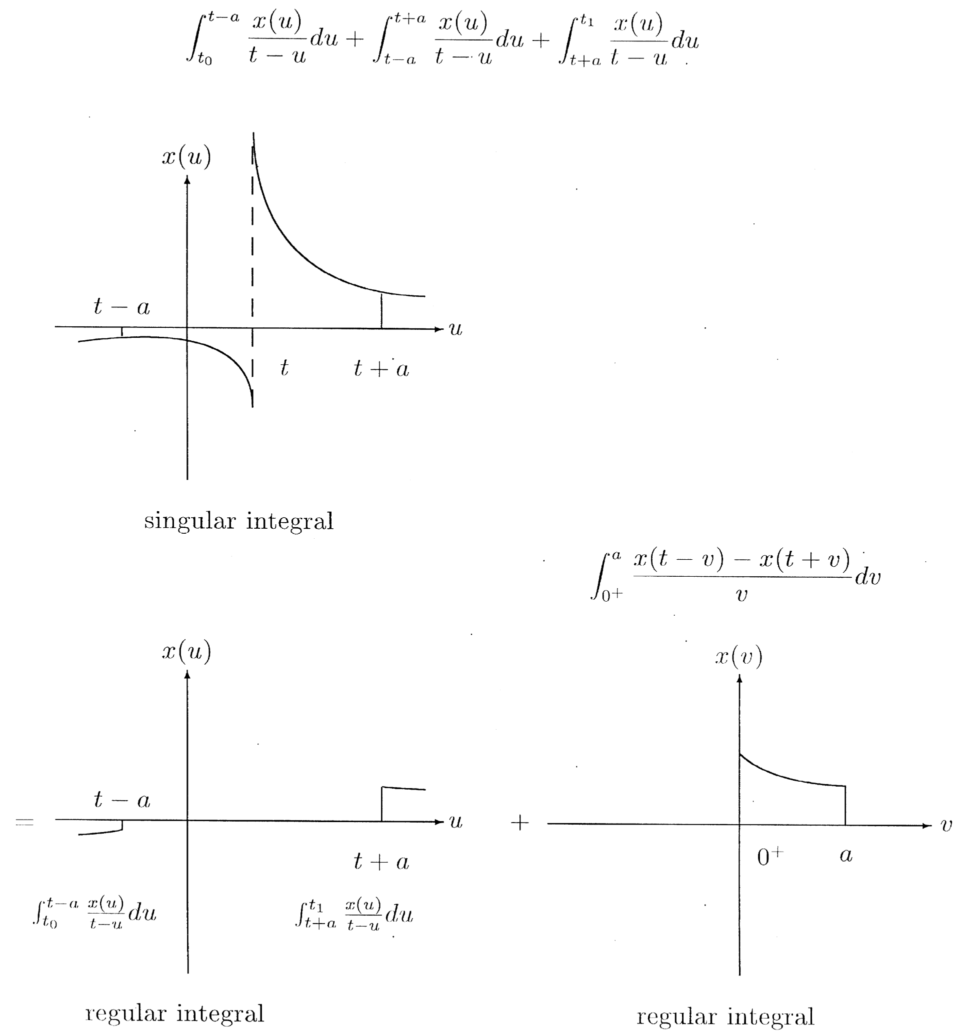

Fig. 1. A folding technique for the Cauchy principal value in the Hilbert transform.

steady-state response, two approaches in the frequency domain are employed as follows:

(1) Method of a real integral

$$x(t) = \frac{1}{\pi} \int_0^{\infty} \frac{(k - m\bar{\omega}^2)\cos \bar{\omega}t + k\eta \sin \bar{\omega}t}{(k - m\bar{\omega}^2)^2 + (k\eta)^2} \, d\bar{\omega} \tag{4}$$

(2) Method of the FFT technique

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\bar{\omega}t}}{(k - m\bar{\omega}^2 \pm ik\eta)} \, d\bar{\omega} \tag{5}$$

2.2. Time domain approach

By taking the inverse Fourier transform of Eq. (3), the governing equation in the time domain which has been derived by Chen [10] and Inaudi and Kelly [5] independently is as follows:

$$m\ddot{x}(t) - \frac{k\eta}{\pi} \int_{-\infty}^{\infty} \frac{x(u)}{(t-u)} \, du + kx(t) = p(t) \tag{6}$$

with the conditions at $t = -\infty$ being

$$x(t)|_{t=-\infty} = 0, \;\; \dot{x}(t)|_{t=-\infty} = 0 \tag{7}$$

Eqs. (6) and (7) can be viewed as the governing equation and initial conditions in the time domain for the hysteretic damping model, respectively. If $p(t)$ in Eq. (6) vanishes, the solution, $x(t)$, becomes the free vibration, which can be compared with the solution obtained by using the method of a real integral in Eq. (4) and FFT in Eq. (5). To construct the hysteresis loop in the time domain, the forcing function, $p(t)$, in Eq. (6) must be harmonic excitation. In order to solve the integral–differential equation for Eq. (6) by using the iteration

