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Determination of the natural frequencies and natural modes of a rod using the dual BEM in conjunction with the domain partition technique

J. R. Chang, W. Yeih, J. T. Chen

Abstract In this paper, the dual BEM in conjunction with the domain partition technique is employed to solve both natural frequencies and natural modes of a rod. In this new approach, there exists no spurious eigenvalue using the complex-valued singular or hypersingular equation alone. In the derivation of the singular and hypersingular integral equations, if only the real parts of the kernel functions are chosen, the resulting eigenequations have spurious eigenvalues. Such spurious eigenvalues stem from adding the dummy links into the interior structures considered. Although the spurious eigenvalues exist in this approach which uses the real-valued kernel functions, the possible indeterminacy of eigenmodes using the conventional real-valued singular or real-valued hypersingular equations disappears when the domain partition technique is adopted. The conventional real-valued multiple reciprocity BEM results in spurious eigenvalues for the mixed boundary conditions and indeterminacy of eigenmodes owing to insufficient rank of the leading coefficient matrix for the Dirichlet and Neumann boundary conditions. Such problems can be solved by combining the singular and hypersingular equations together; however, they also can be treated by using the real-valued singular or hypersingular equation alone if the domain partition technique is adopted. Three examples including the Dirichlet, Neumann and mixed type boundary conditions are investigated to show the validity of current approach.

1 Introduction

The domain decomposition method (DDM) has been successfully used in the finite element method (FEM) in the fields related to structural engineering. Many researchers and engineers have begun to calculate numerically dynamic responses of large or super-large structures, such as the natural frequency, mode shape, or vibration acceleration, by partitioning a large structure into several substructures such that the parallel treatment schemes can be adopted to reduce the computation burden. Especially for the structures with discontinuous physical properties, e.g., several rods with the unequal cross sections in a rod structure system or the discontinuous lateral loading density functions applied to a beam as shown in Fig. 1, the DDM is becoming a popular way to treat such problems. Guyan (1965) first proposed the static condensation method for FEM; the solution of an eigenproblem then can be obtained effectively for the modelized structures with a relative moderate number of degrees of freedom. A second class of the reduction methods, called the dynamic condensation methods, were further proposed by Craig and Bampton (1968), Petersmann (1984), and Leung (1988, 1993), to exploit both modal synthesis and dynamic analysis in the domain decomposition scheme. The proposed static and dynamic condensation approaches both distribute the original stiffness or mass matrices into a master degree of freedom to solve the eigenproblems within the numerical precision. A demonstrative work, using the DDM with the conjugate gradient iteration on a parallel workstation cluster to perform finite element analyses, was recently established for evaluating the parallel performance of large scale computing by Horie and Kuramae (1996). Therefore, the dimensionality reduction and computational efficiency can be achieved owing to the matrix and/or domain partition and later numerical schemes in the FEM sense.

Although the boundary element method (BEM) has been developed for a long time, the dual boundary element method (DBEM) formulations have not been well implemented until Hong and Chen (1988). After that, the DBEM was further applied to the multiple-cracked fatigue propagation and life evaluation by Yan and Nguyen-Dang (1995), the static and dynamic fracture analysis by Aliabadi and Brebbia (1993), Aliabadi (1995, 1997), the exterior problem by Chen et al. (1995), error estimation for adaptive mesh generation by Liang et al. (1999), and the boundary value problems with a degenerate boundary and corner problems by Chen and Hong (1994, 1995). Many such problems can be dealt with more directly and effectively by combining the singular integral equation in conventional BEM with the hypersingular integral equation. Besides, a comprehensive review of DBEMs with emphasis on hypersingular integrals and their applications was recently addressed by Chen and Hong (1999).

For a Helmholtz equation, the complex-valued fundamental solution has been employed to solve eigenproblems e.g., by Kamiya, Ando and Nogae (1996). To avoid complicated computation in the domain of a complex number,

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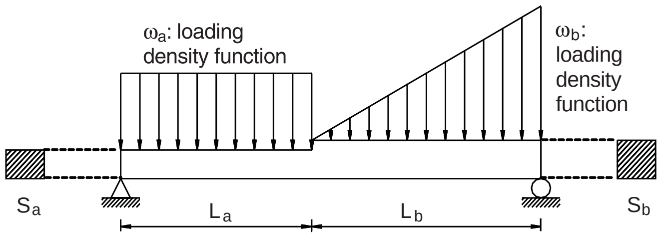


Fig. 1. Discontinuous loading density function applied to a beam with unequal cross sections

the multiple reciprocity boundary element method (MR/BEM) was introduced to solve the eigenproblems in the real domain: Nowak and Brebbia (1989), Kamiya and Ando (1993). Meanwhile, another approach named as the dual reciprocity boundary element method (DR/BEM), originally introduced by Brebbia and Nardini (1983) to solve vibration problems and later used to investigate the accuracy of two-dimensional (2-D) elastodynamic interior problems by Agnantiaris, Polyzer, and Beskos (1996) with the radial basis function $(1 + r)$, was also adopted to deal with the dynamic problem in the real-valued domain. Based on the advantages of DR/BEM, such as the dimensionality reduction, the simple elastostatic fundamental solution, a further extended and more generalized works on a three-dimensional (3-D) dynamic analysis including both free and forced vibrations have also been developed by Agnantiaris, Polyzer, and Beskos (1998). A comparison of the two studies shows that the DR/BEM is more accurate and efficient when used for a 2-D elastodynamic problem than 3-D one. However, when the DR/BEM is applied on such problems, taking the appropriate radial basis functions as the interpolation function for calculating the domain integral of body source term, whichever radial terms should be chosen has attracted the researchers' attention: Golberg, Chen, Bowman, and Power (1998).

