Review of dual boundary element methods with emphasis on hypersingular integrals and divergent series

by

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Reprinted from *Applied Mechanics Reviews*
vol 52, no 1, January 1999
ASME Reprint No AMR265 $18

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This article provides a perspective on the current status of the formulations of dual boundary element methods with emphasis on the regularizations of hypersingular integrals and divergent series. A simple example is given to show the dual integral representation and the dual series representation for a discontinuous function and its derivative and thereby to illustrate the regularization problems encountered in dual boundary element methods. Hypersingularity and the theory of divergent series are put under the framework of the dual representations, their relation and regularization techniques being examined. Applications of the dual boundary element methods using hypersingularity and divergent series are explored. This review article contains 249 references.

1 INTRODUCTION

The boundary element method (BEM), or sometimes called boundary integral equation method (BIEM), has received much attention since Rizzo [188] proposed a numerical treatment of the boundary integral equation for elastostatics [130]. Even earlier, Kinoshita and Mura [130] derived the singular boundary integral equations for elasticity. Most of the efforts have been focused on singular boundary integral equations for primary fields (eg, potential u or displacement $\mathbf{u}$). For most problems, the formulation of a singular boundary integral equation for the primary field provides sufficient conditions to ensure a unique solution. In some cases, eg, those with Hermite polynomial element [220], degenerate boundaries [29, 109, 110], corners [44], the construction of a symmetric matrix [7, 8, 126, 194], the improvement of condition numbers [38], the construction of an image system [38], the tangent flux or hoop stress calculation on the boundary [48], an error indicator in an adaptive BEM [147], fictitious (irregular) eigenfrequencies in an exterior problem [33,116, 117, 134, 135], spurious roots in a multiple reciprocity method (MRM) [50, 51], and the Tikhonov formulation for inverse problems [237], it is found that the integral representation for a primary field cannot provide sufficient conditions, however. Watson [220, 221] presented the normal derivative of the displacement boundary integral equation for the development of a Hermite cubic element where the number of unknowns is larger than the number of equations. For the case of a degenerate boundary, the dual integral representation has been proposed for crack problems in elasticity by Hong and Chen [29, 35, 38, 109, 110], and boundary element researchers [60, 91, 94, 166, 180, 197, 231, 242] have increasingly paid attention to the second equation of the dual representation. The second equation, which is derived for the secondary field (eg, flux t or traction $\mathbf{t}$), is very popular now and is termed the hypersingular boundary integral equation. Hong and Chen presented the theoretical bases of the dual integral equations in a general formulation which incorporates the displacement and traction boundary integral equations. Huang and So [120] extended the concept of the Hadamard principal value in the dual integral equations [109] to determine the dynamic stress intensity factors of multiple cracks. Gray [91, 94] also found the dual integral representations for the Laplace equation and the Navier equation, although he did not coin the formulation *dual*. Martin, Rizzo, and Gonsalves [160] called the new kernel in the dual integral equations *hypersingular*, while Kaya [127] earlier called the kernel a *superstrong singularity*. Since the formulation was derived for the secondary field, by analogy with the term *natural boundary condition*, Feng and Yu [83, 244, 245, 246, 247] called the method *natural BEM* or *canonical integral equations*. Balas, Sladek, and Sladek in their book [9] proposed a unified theory for elasticity problems which contains the displacement boundary integral equation and another integro-differential equation for the traction field.

Based on the dual integral representation, Hong and Chen developed the dual BEM program for crack and potential flow problems. Also, Chen and his coworkers extended the dual BEM program for the Laplace equation to two pro-

Transmitted by Associate Editor Q Du

*Part of this manuscript was presented in a one-hour invited lecture at the Fifth International Colloquium on Numerical Analysis, Plovdiv, Bulgaria, 1996.

grams. One is for the Helmholtz equation by the dual MRM [225]. The other is for the Helmholtz equation by the complex-valued formulation [55]. A general purpose program, BEASY, was developed for crack problems by the WIT (Wessex Institute of Technology) group and termed the *dual boundary element method* (*DBEM*) approach [180]. This program has been extended to crack propagation more efficiently by using the benefit of the single-domain approach [174]. Mi and Aliabadi [166] extended two-dimensional cases to three-dimensional crack problems. A program implemented by Lutz *et al* [155] was also reported.

The dual representations have recently been extended to dynamic problems [45, 47, 248], random vibration [234], deconvolution in site response [59, 114], heat conduction [39], and active control [115] using the concept of modal analysis. In this way, dual integral representation is transformed into dual series representation. The extension of the hypersingular integral to elastodynamics was fully developed by means of direct transient dynamics [137]. According to the literature to date, the closed-form kernel functions related to the fundamental solution in dual integral equations have first to be replaced by Green's function and its derivatives and then expanded into degenerate series by means of spectral decomposition, resulting in a dual series representation. In so doing, hypersingular integrals are transformed into divergent series, while the integrability and the principal values for the hypersingular integrals are changed to the summability and the finite parts of the divergent series [31, 47, 49, 111].

Mathematically speaking, all these problems, either hypersingular integrals or divergent series, stem from taking derivatives of the double layer potentials. In fact, the original idea came from the applications of the continuous and discontinuous properties of the single and double layer potentials and their derivatives when the field point approaches to or passes through the boundary. These properties are classical results, and in the mathematical literature the relationships between the boundary integral operators of various layer potentials are obtainable through the so-called Calderon projector. Detailed discussions can be found in [169, 170, 171, 172]. These mathematical problems were first studied by Hadamard [100] and Mangler [156]. The hypersingular integral equation was treated by Hadamard in solving the cylindrical wave equation using a spherical means of descent. The improper integral was then defined by Tuck [214] as the *Hadamard principal value*. Almost at the same time as Hadamard's work, Mangler derived the same mathematical form in solving the thin airfoil problem. This is the reason why the improper integral of hypersingularity is also called the *Mangler principal value* in theoretical aerodynamics. This nonintegrable integral of hypersingularity [172] arises naturally in the dual boundary integral representations especially for problems with degenerate boundaries, eg, crack problems in elasticity [29, 35, 38, 109, 110], heat flow through a baffle [36], Darcy flow around a cutoff wall [29, 37, 40], a cracked bar under torsion [54], screen impinging in acoustics [51, 52, 55, 56, 152, 209], acoustic holography [112, 113], or aerodynamic problems of a thin airfoil [218]. A general application of the hypersingular integral equation in mechanics was discussed in Martin *et al* [160] and by Chen and Hong [38]. Combining the singular integral equation, eg, Green's identity or Somigliana's identity, with the hypersingular integral equation, we can construct the dual integral equations according to the continuous and discontinuous properties of the potential as the field point moves across the boundary [41]. From the above point of view, the definition of the *dual (boundary) integral equations* is quite different from the *dual integral equations* given by Sneddon and Lowangrub [201] and Buecker [21], which, indeed, come from the same equation but different collocation points in crack problems of elastodynamics. The solution for the conventional dual integral equations was first studied by Beltrami [66]. The dual boundary integral equations for the primary and secondary fields defined and coined by Hong and Chen are generally independent of each other, and only for very special cases are they dependent [31, 33].

Dual integral equations have four kernel functions in total, which make a unified theory encompassing different schemes possible with various derivations and interpretations. The equivalence between the direct and indirect methods has been discussed using four lemmas for the kernels in dual integral equations [109]. The constraint relationships derived from dual integral equations are dependent for boundary unknowns on a nondegenerate boundary and are independent for boundary unknowns on a degenerate boundary. The assertion can be proved by using degenerate kernels [31, 33]. The singularity order of the hypersingular kernel in the dual integral equations is stronger than the Cauchy kernel by one order. The paradox of the nonintegrable hypersingular kernel is introduced due to the wrong change of the integral and trace operators from the viewpoint of the dual integral representation. L'Hospital's rule should be considered for the trace process in the limiting sense. From another point of view in the commutativity diagram as shown in Fig 1a for the dual integral representation, the nonintegrable kernel results from wrongly putting the differentiation operator directly into the integral of the singular Cauchy kernel. In order to ensure a finite value, Leibnitz' rule should be considered when the derivative of the Cauchy principal value (CPV) is taken since the integration boundary is dependent on the differential variable, so that the boundary term cannot be neglected. In