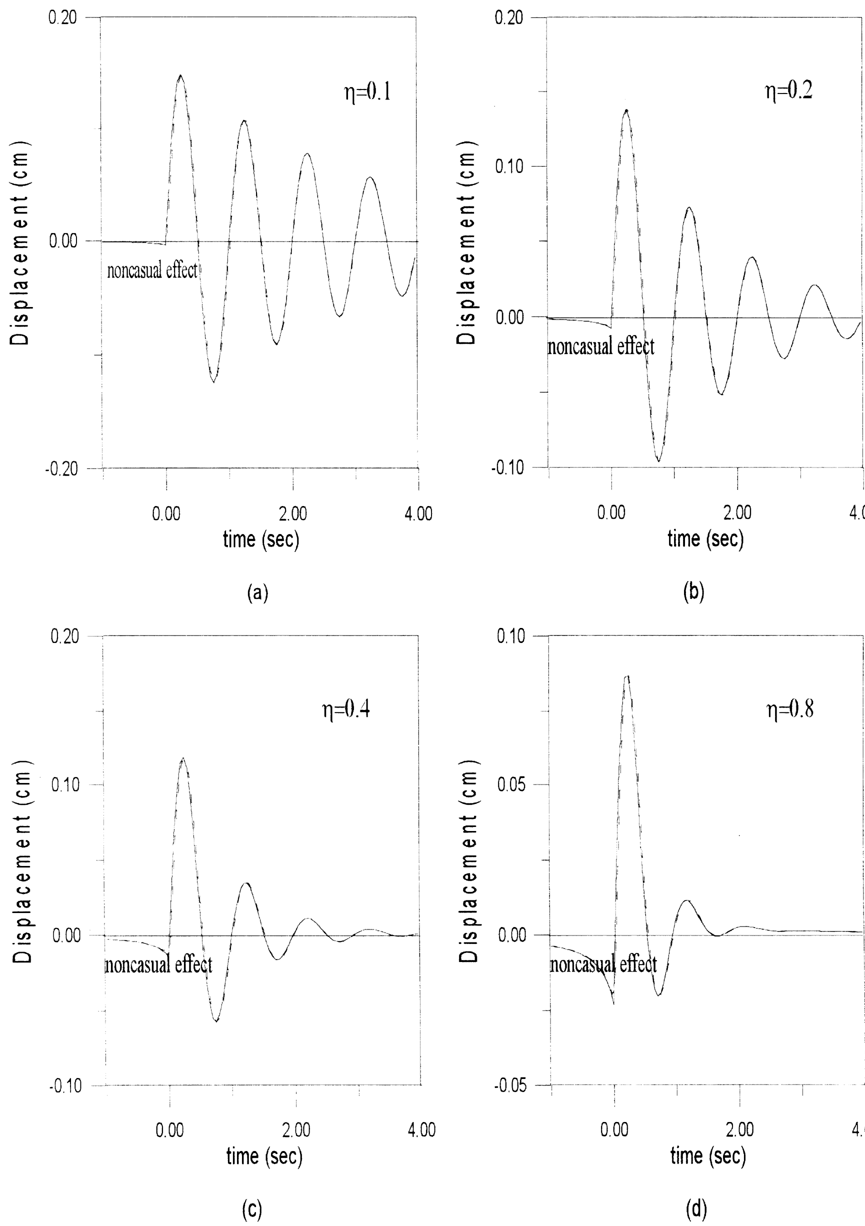

Fig. 2. Free vibration by real integral and FFT in the frequency domain for (a) $\eta = 0.1$, (b) $\eta = 0.2$, (c) $\eta = 0.4$, (d) $\eta = 0.8$. ———, FFT ($N = 1024$); - - -, real integral.