In the algorithms of both DR/BEM and MR/BEM, the Helmholtz equation is treated as a Poisson equation with an external source; thus, the fundamental solution of the Laplace equation is considered. However, the domain integral is present due to integration of the external source. MR/BEM can transform the domain integral into a boundary integral iteratively so that the domain cells are not necessarily formed when the remainder terms of the domain integral can be neglected without loss of precision. Although De Mey (1977) could be the first one that replaced the complex-valued kernel functions by their real parts in eigenvalue analysis and termed it as the simplified technique, only the first eigenvalue calculation was performed that makes the spurious eigenvalue phenomenon undiscovered in the following calculation. Hutchinson (1985) used the same concept and later found a practical way to filter out the spurious modes by observing the mode shapes. Up to date, the conventional (real-valued) singular integral equation has been used only in MR/BEM: Nowak and Neves (1994). In the study of Chen (1998), a combination of the singular (or called UT) and hypersingular (or called LM) equations was further proposed to cope with spurious eigenvalues and possible resulting indeterminacy of eigenmodes for one-dimensional rod and two-dimensional cavity. In addition, the singular value decomposition (SVD) method has been introduced to solve such two problems in a systematic way: Yeih et al. (1999). Nevertheless, the methods mentioned above must combine the real-valued UT and LM equations together to have enough information. Kamiya, Ando and Nogae (1996) found that the kernels in MR/BEM were no more than real parts of the kernels in the complex-valued formulation for two-dimensional cases. Yeih et al. (1998) proved that MR/BEM could be constructed such that it is fully equivalent to the complex-valued formulation by adding a complex constant into the zeroth fundamental solution for the Laplace operator when the radiation condition is satisfied. Furthermore, they clearly explained why the spurious eigenvalue problem was encountered in the conventional real-valued MR/BEM.

In this paper, the problem of determining the natural frequencies and natural modes for a rod is revisited. The concept of domain partition with the complex-valued dual BEM is first introduced. It is found that there exists no spurious eigenvalue in the complex-valued formulation, whatever complex-valued singular (complex-valued UT) or hypersingular (complex-valued LM) equation is used. The real-valued singular (real-valued UT) and hypersingular (real-valued LM) equations are derived using the real part of the complex-valued kernel functions in the complex-valued formulation. The spurious eigenvalues, which result from taking the real part of complex-valued kernel functions in the real-valued singular or hypersingular equation, disappear after the domain partition technique is introduced. It is also found that the real-valued singular (real-valued UT) equation or hypersingular (real-valued LM) equations in conjunction with the domain partition concept results in other spurious eigenvalues. The mechanism of such phenomena is due to the introduction of domain partition where dummy links are added and the real parts of the complex-valued kernel functions are adopted. By moving the positions of dummy links, we can effectively filter out the spurious eigenvalues problem in the domain partition scheme. Meanwhile, another important result of the current research is that the problem of possible indeterminacy of eigenmodes for the conventional methods (real-valued UT equation or real-valued LM equation without the domain partition concept) disappears when the proposed technique is adopted. The domain partition technique in conjunction with the real-valued UT or LM equation then can be used to solve the eigenproblem within its own formulation and does not require any help from the complementary equation of the dual real-valued MR/BEM. This concept avoids the need to combine real-valued UT and LM equations in the conventional real-valued dual MRM formulation, in which the domain partition technique is not used.

2 Problem statement and analytical derivations

Consider a one-dimensional subdomain using the global coordinate system. The governing equation of vibration problems in the wave number domain can be expressed as

$$\frac{d^2u(x)}{dx^2} + \lambda u(x) = 0, \quad a \le x \le b \ , \tag{1}$$

where λ and $u(x)$ denote the eigenvalue and eigenmode, respectively.

Select an auxiliary system with a fundamental solution satisfying

$$\frac{d^2U(x,s)}{dx^2} + \lambda U(x,s) = \delta(x,s), \quad -\infty < x < \infty \ , \quad (2)$$

where $U(x,s)$ is a fundamental solution expressed as

$$U(x,s) = \frac{e^{i\sqrt{\lambda}r}}{2i\sqrt{\lambda}} \text{ with } r \equiv |x-s| \ . \quad (3)$$

Introducing Green's third identity, we have

$$u(s) = \int_a^b \left[\frac{d^2U(x,s)}{dx^2} + \lambda U(x,s)\right] u(x)dx = \left[\frac{dU(x,s)}{dx} u(x) - U(x,s)\frac{du(x)}{dx}\right]_a^b \ . \quad (4)$$

Substituting the fundamental solution, $U(x,s) = e^{i\sqrt{\lambda}r}/2i\sqrt{\lambda}$, into Eq. (4) yields the equation for the primary field, $u(s)$, as

$$u(s) = \frac{e^{i\sqrt{\lambda}(s-a)}}{2} u(a) + \frac{e^{i\sqrt{\lambda}(b-s)}}{2} u(b) + \frac{e^{i\sqrt{\lambda}(s-a)}}{2i\sqrt{\lambda}} t(a) - \frac{e^{i\sqrt{\lambda}(b-s)}}{2i\sqrt{\lambda}} t(b) \ , \quad (5)$$

where $t(x_0) \equiv du(x)/dx|_{x=x_0}$.