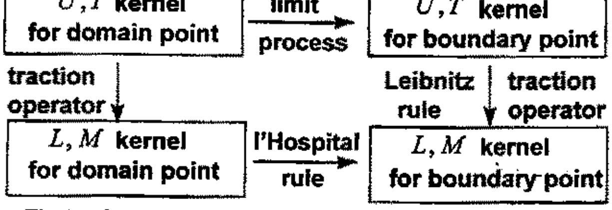

Fig 1a. Commutativity diagram in dual integral equations

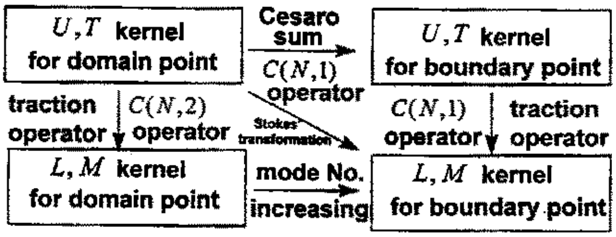

Fig 1b. Commutativity diagram in dual series representation

the dual series representation as shown in Fig 1b, the divergent series results from the illegal operation of termwise differentiation in a similar sense. In other words, the differentiation operator is wrongly put inside the summation operator. By using the legal method of series differentiation, Stokes' transformation can extract the finite part in a similar way as Stokes' theorem due to integration by parts. Stokes' transformation will introduce a boundary term which is similar to the boundary term derived by Leibnitz' rule since the integration boundary for the Cauchy principal value depends on the differential variable. This may be the reason why Stokes' transformation has been called summation by parts [129]. It is interesting to find that Stokes' transformation is very similar to the alternative series in the textbook of engineering mathematics [230]. Another regularization technique for a divergent series using the Cesàro sum with an increasing number of modes is similar to l'Hospital's rule in the limiting process by moving the field point to the boundary point. The reconstructing function can be proved to converge to the original function by using the reproducing kernel. Therefore, the finite value can be extracted in different ways as shown in Fig 1a for the dual integral equations and in Fig 1b for the dual series representation.

In this article, the dual representations including dual integral representations and dual series representations will be reviewed first. A simple example to represent a function with discontinuity by three different methods is given in Section 2. The hypersingularity and divergent series are linked from the viewpoint of dual representations in Section 3. Sections 4 and 5 summarize the regularization techniques for hypersin-

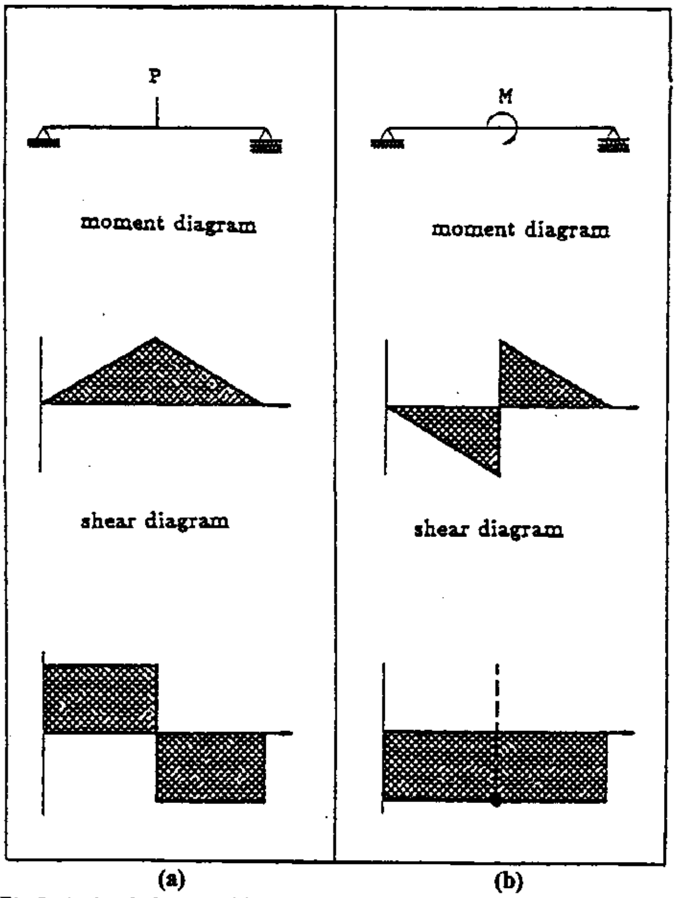

Fig 2. A simple beam subjected to: a) a concentrated force b) a concentrated moment and the corresponding moment and shear diagrams in closed-form solutions

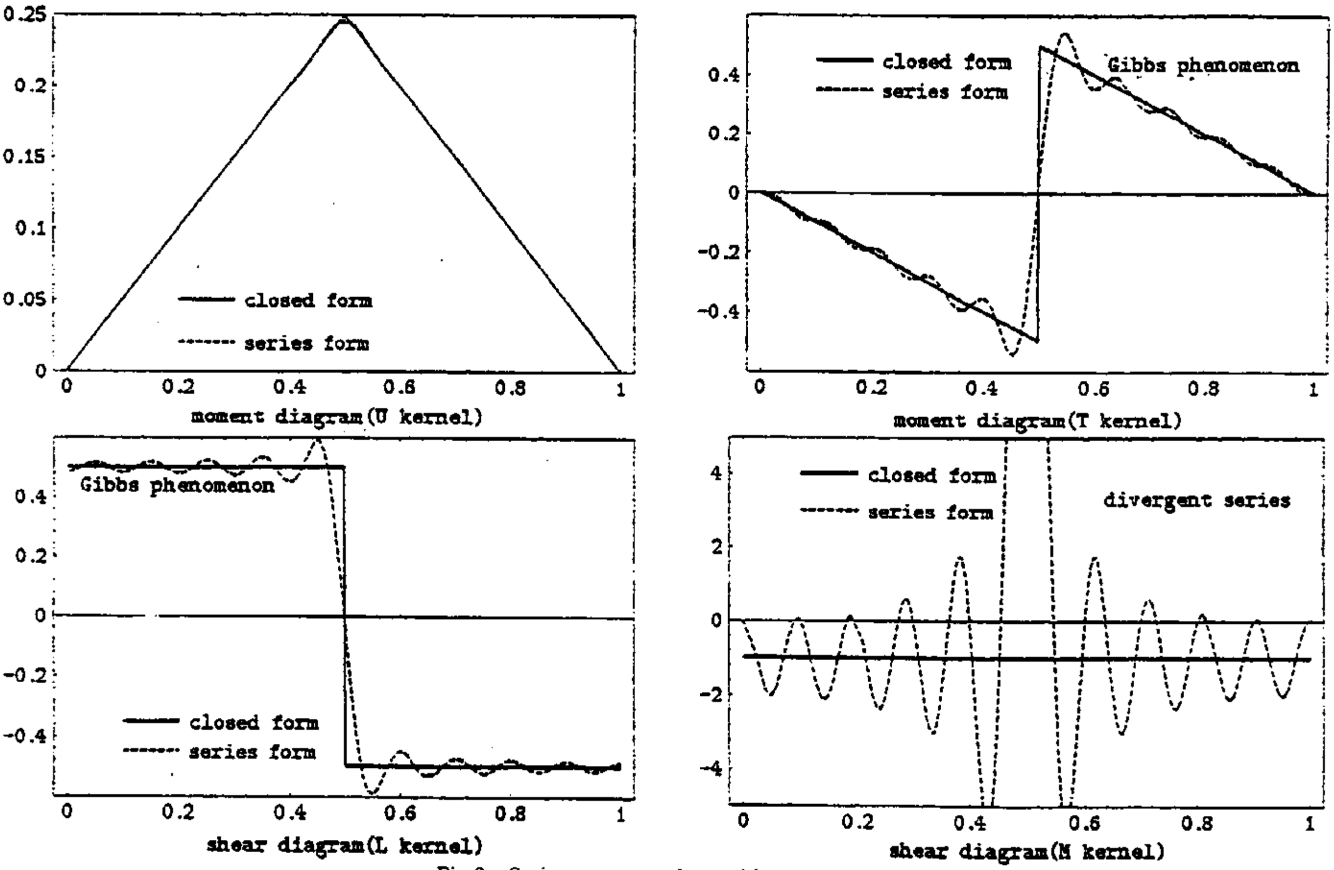

Fig 3a. Series representations without treatment

gularity and divergent series, respectively. Sections 6 and 7 discuss the applications of hypersingularity and divergent series in the boundary element method, respectively.