technique, Eq. (6) can be reduced to the following form:

$$\ddot{x}_{n+1}(t) + 2\xi\omega\dot{x}_{n+1}(t) + \omega^2 x_{n+1}(t) = 2\xi\omega\dot{x}_n(t) + \frac{\omega^2\eta}{\pi}\int_{-\infty}^{\infty}\frac{x_n(u)}{(t-u)}\,\mathrm{d}u + \frac{p(t)}{m} \quad (8)$$

where $x_n(t)$ denotes the nth iteration state for $x(t)$, ξ is the artificial viscous damping ratio, and $\omega = \sqrt{k/m}$. By using the Duhamel integral and treating the terms on the right-hand side of the equal sign in Eq. (8) as external forces, Eq. (8) can be reduced to the following iteration form:

$$x_{n+1}(t) = \frac{1}{\omega\sqrt{1-\xi^2}}\int_{-\infty}^{\infty} \mathrm{e}^{-\xi\omega(t-\tau)}\sin(\omega\sqrt{1-\xi^2}(t-\tau)) \times\left\{\frac{\omega^2\eta}{\pi}\int_{-\infty}^{\infty}\frac{x_n(u)}{(\tau-u)}\,\mathrm{d}u + 2\xi\omega x_n(\tau) + \frac{p(\tau)}{m}\right\}\mathrm{d}\tau \quad (9)$$

By iterating $x_n(t)$ in Eq. (9), a hysteresis loop can be constructed after setting harmonic loading for $p(t)$ and the convergent solution can be obtained using the following

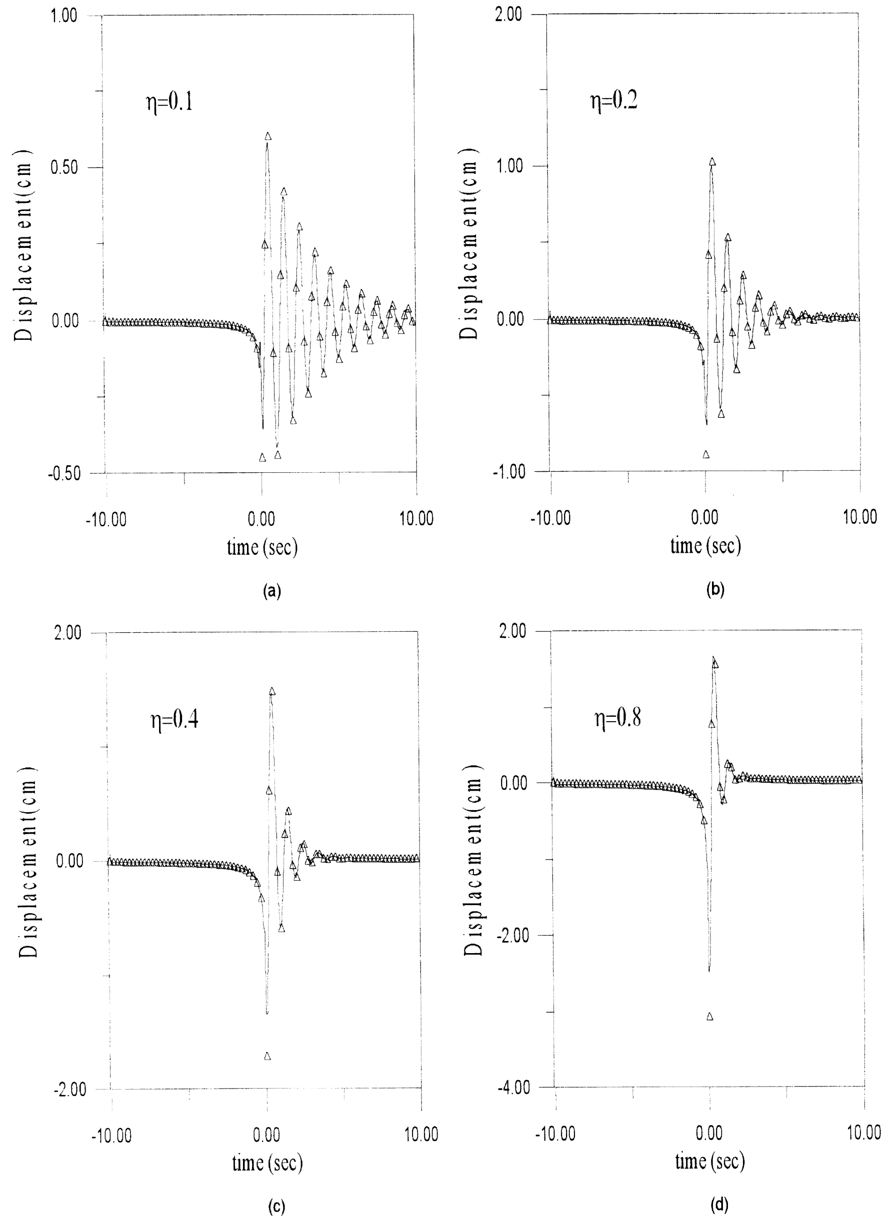

Fig. 3. Diagram for checking the solution obtained by using the frequency domain approach to satisfy the governing equation in the time domain: (a) $\eta = 0.1$, (b) $\eta = 0.2$, (c) $\eta = 0.4$, (d) $\eta = 0.8$. ———, $\frac{k\eta}{\pi}\int_{-\infty}^{\infty}\frac{x(u)}{(t-u)}\,\mathrm{d}u$; △△△, $m\ddot{x}(t) + kx(t)$.