Differentiating Eq. (5), the equation for the secondary field, $t(s) \equiv du(x)/dx|_{x=s}$, is expressed as

$$t(s) = \frac{i\sqrt{\lambda}e^{i\sqrt{\lambda}(s-a)}}{2} u(a) - \frac{i\sqrt{\lambda}e^{i\sqrt{\lambda}(b-s)}}{2} u(b) + \frac{e^{i\sqrt{\lambda}(s-a)}}{2} t(a) + \frac{e^{i\sqrt{\lambda}(b-s)}}{2} t(b) \ . \quad (6)$$

In the matrix form, Eqs. (4) and (5) can be expressed, respectively, as (Complex-valued UT equation)

$$u(s) = \left[\frac{e^{i\sqrt{\lambda}(s-a)}}{2}, \frac{e^{i\sqrt{\lambda}(b-s)}}{2}, \frac{e^{i\sqrt{\lambda}(s-a)}}{2i\sqrt{\lambda}}, -\frac{e^{i\sqrt{\lambda}(b-s)}}{2i\sqrt{\lambda}}\right] \times \begin{bmatrix} u(a) \\ u(b) \\ t(a) \\ t(b) \end{bmatrix} ; \quad (7)$$

(Complex-valued LM equation)

$$t(s) = \left[\frac{i\sqrt{\lambda}e^{i\sqrt{\lambda}(s-a)}}{2}, -\frac{i\sqrt{\lambda}e^{i\sqrt{\lambda}(b-s)}}{2}, \frac{e^{i\sqrt{\lambda}(s-a)}}{2}, \frac{e^{i\sqrt{\lambda}(b-s)}}{2}\right] \times \begin{bmatrix} u(a) \\ u(b) \\ t(a) \\ t(b) \end{bmatrix} . \quad (8)$$

Equations. (7) and (8) are both exact forms, and the leading coefficient matrices originate from complex-valued kernels. Once the coefficients are expanded in series form, they become the complete MR/BEM: Yeih et al. (1998). When the real-parts of complex-valued kernels are chosen, they form the conventional dual real-valued MR/BEM. In addition, the above two equations can derive the exact dynamic stiffness. Comparison of this approach with traditional approximate dynamic condensation methods will be left for another paper.

By taking out the real-part of the leading coefficient in Eqs. (7) and (8), the following equations are derived: (Real-valued UT equation)

$$u(s) = \left[\frac{\cos[\sqrt{\lambda}(s-a)]}{2}, \frac{\cos[\sqrt{\lambda}(b-s)]}{2}, \frac{\sin[\sqrt{\lambda}(s-a)]}{2\sqrt{\lambda}}, -\frac{\sin[\sqrt{\lambda}(b-s)]}{2\sqrt{\lambda}}\right] \begin{bmatrix} u(a) \\ u(b) \\ t(a) \\ t(b) \end{bmatrix} ; \quad (9)$$

(Real-valued LM equation)

$$t(s) = \left[-\frac{\sqrt{\lambda}\sin[\sqrt{\lambda}(s-a)]}{2}, \frac{\sqrt{\lambda}\sin[\sqrt{\lambda}(b-s)]}{2}, \frac{\cos[\sqrt{\lambda}(s-a)]}{2}, -\frac{\cos[\sqrt{\lambda}(b-s)]}{2}\right] \begin{bmatrix} u(a) \\ u(b) \\ t(a) \\ t(b) \end{bmatrix} . \quad (10)$$

3
Free vibration problems of a rod using the domain partition technique

Consider a one-dimensional rod vibration problem with the following governing equation:

$$\frac{d^2u(x)}{dx^2} + \lambda u(x) = 0, \quad 0 \le x \le 1 \ . \quad (11)$$

It is assumed that the rod has unit length without loss of generality. Three benchmark examples are considered as follows: Chen, Wong (1997)

Case I: $u(0) = 0, u(1) = 0$ (Dirichlet boundary conditions);
Case II: $t(0) = 0, t(1) = 0$ (Neumann boundary conditions);
Case III: $u(0) = 0, t(1) = 0$ (Mixed boundary conditions).

Assuming that the rod considered is arbitrarily partitioned into two subdomains at $x = \xi$, the two subdomains are $x \in \{0,\xi\}$ and $x \in \{\xi, 1\}$, where $0 < \xi < 1$. By moving the field point, s, to boundary points in two subdomains, respectively, and introducing interface conditions,

$$u(\xi^-) = u(\xi^+) \equiv u(\xi) \text{ (displacement continuous)} \ , \quad (12)$$

$$t(\xi^-) = t(\xi^+) \equiv t(\xi) \text{ (force continuous)} \ , \quad (13)$$

the following equations can be formulated,

(Complex-valued UT equation)