2 REPRESENTATIONS AND REGULARIZATIONS FOR DISCONTINUOUS FUNCTIONS

Discontinuous functions are often encountered in mathematical formulations of continuous physical fields. For illustration purposes, let us consider a simple beam subjected to a concentrated force or a concentrated couple (moment) at s = 0.5 as shown in Fig 2. The closed-form solutions are easily derived as follows (see also Fig 2):

$$U(s,x)=\begin{cases}0.5x, & 0.0\le x\le 0.5,\\ 0.5(1-x) & 0.5\le x\le 1.0,\end{cases} \tag{1}$$

$$T(s,x)=\begin{cases}-x, & 0.0\le x<0.5,\\ -x+1, & 0.5<x\le 1.0,\end{cases} \tag{2}$$

$$L(s,x)=\begin{cases}0.5, & 0.0\le x<0.5,\\ -0.5, & 0.5<x\le 1.0,\end{cases} \tag{3}$$

$$M(s,x)=\begin{cases}-1, & 0.0\le x<0.5,\\ -1, & 0.5<x\le 1.0,\end{cases} \tag{4}$$

where $U(s, x)$ and $T(s, x)$ denote the moments at x of the beam subjected to a concentrated force and a concentrated moment at s = 0.5, respectively, while $L(s, x)$ and $M(s, x)$ are the corresponding shear forces at x. However, we may express the moment due to a concentrated force or moment at s = 0.5, in the form of series representations given as follows:

$$U(s,x)=\sum_{n=1}^{N}-2(-1)^n\frac{\sin((2n-1)\pi x/l)}{(2n-1)^2\pi^2}, \tag{5}$$

$$T(s,x)=\sum_{n=1}^{N}(-1)^n\frac{\sin(2n\pi x/l)}{n\pi}. \tag{6}$$

Without rigorous consideration, term by term differentiation of Eqs (5) and (6) yields the following series representations for shear forces:

$$L(s,x)=\sum_{n=1}^{N}-2(-1)^n\frac{\cos((2n-1)\pi x/l)}{(2n-1)\pi}, \tag{7}$$

$$M(s,x)=\sum_{n=1}^{N}2(-1)^n\cos(2n\pi x/l). \tag{8}$$

The series representations of Eqs (5), (6), (7), and (8) are shown in Fig 3a. Their corresponding Cesàro sum (or the Fejèkernel) and Stokes' transformation (or the so-called alternative series) treatments are shown in Figs 3b and 3c, respectively, for comparison with the closed-form solution shown in Fig 2. The Gibbs phenomena of Eqs (6) and (7) in Fig 3a are suppressed by use of the Cesàro sum as shown in Fig 3b while the convergent value is recovered from the divergent (oscillating) result of Eq (8) in Fig 3a by using the Cesàro sum in Fig 3b or by using the Stokes transformation in Fig 3c.

These figures reveal that Stokes' transformation is a better way to approximate the closed-form solution compared with the Cesàro sum treatment. This example vividly warns us that

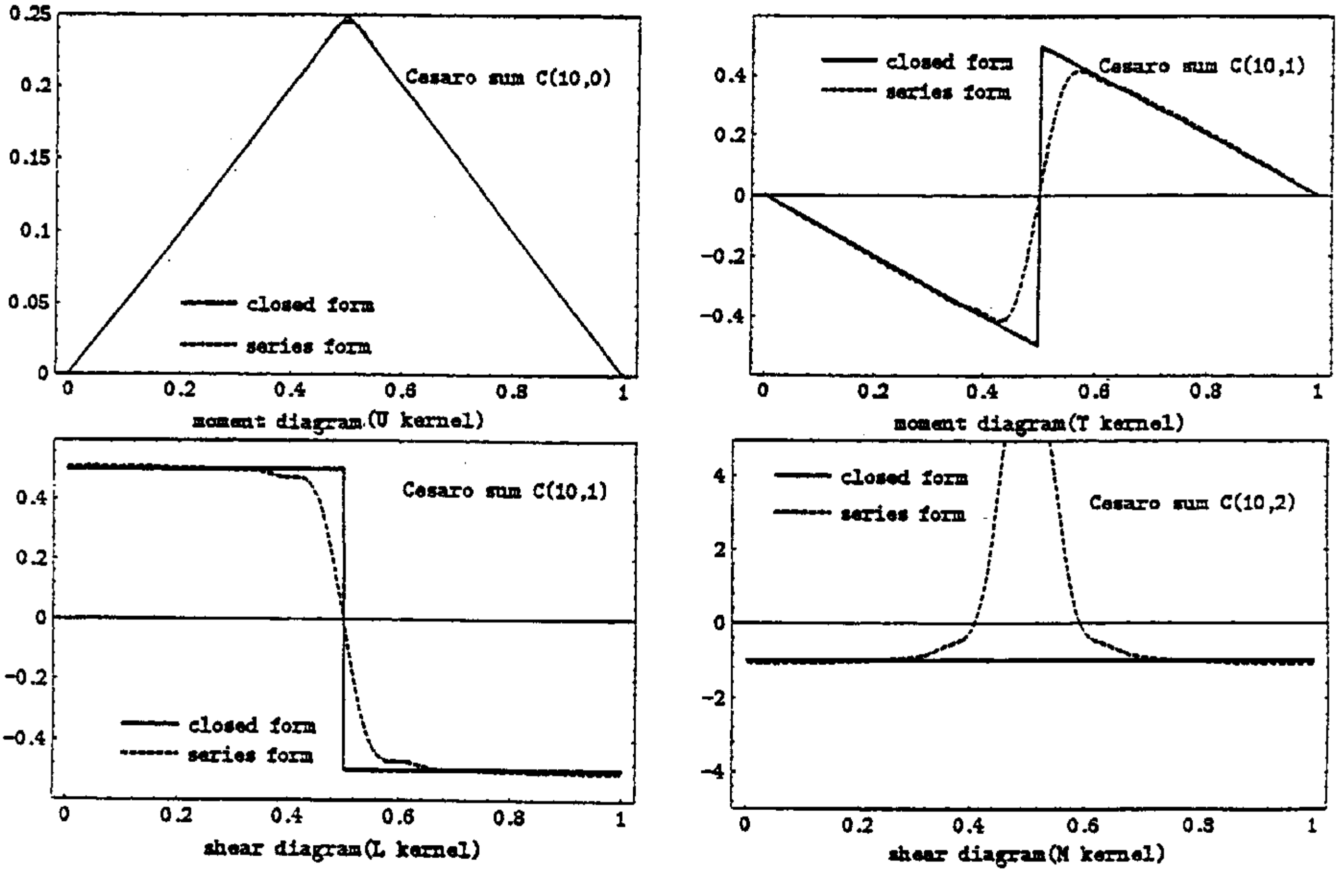

Fig 3b. Series representations by the Cesàro (Fejèr) sum treatment

term by term differentiation of series may not be permissible. If term by term differentiation has already been done, the posterior Cesàro sum treatment can be employed to recover the convergent value as the finite part or to depress the oscillation by using the concept of reproducing kernel. This is the scenario how the regularization method is used to restore correct results even though term by term differentiation may be illegal. Tahan *et al* [206] applied the Cesàro sum technique to recover the correct value of stress in a rectangular plate subjected to colinear compression. It must be noted that the point near the singular loading requires much more terms to ensure convergence as shown in Fig 3b at $x = 0.5$. To obtain a smoother result, the Cesàro (Fejèr) sum of a higher order must be considered. As shown in Fig 3b, the Cesàro operators of orders $C(10, 2)$ and $C(10, 1)$ are applied to Eqs (7) and (8), respectively. It appears that the order of the Cesàro (Fejèr) sum was not discussed in Tahan's paper. From the viewpoint of Stokes' transformation, the series differentiation contains two parts: the term from the termwise differentiation and the boundary term. If the boundary term is lost, the Gibbs phenomenon and a divergent (oscillating) series are present. It is the boundary term which can accelerate convergence against the Gibbs phenomenon in the primary field and can extract the finite part of the divergent (oscillating) series for the secondary field. Various regularization techniques will be discussed in detail in this review article.

3 RELATIONS OF DUAL INTEGRAL EQUATIONS AND DUAL SERIES REPRESENTATIONS

Dual boundary integral equations were developed in 1985-1986 [29] and published in 1988 [109, 110] for crack problems by Hong and Chen. Based on the concept of modal dynamics (eigenfunction expansion), the idea of dual integral equations were extended to dual series representations for structural dynamics [111], random vibration [49], and heat conduction [39]. For simplicity, we will consider a dynamic problem as an illustrative example to describe the relationships between dual integral equations and dual series representations.