criteria:

$$|x_n(t) - x_{n+1}(t)| < \epsilon, \quad \text{if } n > N \tag{10}$$

where ϵ is tolerance and N is the number of iterations. However, it is found that the Hilbert transform is present in the damping force term of Eq. (6) and in the forcing term of Eq. (9), so a numerical folding technique for the integral will be elaborated on in the next section.

3. A numerical folding technique for the Cauchy principal value

To deal with the Cauchy principal value in the Hilbert transform in Eqs. (6) and (9) for free vibration and forced vibration, respectively, we can divide the domain of integral into three regions, two regular integrals and one singular integral, as shown in Fig. 1 and below:

$$\int_{t_0}^{t_1}\frac{x(u)}{(t-u)}\,\mathrm{d}u = \int_{t_0}^{t-a}\frac{x(u)}{(t-u)}\,\mathrm{d}u + \int_{t-a}^{t+a}\frac{x(u)}{(t-u)}\,\mathrm{d}u + \int_{t+a}^{t_1}\frac{x(u)}{(t-u)}\,\mathrm{d}u \tag{11}$$

By folding the domain of the second integral on the right-hand side of the equal sign in Eq. (11), we have

$$\int_{0^+}^{a}\frac{x(t-v) - x(t+v)}{v}\,\mathrm{d}v \tag{12}$$

It is easy to find that the singularity at $t=u$ in Eq. (11) can be transformed so as to be regular at $v=0$ in Eq. (12), since

$$\lim_{v\to 0} \frac{x(t-v)-x(t+v)}{v}\,\mathrm{d}v = -\frac{1}{2}x'(t^{+}) - \frac{1}{2}x'(t^{-}) \tag{13}$$

is bounded; i.e. the integrand in the Hilbert transform is reduced to be nonsingular. Many available numerical techniques can be used to determine the regular integral easily.

4. Numerical results and discussion

4.1. Free vibration by using the frequency domain approach

By setting

$$m = 1\ \mathrm{kg},\quad k = 4\pi^2\ \mathrm{N\ m^{-1}},\quad \omega = 2\pi\ \mathrm{rad\ s^{-1}}$$

the free vibration approach can be obtained by using the frequency domain approach and by using either Eq. (4) or Eq. (5). The free vibration responses are shown in Fig. 2 for four cases: (a) $\eta=0.1$, (b) $\eta=0.2$, (c) $\eta=0.4$ and (d) $\eta=0.8$.