$$\begin{Bmatrix} u(0) \\ u(\xi) \\ u(\xi) \\ u(1) \end{Bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{e^{i\sqrt{\lambda}\xi}}{2} & 0 & \frac{1}{2i\sqrt{\lambda}} & -\frac{e^{i\sqrt{\lambda}\xi}}{2i\sqrt{\lambda}} & 0 \\ \frac{e^{i\sqrt{\lambda}\xi}}{2} & \frac{1}{2} & 0 & \frac{e^{i\sqrt{\lambda}\xi}}{2i\sqrt{\lambda}} & -\frac{1}{2i\sqrt{\lambda}} & 0 \\ 0 & \frac{1}{2} & \frac{e^{i\sqrt{\lambda}(1-\xi)}}{2} & 0 & \frac{1}{2i\sqrt{\lambda}} & -\frac{e^{i\sqrt{\lambda}(1-\xi)}}{2i\sqrt{\lambda}} \\ 0 & \frac{e^{i\sqrt{\lambda}(1-\xi)}}{2} & \frac{1}{2} & 0 & \frac{e^{i\sqrt{\lambda}(1-\xi)}}{2i\sqrt{\lambda}} & -\frac{1}{2i\sqrt{\lambda}} \end{bmatrix} \begin{Bmatrix} u(0) \\ u(\xi) \\ u(1) \\ t(0) \\ t(\xi) \\ t(1) \end{Bmatrix} \tag{14}$$

(Complex-valued LM equation)

$$\begin{Bmatrix} t(0) \\ t(\xi) \\ t(\xi) \\ t(1) \end{Bmatrix} = \begin{bmatrix} \frac{i\sqrt{\lambda}}{2} & -\frac{i\sqrt{\lambda}e^{i\sqrt{\lambda}\xi}}{2} & 0 & \frac{1}{2} & \frac{e^{i\sqrt{\lambda}\xi}}{2} & 0 \\ \frac{i\sqrt{\lambda}e^{i\sqrt{\lambda}\xi}}{2} & -\frac{i\sqrt{\lambda}}{2} & 0 & \frac{e^{i\sqrt{\lambda}\xi}}{2} & \frac{1}{2} & 0 \\ 0 & \frac{i\sqrt{\lambda}}{2} & -\frac{i\sqrt{\lambda}e^{i\sqrt{\lambda}(1-\xi)}}{2} & 0 & \frac{1}{2} & \frac{e^{i\sqrt{\lambda}(1-\xi)}}{2} \\ 0 & \frac{i\sqrt{\lambda}e^{i\sqrt{\lambda}(1-\xi)}}{2} & -\frac{i\sqrt{\lambda}}{2} & 0 & \frac{e^{i\sqrt{\lambda}(1-\xi)}}{2} & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} u(0) \\ u(\xi) \\ u(1) \\ t(0) \\ t(\xi) \\ t(1) \end{Bmatrix} \tag{15}$$

(Real-valued UT equation)

$$\begin{Bmatrix} u(0) \\ u(\xi) \\ u(\xi) \\ u(1) \end{Bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\cos(\sqrt{\lambda}\xi)}{2} & 0 & 0 & -\frac{\sin(\sqrt{\lambda}\xi)}{2\sqrt{\lambda}} & 0 \\ \frac{\cos(\sqrt{\lambda}\xi)}{2} & \frac{1}{2} & 0 & \frac{\sin(\sqrt{\lambda}\xi)}{2\sqrt{\lambda}} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\cos[\sqrt{\lambda}(1-\xi)]}{2} & 0 & 0 & -\frac{\sin[\sqrt{\lambda}(1-\xi)]}{2\sqrt{\lambda}} \\ 0 & \frac{\cos[\sqrt{\lambda}(1-\xi)]}{2} & \frac{1}{2} & 0 & \frac{\sin[\sqrt{\lambda}(1-\xi)]}{2\sqrt{\lambda}} & 0 \end{bmatrix} \begin{Bmatrix} u(0) \\ u(\xi) \\ u(1) \\ t(0) \\ t(\xi) \\ t(1) \end{Bmatrix} \tag{16}$$

(Real-valued LM equation)

$$\begin{Bmatrix} t(0) \\ t(\xi) \\ t(\xi) \\ t(1) \end{Bmatrix} = \begin{bmatrix} 0 & \frac{\sqrt{\lambda}\sin(\sqrt{\lambda}\xi)}{2} & 0 & \frac{1}{2} & \frac{\cos(\sqrt{\lambda}\xi)}{2} & 0 \\ -\frac{\sqrt{\lambda}\sin(\sqrt{\lambda}\xi)}{2} & 0 & 0 & \frac{\cos(\sqrt{\lambda}\xi)}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{\sqrt{\lambda}\sin[\sqrt{\lambda}(1-\xi)]}{2} & 0 & \frac{1}{2} & \frac{\cos[\sqrt{\lambda}(1-\xi)]}{2} \\ 0 & -\frac{\sqrt{\lambda}\sin[\sqrt{\lambda}(1-\xi)]}{2} & 0 & 0 & \frac{\cos[\sqrt{\lambda}(1-\xi)]}{2} & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} u(0) \\ u(\xi) \\ u(1) \\ t(0) \\ t(\xi) \\ t(1) \end{Bmatrix}. \tag{17}$$

Substituting the boundary conditions for the three cases into Eqs. (14), (15), (16) and (17), we have the following.