Consider a homogeneous, isotropic, linear, elastic body with finite domain D bounded by boundary $B = B_t + B_u$. The governing equation for the displacement $\mathbf{u}(\mathbf{x}, t)$ at a domain point $\mathbf{x}$ at time t can be written as

$$\rho\ddot{\mathbf{u}} + (2\alpha\rho + \beta\mathcal{L})\{\dot{\mathbf{u}}\} + \mathcal{L}\{\mathbf{u}\} + \mathbf{f}(\mathbf{x}, t) = 0, \mathbf{x} \in D,\ t \in (0, \infty), \quad (9)$$

where α, β are the damping coefficients, ρ is the mass density, $\mathbf{f}(\mathbf{x}, t)$ is the body force excitation, and the operator $\mathcal{L}$ means

$$\mathcal{L}\{\mathbf{u}\} = \begin{cases} -(\lambda + G)\nabla\nabla\cdot\mathbf{u} - G\nabla^2\mathbf{u}, & \textit{elastic body}, \\ -GA\dfrac{\partial^2 u}{\partial x^2}, & \textit{elastic shear beam} \\ EI\dfrac{\partial^4 u}{\partial x^4}, & \textit{elastic flexural beam}, \end{cases} \quad ,(10)$$

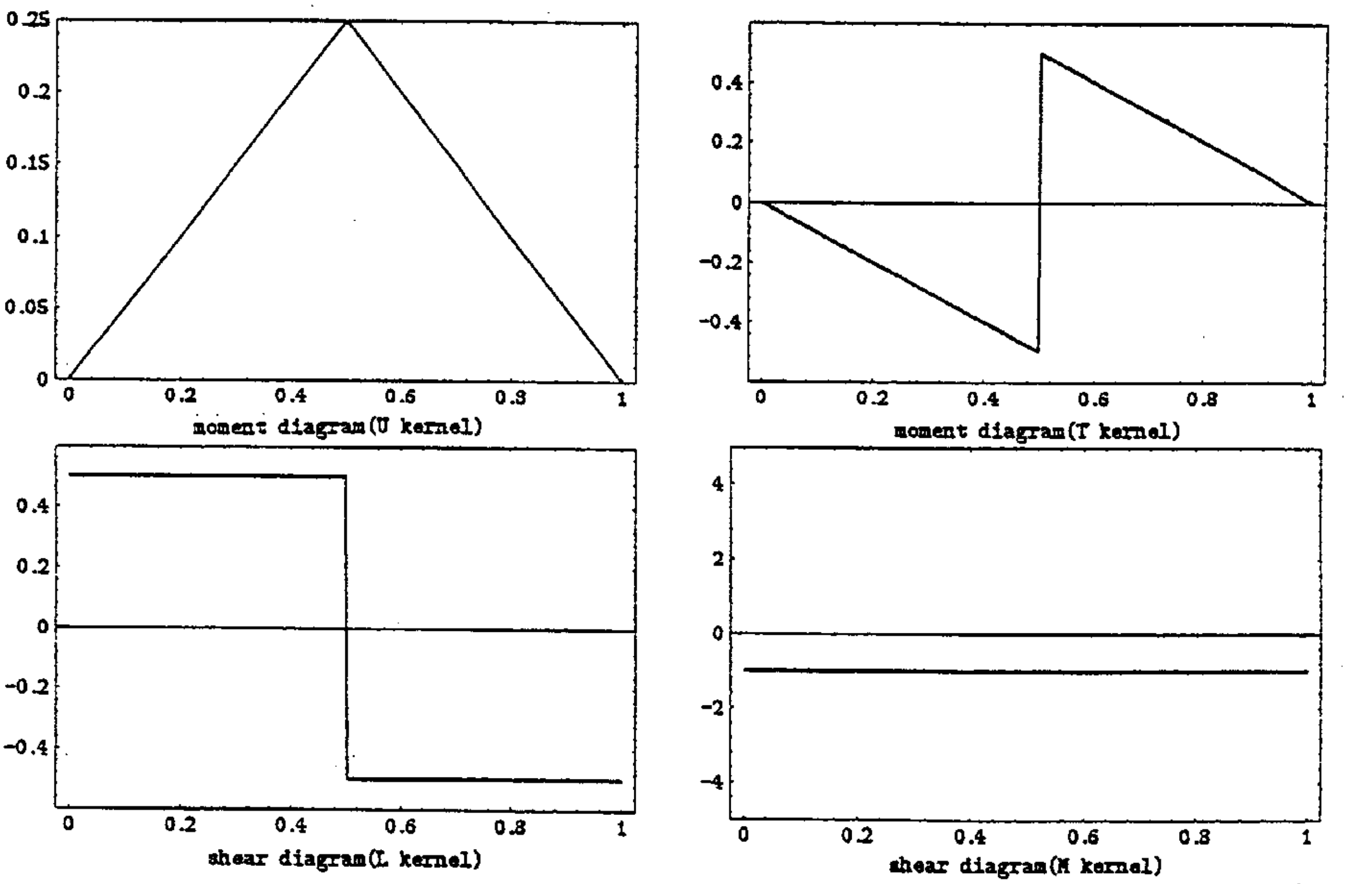

Fig 3c. Series representations by Stokes' transformation treatment

where λ and G are Lamè constants, E is Young's modulus, A is the area of the cross section of the shear beam, and I is the moment of inertia of the cross section of the flexural beam. The time-dependent boundary conditions are

$$\mathcal{T}\{\mathbf{u}(\mathbf{x},t)\} \equiv \mathbf{t}(\mathbf{x},t) = \mathbf{t}_b(\mathbf{x},t),\ \mathbf{x} \in B_t, \tag{11}$$

$$\mathbf{u}(\mathbf{x},t) = \mathbf{u}_b(\mathbf{x},t),\ \mathbf{x} \in B_u, \tag{12}$$

where $\mathbf{u}_b$ is the prescribed displacement on B_u, t is the traction on B, $\mathbf{t}_b$ is the prescribed traction on B_t, and $\mathcal{T}$ is the traction operator defined as

$$\mathcal{T}\{\mathbf{u}\} = \begin{cases} \left[\lambda \mathbf{I}(\nabla \cdot u) + G\nabla\mathbf{u} + G\mathbf{I}\cdot(\mathbf{u}\nabla)\right]\cdot \mathbf{n}, & \textit{elastic body}, \\ -GA\dfrac{\partial u}{\partial x}, & \textit{elastic shear beam}, \\ -EI\dfrac{\partial^3 u}{\partial x^3}, & \textit{elastic flexural beam}. \end{cases} \tag{13}$$

The initial conditions are

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), \tag{14}$$

$$\dot{\mathbf{u}}(\mathbf{x},0) = \mathbf{v}_0(\mathbf{x}). \tag{15}$$

For comparison purposes, both dual integral equations for direct and modal elastodynamics are formulated as follows:

On extending the dual integral representation to transient elastodynamics, the displacement $\mathbf{u}(\mathrm{x}, t)$ and traction $\mathbf{t}(\mathrm{x}, t)$ for a domain point x at time t can be written as

$$\begin{aligned} \mathbf{u}(\mathbf{x},t) = &\int_0^t \int_B \mathbf{U}(\mathbf{s},\mathbf{x};\tau,t)\cdot \mathbf{t}(\mathbf{s},\tau)dB(\mathbf{s})d\tau \\ &- \int_0^t \int_B \mathbf{T}(\mathbf{s},\mathbf{x};\tau,t)\cdot \mathbf{u}(\mathbf{s},\tau)dB(\mathbf{s})d\tau \\ &+ \int_0^t \int_D \mathbf{U}(\mathbf{s},\mathbf{x};\tau,t)\cdot \mathbf{f}(\mathbf{s},\tau)dD(\mathbf{s})d\tau \\ &+ \int_D \mathbf{U}(\mathbf{s},\mathbf{x};0,t)\cdot \rho\mathbf{v}_0(\mathbf{s})dD(\mathbf{s}) + \int_D \dot{\mathbf{U}}(\mathbf{s},\mathbf{x};0,t)\cdot \rho\mathbf{u}_0(\mathbf{s})dD(\mathbf{s}), \end{aligned} \tag{16}$$

$$\begin{aligned} \mathbf{t}(\mathbf{x},t) = &\int_0^t \int_B \mathbf{L}(\mathbf{s},\mathbf{x};\tau,t)\cdot \mathbf{t}(\mathbf{s},\tau)dB(\mathbf{s})d\tau \\ &- \int_0^t \int_B \mathbf{M}(\mathbf{s},\mathbf{x};\tau,t)\cdot \mathbf{u}(\mathbf{s},\tau)dB(\mathbf{s})d\tau \\ &+ \int_0^t \int_D \mathbf{L}(\mathbf{s},\mathbf{x};\tau,t)\cdot \mathbf{f}(\mathbf{s},\tau)dD(\mathbf{s})d\tau \\ &+ \int_D \mathbf{L}(\mathbf{s},\mathbf{x};0,t)\cdot \rho\mathbf{v}_0(\mathbf{s})dD(\mathbf{s}) + \int_D \dot{\mathbf{L}}(\mathbf{s},\mathbf{x};0,t)\cdot \rho\mathbf{u}_0(\mathbf{s})dD(\mathbf{s}), \end{aligned} \tag{17}$$

where $\mathbf{U}(\mathbf{s}, \mathbf{x}; \tau, t)$, $\mathbf{T}(\mathbf{s}, \mathbf{x}; \tau, t)$, $\mathbf{L}(\mathbf{s}, \mathbf{x}; \tau, t)$, and $\mathbf{M}(\mathbf{s}, \mathbf{x}; \tau, t)$ are four closed-form kernel functions.