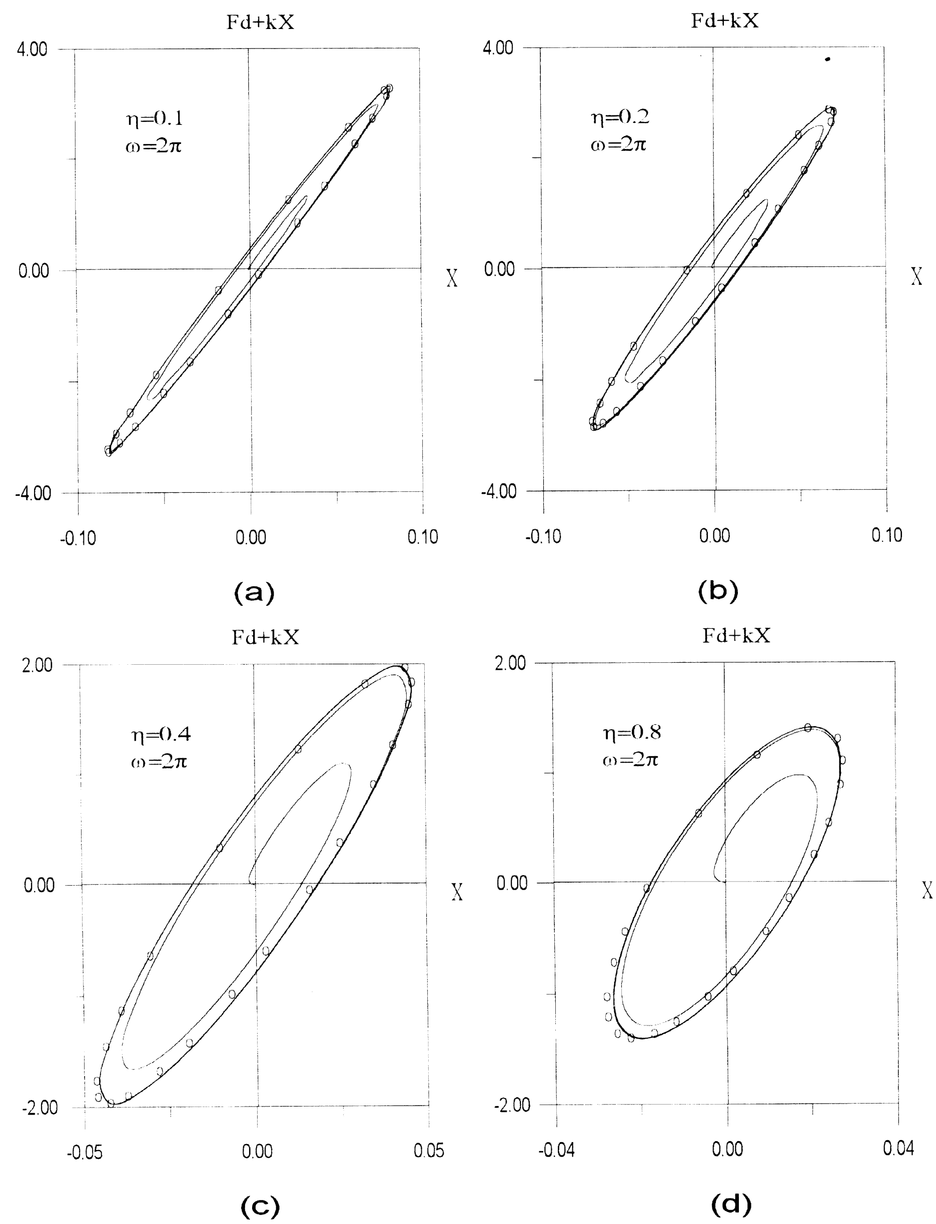

Fig. 4. Hysteresis loop obtained by using the time domain approach and the frequency domain approach. ———, time domain approach; ∘ ∘ ∘, frequency domain approach.

Table 1
Velocity discontinuity for the free vibration of hysteretic damping

	$\eta = 0$	$\eta = 0.1$	$\eta = 0.2$	$\eta = 0.4$	$\eta = 0.8$
$\dot{x}(0^+)$	1	0.974	0.957	0.924	0.871
$\dot{x}(0^-)$	0	-0.0083	-0.022	-0.049	-0.091
$x(0)$	0	-0.0039	-0.00773	-0.0145	-0.0232

The results obtained using both methods, real integral and FFT, are in good agreement. However, the real integral method is more time-consuming since the domain of integration is infinite. Also, the noncasual effect is found in Fig. 2 as expected.

4.2. Check for free vibration in the time domain

In order to check whether or not the governing equation of Eq. (6) in the time domain is equivalent to that of Eq. (3) in the frequency domain, the free vibration obtained by Eq. (3) in the frequency domain approach is substituted into Eq. (6) in the time-domain governing equation. Fig. 3 shows that the forces for the time-domain governing equation are in equilibrium for Eq. (6), where the sum of the inertia force, $m\ddot{x}(t)$, and spring force, $kx(t)$, is equal to the damping force,

$$\frac{k\eta}{\pi}\int_{-\infty}^{\infty}\frac{x(u)}{t-u}\,\mathrm{d}u$$

for η=0.1, 0.2, 0.4 and 0.8. Only a small difference appears near t=0 owing to the presence of the Dirac Delta function there since the velocity is discontinuous across t=0 as shown in Table 1.

4.3. Hysteresis loop by using the frequency domain approach and the time domain approach

The hysteresis loop is solved in the time domain by setting the forcing function, $p(t)$, in Eq. (9), to be harmonic loading. The convergent solution can be obtained by six iterations (N=6) for Eq. (9) when the artificial damping ratio is chosen as 0.8. The transient behavior in the development to steady-state response on the damping ellipse is shown in Fig. 4. The curve originates from the second quadrant since noncasual effect is also present not only in free vibration but also in forced vibration. It can be found that the ellipse for steady state response matches the result obtained by using the frequency domain approach very well.

5. Conclusions

The governing equation in the time domain for free vibration of hysteretic damping has been reviewed in this paper. The free vibration obtained by using the frequency domain approach has been proved to satisfy the governing equation in the time domain. Also, the successive iteration technique has been successfully applied to obtain the hysteresis loop. Obviously, the present formulation can be extended to the hysteretic damping model subject to arbitrary loading in the time domain.

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