Case I. (Dirichlet B.C.s, $u(0) = u(1) = 0$):

(Complex-valued UT equation)

$$\begin{bmatrix} \frac{e^{i\sqrt{\lambda}\xi}}{2} & \frac{1}{2i\sqrt{\lambda}} & -\frac{e^{i\sqrt{\lambda}\xi}}{2i\sqrt{\lambda}} & 0 \\ -\frac{1}{2} & \frac{e^{i\sqrt{\lambda}\xi}}{2i\sqrt{\lambda}} & -\frac{1}{2i\sqrt{\lambda}} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2i\sqrt{\lambda}} & -\frac{e^{i\sqrt{\lambda}(1-\xi)}}{2i\sqrt{\lambda}} \\ \frac{e^{i\sqrt{\lambda}(1-\xi)}}{2} & 0 & \frac{e^{i\sqrt{\lambda}(1-\xi)}}{2i\sqrt{\lambda}} & -\frac{1}{2i\sqrt{\lambda}} \end{bmatrix} \begin{Bmatrix} u(\xi) \\ t(0) \\ t(\xi) \\ t(1) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}; \tag{18}$$

(Complex-valued LM equation)

$$\begin{bmatrix} -\frac{i\sqrt{\lambda}e^{i\sqrt{\lambda}\xi}}{2} & -\frac{1}{2} & \frac{e^{i\sqrt{\lambda}\xi}}{2} & 0 \\ -\frac{i\sqrt{\lambda}}{2} & \frac{e^{i\sqrt{\lambda}\xi}}{2} & -\frac{1}{2} & 0 \\ \frac{i\sqrt{\lambda}}{2} & 0 & -\frac{1}{2} & \frac{e^{i\sqrt{\lambda}(1-\xi)}}{2} \\ \frac{i\sqrt{\lambda}e^{i\sqrt{\lambda}(1-\xi)}}{2} & 0 & \frac{e^{i\sqrt{\lambda}(1-\xi)}}{2} & -\frac{1}{2} \end{bmatrix} \begin{Bmatrix} u(\xi) \\ t(0) \\ t(\xi) \\ t(1) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}; \tag{19}$$

(Real-valued UT equation)

$$\begin{bmatrix} \frac{\cos(\sqrt{\lambda}\xi)}{2} & 0 & -\frac{\sin(\sqrt{\lambda}\xi)}{2\sqrt{\lambda}} & 0 \\ -\frac{1}{2} & \frac{\sin(\sqrt{\lambda}\xi)}{2\sqrt{\lambda}} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & -\frac{\sin[\sqrt{\lambda}(1-\xi)]}{2\sqrt{\lambda}} \\ \frac{\cos[\sqrt{\lambda}(1-\xi)]}{2} & 0 & \frac{\sin[\sqrt{\lambda}(1-\xi)]}{2\sqrt{\lambda}} & 0 \end{bmatrix} \times \begin{Bmatrix} u(\xi) \\ t(0) \\ t(\xi) \\ t(1) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}; \tag{20}$$

(Real-valued LM equation)

$$\begin{bmatrix} \frac{\sqrt{\lambda}\sin(\sqrt{\lambda}\xi)}{2} & -\frac{1}{2} & \frac{\cos(\sqrt{\lambda}\xi)}{2} & 0 \\ 0 & \frac{\cos(\sqrt{\lambda}\xi)}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{\cos[\sqrt{\lambda}(1-\xi)]}{2} \\ -\frac{\sqrt{\lambda}\sin[\sqrt{\lambda}(1-\xi)]}{2} & 0 & \frac{\cos[\sqrt{\lambda}(1-\xi)]}{2} & -\frac{1}{2} \end{bmatrix} \times \begin{Bmatrix} u(\xi) \\ t(0) \\ t(\xi) \\ t(1) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{21}$$

Case II. (Neumann B.C.s, $t(0) = t(1) = 0$):
(Complex-valued UT equation)

$$\begin{bmatrix} -\frac{1}{2} & \frac{e^{i\sqrt{\lambda}\xi}}{2} & 0 & -\frac{e^{i\sqrt{\lambda}\xi}}{2i\sqrt{\lambda}} \\ \frac{e^{i\sqrt{\lambda}\xi}}{2} & -\frac{1}{2} & 0 & -\frac{1}{2i\sqrt{\lambda}} \\ 0 & -\frac{1}{2} & \frac{e^{i[\sqrt{\lambda}(1-\xi)]}}{2} & \frac{1}{2i\sqrt{\lambda}} \\ 0 & \frac{e^{i[\sqrt{\lambda}(1-\xi)]}}{2} & -\frac{1}{2} & \frac{e^{i\sqrt{\lambda}(1-\xi)}}{2i\sqrt{\lambda}} \end{bmatrix} \begin{Bmatrix} u(0) \\ u(\xi) \\ u(1) \\ t(\xi) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}; \tag{22}$$

(Complex-valued LM equation)

$$\begin{bmatrix} \frac{i\sqrt{\lambda}}{2} & -\frac{i\sqrt{\lambda}e^{i\sqrt{\lambda}\xi}}{2} & 0 & \frac{e^{i\sqrt{\lambda}\xi}}{2} \\ \frac{i\sqrt{\lambda}e^{i\sqrt{\lambda}\xi}}{2} & -\frac{i\sqrt{\lambda}}{2} & 0 & -\frac{1}{2} \\ 0 & \frac{i\sqrt{\lambda}}{2} & -\frac{i\sqrt{\lambda}e^{i\sqrt{\lambda}(1-\xi)}}{2} & -\frac{1}{2} \\ 0 & \frac{i\sqrt{\lambda}e^{i\sqrt{\lambda}(1-\xi)}}{2} & -\frac{i\sqrt{\lambda}}{2} & \frac{e^{i\sqrt{\lambda}(1-\xi)}}{2} \end{bmatrix} \begin{Bmatrix} u(0) \\ u(\xi) \\ u(1) \\ t(\xi) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}; \tag{23}$$