If the closed-form kernel functions in the dual integral equations are changed to the degenerate series forms as

$$\mathbf{U}(\mathbf{s},\mathbf{x};\tau,t) \rightarrow \text{closed-form Green's function} \rightarrow \sum_{m=1}^{\infty} \frac{1}{N_m \omega_m^d} e^{-\xi_m \omega_m (t-\tau)} \sin\left(\omega_m^d (t-\tau)\right) \mathbf{u}_m(\mathbf{x}) \otimes \mathbf{u}_m(\mathbf{s}),$$

$$\mathbf{T}(\mathbf{s},\mathbf{x};\tau,t) \rightarrow \text{closed-form Green's function} \rightarrow \sum_{m=1}^{\infty} \frac{1}{N_m \omega_m^d} e^{-\xi_m \omega_m (t-\tau)} \sin\left(\omega_m^d (t-\tau)\right) \mathbf{u}_m(\mathbf{x}) \otimes \mathbf{t}_m(\mathbf{s}),$$

$$\mathbf{L}(\mathbf{s},\mathbf{x};\tau,t) \rightarrow \text{closed-form Green's function} \rightarrow \sum_{m=1}^{\infty} \frac{1}{N_m \omega_m^d} e^{-\xi_m \omega_m (t-\tau)} \sin\left(\omega_m^d (t-\tau)\right) \mathbf{t}_m(\mathbf{x}) \otimes \mathbf{u}_m(\mathbf{s}),$$

$$\mathbf{M}(\mathbf{s},\mathbf{x};\tau,t) \rightarrow \text{closed-form Green's function} \rightarrow \sum_{m=1}^{\infty} \frac{1}{N_m \omega_m^d} e^{-\xi_m \omega_m (t-\tau)} \sin\left(\omega_m^d (t-\tau)\right) \mathbf{t}_m(\mathbf{x}) \otimes \mathbf{t}_m(\mathbf{s}),$$

where N_m denotes the m-th normalized mass, ω_m^d denotes the damped natural frequency, $\mathbf{u}_m(\mathbf{x})$ and $\mathbf{t}_m(\mathbf{x})$ are the m-th modal displacements and modal tractions and $\otimes$ indicates the dyadic operation, then the dual integral equations are transformed into dual series representations, which, indeed, has the usual meaning of modal elastodynamics.

Comparison of the above two formulations shows that the series-type kernels in modal elastodynamics come from spectral decomposition of the closed-form kernel functions in direct elastodynamics. Based on eigenfunction expansions, the series-type kernel functions are series expansions of Green's functions instead of the **U**, **T**, **L**, and **M**. The fundamental solution in the dual integral equations should be first changed to Green's function for an appropriate, finite region containing D. Therefore, direct integration by time marching is feasible, and matrix inversion can be omitted since the series-type kernel functions satisfy homogeneous boundary conditions. The relations between the dual integral equations and the dual series representation are summarized in Fig 1.

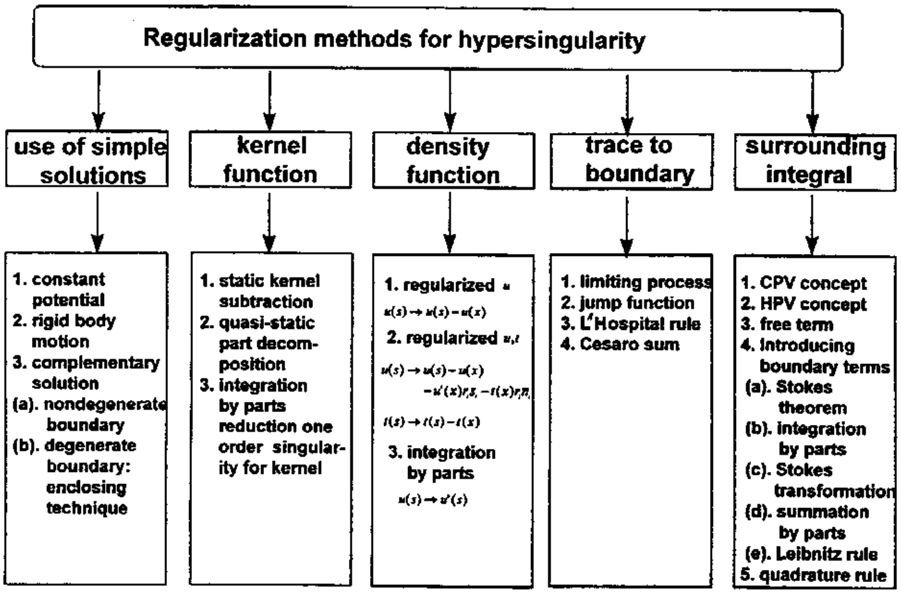

Fig 4. Regularization methods for hypersingular integrals

4 REGULARIZATIONS FOR HYPERSINGULARITY

An increasing number of researchers have focused on hypersingularity, thus proposing quite a few analytical and numerical techniques to handle it. These are summarized in Fig 4 and are discussed below.

4.1 Trace to boundary

Analytical integrations of singular and hypersingular kernels for constant elements have been developed by Hong and Chen [38], Cruse [74], and Gray *et al* [94] in crack problems by bringing the collocated point into a boundary point. The analytical formula reveals the jump behavior of the double layer potential by a jump function, for example, the arctangent function. The normal derivative of the double layer potential for hypersingularity can be analytically derived using l'Hospital's rule and the relations of the inverse trigonometric function. An example has been given in [31, 38] for the Laplace equation. Gray [94] used an analytical method to integrate the hypersingular kernel by means of the polar coordinate transformation, and all the components in the M kernel for a crack problem were derived. For the cases of general interpolation functions or curved boundary elements, analytical formulae do not exist and other methods are needed.

4.2 Use of simple solutions

The simple solution is defined as a complementary solution for a homogeneous partial differential equation. For example, a rigid body motion in elasticity or a constant potential in the Laplace equation is the simplest. This method can be used to check the correctness of the $[T]$ matrix and has been implemented in the BEM program as a check procedure when alternative methods are employed to calculate the diagonal coefficients. For a nondegenerate boundary, the $[M]$ matrix can also be tested by using a rigid body motion. However, the rigid body motion test has its limitations when it is applied to determine the diagonal coefficients of the $[T]$ and $[M]$ matrices of a degenerate boundary since the sum of two hypersingular integrals in the same row is automatically zero. Trivial information for the diagonal elements of the $[T]$ and $[M]$ matrices will be obtained. In order to check the diagonal terms of the $[T]$ and $[M]$ matrices, an artificial boundary enclosing the degenerate boundary has been proposed by Chen and Hong [38], Lutz *et al* [155], Chen and Chen [60], and Chen [31]. Another simple solution for a degenerate boundary, which is the complementary solution, has been applied to test the $[M]$ matrix by Chen and Hong [38] free from introducing the enclosing boundaries.

4.3 Surrounding technique

Since singularity is present on the boundary, a contour integration around the singularity is considered to obtain the free terms and the principal values, eg, the Cauchy principal value and Hadamard principal value. The finite part can be extracted by introducing the boundary term using Stokes' theorem (integration by parts) in dual integral equations, which is similar to Stokes' transformation (summation by parts) in a dual series representation as will be elaborated on in the next section. The final free terms will be the same as those in the limiting process although the intermediate results from the L and M kernels are different. The derivations can be found in [31, 38, 44]. As another point of view, the commutativity diagram in Fig 1 shows the boundary term comes from the Leibnitz rule since differentiation with respect to the Cauchy principal value is used. Singular integrations for the principal values of the Riemann (logarithm) type, the Cauchy type and the Hadamard type have also been developed by Pina and Fernandes [179] and Kutt [141] using the quadrature rule.

4.4 Partial integration

In order to reduce the order of singularity, Cruse [73], Bui [22], Weaver [222], and Sladek and Sladek [198] integrated the kernels by parts. The key step shifts one order of singularity from the kernel to the density function. Therefore, the hypersingular kernel is reduced to the Cauchy type.

4.5 Adding and subtracting technique

For time-dependent boundary-value problems, Mindlin and Goodman [168] proposed a quasi-static decomposition method to calculate the dynamic response of a structure and to make finite the traction response in the series solution. Using a similar concept, the subtraction and addition technique has been applied to calculate the finite part of hypersingularity. There are two ways to apply this technique; one uses the density function, and the other uses the kernel function. Rizzo *et al* [187] subtracted the static kernel from the T kernel and calculated the remaining integral using the Gaussian quadrature while the additional static kernel can be integrated analytically. Recently, the subtraction technique has been applied to the density function by means of Taylor's expansion. For example, Shiau [195], Cruse *et al* [75], Matsumoto and Tanaka [164] and Sladek *et al* [199] have applied this technique to regularize hypersingularity into a nonsingular integration. Another advantage of the technique is that the boundary effect can be avoided since the jump value across the boundary vanishes after regularization. Three regularized versions of dual integral equations have been derived, and an example has been given to illustrate the suppression of the boundary effect in [31, 44].

5 REGULARIZATIONS FOR OSCILLATING AND DIVERGENT SERIES

There are various methods available for regularizing divergent series, for example, the Shanks transformation, Cesàro sum (Fejér sum), Holder sum, Abel sum, Euler sum, Borel sum, and Stokes' transformation. Here, we shall focus on the Cesàro sum and Stokes' transformation since they have similar behavior in comparison with the regularization techniques for hypersingularity of dual integral equations, as shown in Fig 1b. Regularization methods for oscillating and divergent series are shown in Fig 5.

5.1 Quasi-static decomposition method

The quasi-static decomposition method for continuous systems was first presented by Mindlin and Goodman [168]. Clough and Penzien [65] extended it to discrete systems.

Based on the dual representations, it can be deemed as one of the regularization methods for divergent series. In [31, 111], a detailed discussion was given.

5.2 Cesàro mean

The general Cesàro mean is defined as [86, 105]

$$S_k = C(k,r)\left\{\sum_{n=0}^{k} a_n\right\} \equiv \frac{C_{r-1}^{k+r-1}s_0 + C_{r-1}^{k+r-2}s_1+\cdots+C_{r-1}^{r}s_{k-1}+C_{r-1}^{r-1}s_k}{C_r^{k+r}}, \tag{18}$$

where $C(k, r)$ is the operator of the Cesàro mean of the r-th order, r is an integer, $C_r^k = k!/(r!(k-r)!)$ and the partial sum is

$$s_k = \sum_{n=0}^{k} a_n(x,t). \tag{19}$$

The $C(k, 1)$ mean reduces to the conventional Cesàro mean:

$$S_k = C(k,1)\left\{\sum_{n=0}^{k} a_n\right\} \equiv \frac{s_0 + s_1+\cdots+s_{k-1}+s_k}{k+1}. \tag{20}$$

For efficiency of computation, the s_i terms may be changed to the a_i terms; thus, Eq (18) is expressed as

$$S_k = C(k,1)\left\{\sum_{n=0}^{k} a_n\right\} \equiv \frac{1}{k+1}\sum_{n=0}^{k}(k-n+1)a_n. \tag{21}$$

Similarly, the $C(k, 2)$ mean is

$$\begin{aligned} S_k &= C(k,2)\left\{\sum_{n=0}^{k} a_n\right\} \\ &\equiv \frac{C_1^{k+1}s_0 + C_1^{k}s_1 + C_1^{k-1}s_2+\cdots+C_1^{2}s_{k-1}+C_1^{1}s_k}{C_2^{k+2}} \\ &= \frac{(k+1)s_0 + ks_1+\cdots+2s_{k-1}+s_k}{0.5(k+1)(k+2)}, \end{aligned} \tag{22}$$

or, in terms of a_i,

$$\begin{aligned} S_k &= C(k,2)\left\{\sum_{n=0}^{k} a_n\right\} \\ &\equiv \frac{1}{(k+1)(k+2)}\sum_{n=0}^{k}(k-n+1)(k-n+2)a_n. \end{aligned} \tag{23}$$

For a general integer order r, we have

$$\begin{aligned} S_k &= C(k,r)\left\{\sum_{n=0}^{k} a_n\right\} \\ &= \sum_{n=0}^{k}\frac{(k)!(k+r-n)!}{(k-n-1)!(k+r)!}a_n. \end{aligned} \tag{24}$$

For the case of noninteger order, the Cesàro mean is defined as

$$S_k \equiv C(k,r)\left\{\sum_{n=1}^{k} a_n\right\} = \sum_{n=1}^{k} w_n^r a_n, \tag{25}$$

in which the weight is represented by

$$w_n^r = \frac{\Gamma(k+1)\Gamma(k+r-n+1)}{\Gamma(k-n)\Gamma(k+r+1)}. \tag{26}$$

The regularization operator was applied to extract the finite part of a series representation in [31, 59].

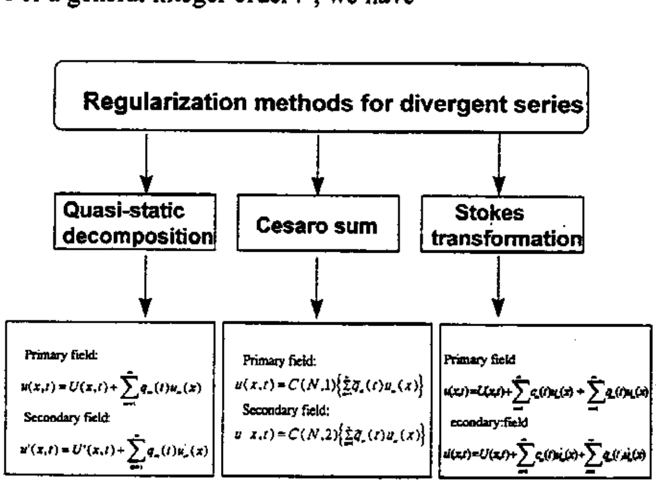

Fig 5. Regularization methods for divergent series

5.3 Stokes' transformation (alternative series)

Stokes' transformation has been utilized as a tool for deriving an analytical solution in terms of series solutions [19, 20, 64, 89, 99, 204, 211]. Recently, the double Fourier series using Stokes' transformation has been employed to solve double-curved panel problems [26]. If the analytical solution for the primary field can be expressed in terms of series representation as

$$f(x,t) = \sum_{k=1}^{\infty} c_k(t)u_k(x), \tag{27}$$

then the differentiation of f with respect to x yields the secondary field

$$f'(x,t) = \sum_{k=1}^{\infty} b_k(t)u_k'(x) + \sum_{k=1}^{\infty} c_k(t)u_k'(x), \tag{28}$$

where the first term on the right-hand side of the equal sign is the boundary term, and the second term results from termwise differentiation. In the initial-boundary-value problem with time-dependent essential boundary conditions, the boundary term is always present and can not be neglected. Instead of using Stokes' transformation to recover the boundary terms, a method of alternative series was employed in [230]. By adding the boundary term, the infinite value can be cancelled out, and the finite part can be extracted. Also, this technique has been applied in [31, 49].

6 APPLICATIONS OF HYPERSINGULARITY IN DUAL BOUNDARY INTEGRAL EQUATIONS

In dual integral equations, hypersingularity is present in the integral with the M kernel. In this section, we discuss why hypersingularity is important in treating certain problems and summarize in Fig 6 the roles it plays in the boundary element methods. One important role of the hypersingular equation is that it can provide additional constraints to ensure a unique solution as shown in the first five items in Fig 6. The last seven items in Fig 6 show other different roles they play in computational mechanics.

6.1 Higher order element

In order to improve accuracy with fewer elements, Watson [220, 221] chose the Hermite cubic element. An apparent gain is the interelement continuity of the first derivative of the primary variable. Another gain is that the consistency of the density function for the primary variable will improve the condition of existence for the M integral. However, since the number of unknown data doubles at the same time, taking the gradient of the displacement integral equation, which introduces hypersingularity, is necessary.

6.2 Degenerate boundary

In a degenerate boundary problem, the spatial coincidence of the two sides of the degenerate boundary leads to the result that the singular integral equation on one side is indistinguishable from that on the other side even though the displacements on the two sides are different. Although the hypersingular integral equations are different between the two collocation points on the two sides by their corresponding normal vectors, they are dependent since the two normal vectors differ only by a minus sign. (Nevertheless, the normal vectors before and after the corner are independent; therefore, the hypersingular equations can establish effective constraints, as shown in Subsection 6.3.)

To obtain enough independent equations, both singular and hypersingular equations, collocated on the degenerate boundary, are necessary [29, 35, 36, 37, 109, 110]. They were first given the name of the dual (boundary) integral equations for elasticity [29, 108, 109, 110] and were later implemented into the BEPO2D program for potential flow [37]. Lutz *et al* also implemented the concept in their program [155]. Cruse [72] formulated this degenerate boundary problem in terms of the density functions of the displacement difference and traction summation on the two sides of the degenerate boundary. Cruse noticed that this formulation introduced double unknowns, and additional equations were required, he did not go on further to survey the required equation. Although Watson [220, 221] proposed another type of additional equation, the kernels he derived were different from the kernels in the dual boundary integral equations [109], and the properties of his kernels had not been investigated thoroughly. Chen [29] as well as Hong in 1986 established the unified dual formulation, which incorporates the displacement and traction boundary integral equations. Gray [91, 94] independently found the formulation of dual integral equations. A review paper in fracture mechanics using dual BEMs can be found in [5]. In mathematical physics, a degenerate boundary is often present in, eg, a cutoff wall in potential flow, a crack problem in elasticity, a thin airfoil in aerodynamics, a baffle in heat conduction, a screen barrier in acoustics, and a magnetic wave across an antenna. All these problems can be successfully solved by using dual integral equations; eg, a seepage flow with sheet piles was considered in [31, 37, 41]. Also, the screen impinging in acoustics can be solved by the dual representations [51, 52, 55, 56, 152, 209].