(Real-valued UT equation)

$$\begin{bmatrix} -\frac{1}{2} & \frac{\cos(\sqrt{\lambda}\xi)}{2} & 0 & -\frac{\sin(\sqrt{\lambda}\xi)}{2\sqrt{\lambda}} \\ \frac{\cos(\sqrt{\lambda}\xi)}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\cos[\sqrt{\lambda}(1-\xi)]}{2} & 0 \\ 0 & \frac{\cos[\sqrt{\lambda}(1-\xi)]}{2} & -\frac{1}{2} & \frac{\sin[\sqrt{\lambda}(1-\xi)]}{2\sqrt{\lambda}} \end{bmatrix} \times \begin{Bmatrix} u(0) \\ u(\xi) \\ u(1) \\ t(\xi) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}; \tag{24}$$

(Real-valued LM equation)

$$\begin{bmatrix} 0 & \frac{\sqrt{\lambda}\sin(\sqrt{\lambda}\xi)}{2} & 0 & \frac{\cos(\sqrt{\lambda}\xi)}{2} \\ -\frac{\sqrt{\lambda}\sin(\sqrt{\lambda}\xi)}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{\sqrt{\lambda}\sin[\sqrt{\lambda}(1-\xi)]}{2} & -\frac{1}{2} \\ 0 & -\frac{\sqrt{\lambda}\sin[\sqrt{\lambda}(1-\xi)]}{2} & 0 & \frac{\cos[\sqrt{\lambda}(1-\xi)]}{2} \end{bmatrix} \times \begin{Bmatrix} u(\xi) \\ t(0) \\ t(\xi) \\ t(1) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{25}$$

Case III. (Mixed B.C.s, $u(0) = t(1) = 0$):
(Complex-valued UT equation)

$$\begin{bmatrix} \frac{e^{i\sqrt{\lambda}\xi}}{2} & 0 & \frac{1}{2i\sqrt{\lambda}} & -\frac{e^{i\sqrt{\lambda}\xi}}{2i\sqrt{\lambda}} \\ -\frac{1}{2} & 0 & \frac{e^{i\sqrt{\lambda}\xi}}{2i\sqrt{\lambda}} & -\frac{1}{2i\sqrt{\lambda}} \\ -\frac{1}{2} & \frac{e^{i[\sqrt{\lambda}(1-\xi)]}}{2} & 0 & \frac{1}{2i\sqrt{\lambda}} \\ \frac{e^{i[\sqrt{\lambda}(1-\xi)]}}{2} & -\frac{1}{2} & 0 & \frac{e^{i\sqrt{\lambda}(1-\xi)}}{2i\sqrt{\lambda}} \end{bmatrix} \times \begin{Bmatrix} u(\xi) \\ u(1) \\ t(0) \\ t(\xi) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}; \tag{26}$$

(Complex-valued LM equation)

$$\begin{bmatrix} -\frac{i\sqrt{\lambda}e^{i\sqrt{\lambda}\xi}}{2} & 0 & -\frac{1}{2} & \frac{e^{i\sqrt{\lambda}\xi}}{2} \\ -\frac{i\sqrt{\lambda}}{2} & 0 & \frac{e^{i\sqrt{\lambda}\xi}}{2} & -\frac{1}{2} \\ \frac{i\sqrt{\lambda}}{2} & -\frac{i\sqrt{\lambda}e^{i\sqrt{\lambda}(1-\xi)}}{2} & 0 & -\frac{1}{2} \\ \frac{i\sqrt{\lambda}e^{i\sqrt{\lambda}(1-\xi)}}{2} & -\frac{i\sqrt{\lambda}}{2} & 0 & \frac{e^{i\sqrt{\lambda}(1-\xi)}}{2} \end{bmatrix} \begin{Bmatrix} u(\xi) \\ u(1) \\ t(0) \\ t(\xi) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}; \tag{27}$$

Table 1. Eigensolutions, eigenvalues and eigenmodes for case I using the domain partition technique with the UT and LM equations

		Eigenequation			
		With domain partition			Without domain partition
		Complex-valued formulation	Real-valued formulation		Real-valued formulation Chen and Wong (1997)
			True	Spurious	
Case I: (Dirichlet) $u(0) = 0$; $u(1) = 0$	UT	$\sin\sqrt{\lambda} = 0$	$\sin\sqrt{\lambda} = 0$	$\frac{1}{4\lambda^{3/2}} \cdot \sin(\sqrt{\lambda}\xi) \cdot \sin[\sqrt{\lambda}(1-\xi)] = 0$	$\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} = 0$
	LM	$\cos\sqrt{\lambda} \cdot \sin\sqrt{\lambda} = 0$: real part $\cos\sqrt{\lambda} + 1 = 0$, $\cos\sqrt{\lambda} - 1 = 0$: imaginary part $\Rightarrow \sin\sqrt{\lambda} = 0$	$\sin\sqrt{\lambda} = 0$	$\frac{\sqrt{\lambda}}{16} \cdot \sin(\sqrt{\lambda}\xi) \cdot \sin[\sqrt{\lambda}(1-\xi)] = 0$	(a) $\cos\sqrt{\lambda} + 1 = 0$ (b) $\cos\sqrt{\lambda} - 1 = 0$