6.3 Corner problem

The corner problem with the Dirichlet boundary condition is another problem in which the number of equations is not sufficient for the conventional BEM. The double node technique was utilized to tackle this problem [12]. Scholars have tried to find better, additional constraints. Again, the hypersingular integral formulation plays a role in providing independent constraints for the boundary unknowns. For the case that the displacement (or potential) is specified at the corner, the traction (or potential flux) unknowns are doubled due to the different normal vectors. Unfortunately, the singular equations alone can not distinguish the normal vectors of the collocation points at the corner. The second equation of the dual integral representation can be collocated to the points before the corner and after the corner with two different independent normal vectors, causing the equations to be independent, as shown below for the two-dimensional Laplace equation [44]:

$$\alpha t^-(x)+\sin(\alpha)t^+(x)=\mathrm{HPV}\int_B M^-(s,x)u(s)dB(s) - \mathrm{CPV}\int_B L^-(s,x)t(s)dB(s), \tag{29}$$

$$\alpha t^+(x)+\sin(\alpha)t^-(x)=\mathrm{HPV}\int_B M^+(s,x)u(s)dB(s) - \mathrm{CPV}\int_B L^+(s,x)t(s)dB(s), \tag{30}$$

where HPV denotes the Hadamard principal value, t^- and t^+ are the normal fluxes before and after the corner, α is the interior angle, and M^- and M^+ denote the kernels with the different normal vectors collocated before and after the corner. The detailed derivations can be found in [31, 44]. Therefore, a unique solution can be achieved by balancing the number of equations and unknowns after choosing any two of the three independent equations (the singular equation and Eqs (18) and (19)). The three methods can all match the exact

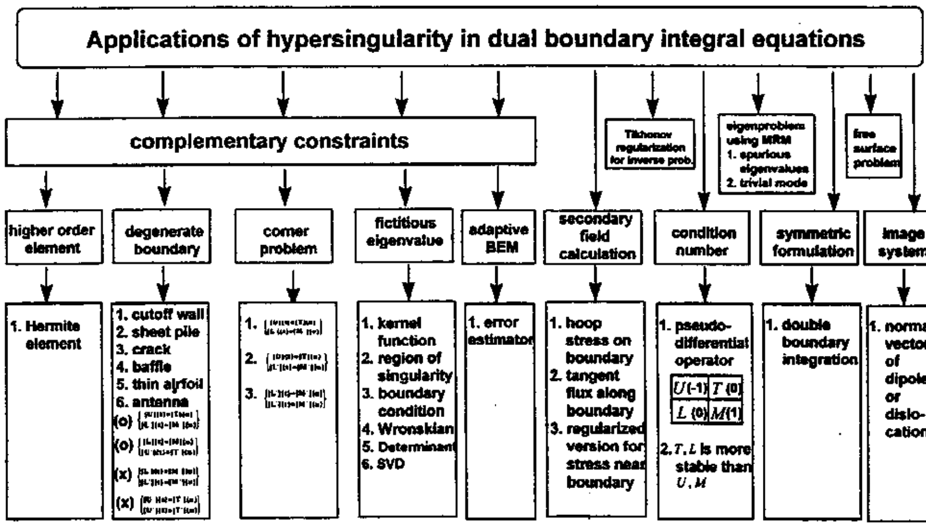

Fig 6. Applications of hypersingularity in dual boundary integral equations

solution well. However, it has been reported [195] that using two hypersingular equations for the two points before and after the corner results in lower accuracy than does using one singular and one hypersingular equation.

6.4 Fictitious eigenfrequencies

It is worth noting that the dual integral equations in acoustic applications received much attention earlier than did the Laplace problem since the exterior problem using the singular integral equation with the U and T kernels introduced fictitious eigenfrequencies. Although Schenck [190] found a unique solution by using the CHIEF method, which is generally preferred by the engineering community, this method has limitations; for example, it can not be used to solve the exterior problem with a degenerate boundary (such as a noise barrier) since an interior point is not available. Burton and Miller [23] were first to propose the combined use of dual integral representation for the acoustic problem with all wave numbers. Terai [209] applied dual integral equations to the acoustic problem with the degenerate boundary of a screen. Wu and Wan [226] also applied dual integral equations to the acoustic radiation and scattering problems for thin bodies. Several researchers [83, 169, 170, 171, 172, 223, 224, 246] have dealt with hypersingularity in this formulation. The available methods have been summarized in [31, 33] from the viewpoint of the dual integral representation. From this viewpoint, the fictitious eigenvalues depend on the kernel of the integral representation of the solution and on the region where the singularity is distributed. In another words, the boundary condition can not change the position of fictitious eigenvalues once the integral representation is chosen. To demonstrate that these statements are true, Chen [31, 33] has given three examples for one-, two- and three-dimensional problems by using the generalized indirect method and the direct method. In the three examples, the degenerate kernels in the frequency domain have been employed to represent the potentials in the interior and exterior domains in [31, 33]. The analytical results can be obtained and the mechanism of fictitious eigenvalues can be easily understood upon considering the difference of the stiffness matrix between the exterior and interior problems. Therefore, some misleading comments by Shaw [192] and Rizzo [186, 187] have been corrected in [31, 33].

6.5 Adaptive boundary element methods

An essential ingredient for all adaptive boundary element methods is a reliable estimate of the local error. The hypersingular integral equation is a complementary equation available for error estimation. Using this concept [147], the error indicator can successfully track the form of the exact error curve. Papers on error estimation and adaptive BEMs can be found in [223, 224].

6.6 Calculation of the tangent flux or the hoop stress on and near the boundary

The hypersingular integral equation can be used to directly calculate the boundary stress instead of using the numerical derivative of the obtained displacement field through Hooke's law. The tangent derivative along the boundary has been formulated in terms of both the boundary potential and the boundary normal flux in [41]. For elasticity problems, Huber *et al* [122] have shown that the accuracy of the numerical derivative is lower than that of the direct calculation of the boundary stress using the hypersingular formulation. Since the integral representation of the solution exhibits the jump behavior across the boundary, the stress or flux near the boundary often displays the Gibbs phenomenon. By using the regularized version of dual integral equations, accuracy near the boundary can be ensured. Numerical examples have been provided in [31, 48].

6.7 Symmetric formulation

In the coupled use of FEM and BEM, the symmetry requirement of the stiffness matrix is especially useful. The four kernel functions in the dual integral equations display the elegant structure of potential theory. The symmetry and transpose symmetry properties for the four kernel functions have been found by Hong and Chen [29, 38, 109]. The dual integral representations can be used to assemble the four kernel functions of the dual internal equations into a global symmetric matrix using the symmetry and transpose symmetry properties of the kernel functions [7, 8, 126]. In order to establish the symmetry for the interpolation function, the quadratic energy form of double integration was needed in the Symmetric-Galerkin formulation by Shiau [194] as well as Hong, Bonnet [17], Kane [126], Parreira [177], and Sirtori [196]. The numerical implementation has been tested successfully by Shiau [194]. However, all the symmetric formulations in the literature need double boundary integrations and, thus, are time-consuming. For reduction to a single boundary integration, degenerate kernels can be employed. Construction of symmetric matrices has been investigated in [7, 8, 31].

6.8 Improvement of condition numbers

In the dual integral representation, the potentials resulted from integrating the U and M kernels are continuous when the field point moves across the boundary, while those from integrating the T and L kernels show the jump behavior. The jump terms will make the $[T]$ and $[L]$ matrices diagonally dominant and preferably lower their condition numbers. For the case of the Dirichlet problem, an inversion of the $[U]$ matrix is needed when the first equation of the dual integral representation is considered. If we adopt the second equation, an inversion of the $[L]$ matrix is preferred since it is more well-conditioned. From the viewpoint of the orders of the pseudo-differential operators, the T and L kernels are of zero order, which are numerically stabler than the U kernel of order one and the M kernel of order minus one [28]. This agrees with the above statement that the inversion of a matrix with diagonal dominance is numerically stabler. The example shown in Fig 7 demonstrates this.

6.9 Detection of spurious roots in the dual multiple reciprocity method (MRM)

The conventional MRM has the problem of spurious eigenvalues. About this, the dual MRM provides an ideal framework to solve the eigenproblems in real domain. To distinguish whether the eigenvalue is true or not, Chen and Wong [50] applied a hypersingular MRM formulation to obtain sufficient constraints for the eigenequations. The dual formulation for the MRM has also been successfully extended to solve the acoustic modes for a two-dimensional cavity with an incomplete partition [51]. The singular value decomposition (SVD) technique can also be used to filter out spurious eigenvalues for an overdeterminate system in the dual MRM. Another advantage using the SVD for the overdeterminate system in the dual MRM was its ability to determine the multiplities of the eigenvalues.

Moreover, a series-type complex-valued dual BEM called the complete MRM was derived in [239]. Four methods, the complete MRM [239], the complex-valued dual BEM [56], the real part of the complex-valued dual BEM [152], and the conventional dual MRM [50, 51], were summarized in [32].

6.10 The Tikhonov formulation for inverse problems with overspecified boundary conditions

In solving an ill-posed inverse problem with overspecified boundary conditions by the Tikhonov formulation, double boundary integrals occur naturally. The inner integrals in the double integrals are hypersingular. To avoid hypersingularity, Yeih, Koya, and Mura [139, 237, 238] applied a fictitious BEM to deal with the inverse problem. Yeih's technique is not absolutely necessary since the hypersingular integral can be evaluated by one of the regularization techniques.

6.11 Construction of the image system

In the half-space [167], quarter-plane [102] or quarter space [103] problems, special Green's functions subjected to certain boundary conditions are often used as auxiliary systems to establish integral equations which can eliminate integrations on the rectilinear or plane boundaries such as the ground surface. Conventionally, we have only the strength of the source to adjust for satisfaction of the boundary conditions. Based on the physical meaning of the dual integral equations of potential theory, an additional degree of freedom is available, which is the normal vector of the dipole or dislocation source. Illustrative examples have been given in the book by Chen and Hong [38].

6.12 Free-surface problems

The free-surface problem can be treated as a moving boundary problem with overspecified boundary conditions. An iterative scheme for the free-surface seepage was proposed by Niwa *et al* [175] using the conventional BEM. By employing the hypersingular integral equation, the rate of convergence can be accelerated.

7 APPLICATIONS OF DIVERGENT SERIES IN DUAL SERIES REPRESENTATIONS

The growing importance of divergent series in both pure and applied mathematics has been justified through investigation of the theory of summability. Since the Fejér reproducing kernel [86] and the Cesàro sum [105] have been shown to be effective methods for summing divergent series, some applications in applied mechanics have been developed, eg, random vibration [234], dynamic responses of a string [47], a shear beam [40, 111] and a flexural beam [45], heat conduction [39], and stress analysis of a plate [206]. More applications can be found in [31, 111].

In mathematical terminology, the Cesàro operator for the series is similar to the reproducing kernel for the function representation. The Fejér kernel is one of the reproducing kernels. By the aid of the Cesàro operator or the Fejér kernel, the series representation of a solution can be approximated closer to the analytical solution in the L_2 sense. By employing the same concept, the wavelet technique [27] using the reproducing kernel to represent a function by means of two parameters, time and frequency, is now a promising technique in signal analysis.

Although the BEMs have been under development for two decades, it is only recent that the theory of divergent series has been correlated with hypersingularity from the viewpoint of the dual representations [31, 111]. In the regularization methods for hypersingularity [200, 207], we have a limiting process and Stokes' theorem, which can be viewed as integration by parts. Correspondingly, summability of divergent series has two alternative techniques: one is the Cesàro operation, and the other is Stokes' transformation, which can also be viewed as summation by parts. The former method behaves like the limiting process for hypersingularity by increasing the number of terms (modes) when the secondary field near the boundary is solved, as shown in Fig 1b. The theory of divergent series in the BEM stems from modal analysis of dynamic problems with an essential time-dependent boundary condition. As Mindlin and Goodman mentioned [168], the quasi-static solution can be decomposed from the total solution first, and the divergent series in the representation for the secondary field can be avoided. This decomposition method, also known as the subtraction method, can be regarded as one of the regularization techniques for extracting the finite part in the series solution. Nevertheless, calculation of the quasi-static solution is a dif-

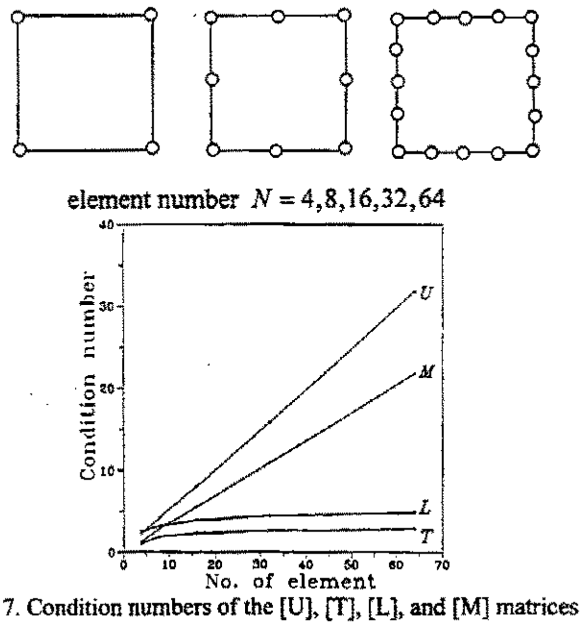

Fig 7. Condition numbers of the [U], [T], [L], and [M] matrices

ficult task since it requires solving the partial differential equation [82]. By means of the dual series representation, the quasi-static solution for displacement is transformed into the integration of the secondary field derived from Stokes' transformation. It is obvious that integrating a known function is easier than solving a partial differential equation (PDE) directly under the same convergence rate of the mode acceleration method [42, 70]. For the secondary field, the quasi-static content is implicitly contained in the boundary terms if Stokes' transformation is utilized.

It is noted that the complete spectral information, including the modal frequencies (eigenvalues), modal displacements (eigenfunctions) and modal reactions, should be known *a priori*, either by means of analysis or experiment, since it is the base in the dual series representation. The modal reaction method to calculate the modal participation factor for support excitations has been proposed in [43]. In the eigen solver, the modal reaction data are often overlooked by engineers and scholars. To the authors' knowledge, only the ABAQUS program among large-scale commercial software has the output option in the eigen solver. But the data can be efficiently utilized for multi-support motion problems. When the kernel function is expanded by eigenfunctions with an associated homogeneous boundary condition, the direct integration scheme is feasible free from matrix inversion.

Another important role the theory of divergent series plays is the function representation in computational mechanics. In solving a PDE, we often represent the solution in terms of a series or integral representation, eg, eigenfuction expansion or the Fourier integral, and then substitute the representation into the PDE to obtain an easier governing equation using operational mathematics. Without considering the theory of divergent series or integrals, this formulation will introduce a paradox since differentiation of the representation has been wrongly used as Stakgold mentioned [202]. By employing the regularization techniques, the paradox can be avoided. In FFT (Fast Fourier Transform), Körner [138] also used the reproducing kernel to represent the Fourier integral more accurately by means of regularized representation, that is,

$$u_b(t)=\sum_{n=-N}^{n=N} w_n^r U(f_n) e^{i2\pi f_n t}, \tag{31}$$

which is different from the conventional representation:

$$u_b(t)=\sum_{n=-N}^{n=N} U(f_n) e^{i2\pi f_n t}. \tag{32}$$

For either an integer or noninteger order r and the Cesàro order k, the term w_n^r in Eq (20) is the weight defined as follows:

$$w_n^r=\frac{\Gamma(k+1)\Gamma(k+r-n+1)}{\Gamma(k-n)\Gamma(k+r+1)}, \tag{33}$$

where $\Gamma(\cdot)$ is the Gamma function. From the convolution point of view, Eq (20) can be transformed into

$$u_b(t)=\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(t-\tau)u_b(\tau)d\tau, \tag{34}$$

where the reproducing kernel is

$$K_N(t-\tau)=\sum_{n=-N}^{n=N} w_n^r e^{i2\pi f_n t}. \tag{35}$$

For the case of Cesàro order 1, the reproducing kernel is reduced to the Fejèr kernel as shown below:

$$K_N(t-\tau)=\frac{1}{(N+1)}\frac{\sin^2((N+1)(t-\tau)/2)}{\sin^2((t-\tau)/2)}. \tag{36}$$

The diagram of the reproducing kernel has been shown in [31, 138] for increasing values of N. In applications in the frequency domain, Eq (20) is used instead of Eq (21). By the reproducing technique, divergence will be filtered out and the finite part can be extracted. In [31, 59, 114] the Cesàro sum in conjunction with the L-curve was used to overcome the divergence in the deconvolution for site response analysis.

8 CONCLUDING REMARKS

In this article, the dual boundary element methods (DBEMs) under the unified framework of dual representations have been reviewed. The relations between the dual integral equations and the dual series representations were discussed. The regularization techniques for hypersingular integrals and divergent series were summarized. The roles the DBEMs with hypersingularity and divergent series play in computational mechanics were explored, and applications to various problems of the general three kinds: the static-boundary-value problems, initial-boundary-value problems and time-harmonic boundary-value problems, were cited. There still remain some interesting problems in applying the DBEMs with hypersingularity and divergent series.

ACKNOWLEDGEMENTS

The continuing support from the National Science Council, Taiwan, under Grants NSC 83-0410-E-002-040, NSC 85-2211-E-002-003, NSC 86-2211-E-019-006, NSC 87-2211-E-019-017 and NSC 88-2211-E-019-005, are gratefully acknowledged. The references provided by Prof QH Du and Dr DH Yu, and the comments by Prof AW Leissa are sincerely acknowledged.

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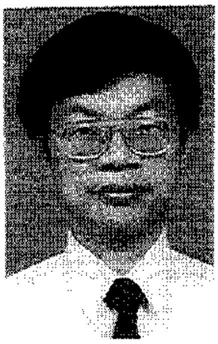

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