* α, β are arbitrary constants since the matrix is a null matrix
$^+$ All methods obtain the same $u(s)$ except the real-valued UT equation without the domain partition

Table 2. Eigensolutions, eigenvalues and eigenmodes for case II using the domain partition technique with the UT and LM equations

		Eigenequation			
		With domain partition			Without domain partition
		Complex-valued formulation	Real-valued formulation		Real-valued formulation: Chen and Wong (1997)
			True	Spurious	
Case II: (Neumann) $t(0) = 0$; $t(1) = 0$	UT	$\cos\sqrt{\lambda} + 1 = 0$, $\cos\sqrt{\lambda} - 1 = 0$: real part $\cos\sqrt{\lambda} \cdot \sin\sqrt{\lambda} = 0$: imaginary part $\Rightarrow \sin\sqrt{\lambda} = 0$	$\sin\sqrt{\lambda} = 0$	$\frac{1}{16\sqrt{\lambda}} \cdot \sin(\sqrt{\lambda}\xi) \cdot \sin[\sqrt{\lambda}(1-\xi)] = 0$	(a) $\cos\sqrt{\lambda} + 1 = 0$ (b) $\cos\sqrt{\lambda} - 1 = 0$
	LM	$\cos\sqrt{\lambda} \cdot \sin\sqrt{\lambda} = 0$: real part $\cos\sqrt{\lambda} + 1 = 0$, $\cos\sqrt{\lambda} - 1 = 0$: imaginary part $\Rightarrow \sin\sqrt{\lambda} = 0$	$\sin\sqrt{\lambda} = 0$	$\frac{\lambda^{3/2}}{16} \cdot \sin(\sqrt{\lambda}\xi) \cdot \sin[\sqrt{\lambda}(1-\xi)] = 0$	$-\sqrt{\lambda}\sin\sqrt{\lambda} = 0$

* α, β are arbitrary constants since the matrix is a null matrix
$^+$ All methods obtain the same $u(s)$ except the real-valued UT equation without the domain partition

Table 1. continued

Eigenvalue				Boundary higenmode at true eigenvalues			Normalized $u(s)^+$
With domain partition			Without domain partition	With domain partition		Without domain partition	
Complex-valued formulation	Real-valued formulation		Real-valued formulation: Chen and Wong (1997)	Complex-valued formulation	Real-valued formulation	Real-valued formulation: Chen and Wong (1997)	
	True	Spurious					
$\sqrt{\lambda}=n\pi$	$\sqrt{\lambda}=n\pi$	$\sqrt{\lambda}=\frac{n\pi}{\xi}$; $\sqrt{\lambda}=\frac{n\pi}{1-\xi}$	$\sqrt{\lambda}=n\pi$	$\begin{Bmatrix} u(\xi)\\ t(\xi)\\ t(0)\\ t(1)\end{Bmatrix}=\begin{Bmatrix}\frac{\cos\sqrt{\lambda}}{\sqrt{\lambda}}\cdot\sin(\sqrt{\lambda\xi})\\ \cos\sqrt{\lambda}\cdot\cos(\sqrt{\lambda\xi})\\ \cos\sqrt{\lambda}\\ 1\end{Bmatrix}$		$\begin{Bmatrix} t(0)\\ t(1)\end{Bmatrix}=\begin{Bmatrix}\alpha\\ \beta\end{Bmatrix}^*$	$\sin(\sqrt{\lambda}s)$
$\sqrt{\lambda}=n\pi$	$\sqrt{\lambda}=n\pi$	$\sqrt{\lambda}=\frac{n\pi}{\xi}$; $\sqrt{\lambda}=\frac{n\pi}{1-\xi}$	$\sqrt{\lambda}=n\pi$	$\begin{Bmatrix} u(\xi)\\ t(\xi)\\ t(0)\\ t(1)\end{Bmatrix}=\begin{Bmatrix}\frac{\cos\sqrt{\lambda}}{\sqrt{\lambda}}\cdot\sin(\sqrt{\lambda}\xi)\\ \cos\sqrt{\lambda}\cdot\cos(\sqrt{\lambda}\xi)\\ \cos\sqrt{\lambda}\\ 1\end{Bmatrix}$		$\begin{Bmatrix} t(0)\\ t(1)\end{Bmatrix}=\begin{Bmatrix}\cos\sqrt{\lambda}\\ 1\end{Bmatrix}$	$\sin(\sqrt{\lambda}s)$

Table 2. continued

Eigenvalue				Boundary eigenmode at true eigenvalues			Normalized $u(s)^+$
With domain partition			Without domain partition	With domain partition		Without domain partition	
Complex-valued formulation	Real-valued formulation		Real-valued formulation: Chen and Wong (1997)	Complex-valued formulation	Real-valued formulation	Real-valued formulation: Chen and Wong (1997)	
	True	Spurious					
$\sqrt{\lambda}=n\pi$	$\sqrt{\lambda}=n\pi$	$\sqrt{\lambda}=\frac{n\pi}{\xi}$; $\sqrt{\lambda}=\frac{n\pi}{1-\xi}$	$\sqrt{\lambda}=n\pi$	$\begin{Bmatrix} u(0)\\ u(\xi)\\ u(1)\\ t(\xi)\end{Bmatrix}=\begin{Bmatrix}\cos\sqrt{\lambda}\\ \cos\sqrt{\lambda}\cdot\cos(\sqrt{\lambda}\xi)\\ 1\\ -\sqrt{\lambda}\cdot\cos\sqrt{\lambda}\cdot\sin(\sqrt{\lambda}\xi)\end{Bmatrix}$		$\begin{Bmatrix} u(0)\\ u(1)\end{Bmatrix}=\begin{Bmatrix}\cos\sqrt{\lambda}\\ 1\end{Bmatrix}$	$\cos(\sqrt{\lambda}s)$
$\sqrt{\lambda}=n\pi$	$\sqrt{\lambda}=n\pi$	$\sqrt{\lambda}=\frac{n\pi}{\xi}$; $\sqrt{\lambda}=\frac{n\pi}{1-\xi}$	$\sqrt{\lambda}=n\pi$	$\begin{Bmatrix} u(0)\\ u(\xi)\\ u(1)\\ t(\xi)\end{Bmatrix}=\begin{Bmatrix}\cos\sqrt{\lambda}\\ \cos\sqrt{\lambda}\cdot\cos(\sqrt{\lambda}\xi)\\ 1\\ -\sqrt{\lambda}\cdot\cos\sqrt{\lambda}\cdot\sin(\sqrt{\lambda}\xi)\end{Bmatrix}$		$\begin{Bmatrix} u(0)\\ u(1)\end{Bmatrix}=\begin{Bmatrix}\alpha\\ \beta\end{Bmatrix}^*$	$\cos(\sqrt{\lambda}s)$

Table 3. Eigensolutions, eigenvalues and eigenmodes for case III using the domain partition technique with the UT and LM equations

		Eigen equation			
		With domain partition			Without domain partition
		Complex-valued formulation	Real-valued formulation		Real-valued formulation: Chen and Wong (1997)
			True	Spurious	
Case III: (mixed) $u(0) = 0$; $t(1) = 0$	UT	$\cos^2\sqrt{\lambda} = 0$: real part $\cos\sqrt{\lambda}\cdot\sin\sqrt{\lambda} = 0$: imaginary part $\Rightarrow \cos\sqrt{\lambda} = 0$	$\cos\sqrt{\lambda} = 0$	$\frac{-1}{4\lambda^{3/2}}\cdot\sin(\sqrt{\lambda}\xi)\cdot\sin[\sqrt{\lambda}(1-\xi)] = 0$	$\cos\sqrt{\lambda} = 0$; $\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} = 0$: (spurious)
	LM	$\cos\sqrt{\lambda} = 0$: real part $\cos\sqrt{\lambda} = 0$: imaginary part $\Rightarrow \cos\sqrt{\lambda} = 0$	$\cos\sqrt{\lambda} = 0$	$\frac{\lambda}{4}\cdot\sin(\sqrt{\lambda}\xi)\cdot\sin[\sqrt{\lambda}(1-\xi)] = 0$	$\cos\sqrt{\lambda} = 0$; $\sqrt{\lambda}\sin\sqrt{\lambda} = 0$: (spurious)

+ All methods obtain the same $u(s)$ except the real-valued UT equation without the domain partition

(Real-valued UT equation)

$$\begin{bmatrix} \frac{\cos(\sqrt{\lambda}\xi)}{2} & 0 & 0 & -\frac{\sin(\sqrt{\lambda}\xi)}{2\sqrt{\lambda}} \\ -\frac{1}{2} & 0 & \frac{\sin(\sqrt{\lambda}\xi)}{2\sqrt{\lambda}} & 0 \\ -\frac{1}{2} & \frac{\cos[\sqrt{\lambda}(1-\xi)]}{2} & 0 & 0 \\ \frac{\cos[\sqrt{\lambda}(1-\xi)]}{2} & -\frac{1}{2} & 0 & \frac{\sin[\sqrt{\lambda}(1-\xi)]}{2\sqrt{\lambda}} \end{bmatrix} \times \begin{Bmatrix} u(\xi) \\ u(1) \\ t(0) \\ t(\xi) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}; \quad (28)$$

(Real-valued LM equation)

$$\begin{bmatrix} \frac{\sqrt{\lambda}\sin(\sqrt{\lambda}\xi)}{2} & 0 & -\frac{1}{2} & \frac{\cos(\sqrt{\lambda}\xi)}{2} \\ 0 & 0 & \frac{\cos(\sqrt{\lambda}\xi)}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{\lambda}\sin[\sqrt{\lambda}(1-\xi)]}{2} & 0 & -\frac{1}{2} \\ -\frac{\sqrt{\lambda}\sin[\sqrt{\lambda}(1-\xi)]}{2} & 0 & 0 & \frac{\cos[\sqrt{\lambda}(1-\xi)]}{2} \end{bmatrix} \times \begin{Bmatrix} u(\xi) \\ u(1) \\ t(0) \\ t(\xi) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}. \quad (29)$$

To solve the eigenproblems for the three cases, the determinants of the leading coefficient matrices in Eqs. (18) to (29) must be zero in order to obtain the non-trivial eigenvectors. The eigenequations, eigenvalues and eigenmodes obtained using UT or LM equations in conjunction with the domain partition concept are tabulated in Tables. 1 to 3 for cases I to III, respectively. From the results obtained using the domain partition technique, it is interesting to find that all three cases have spurious eigenvalues when the real parts of complex-valued kernel functions are considered. For the conventional MR/BEM, the result is spurious eigenvalues only under mixed boundary conditions owing to use of the real-valued kernel functions: Chen and Wong (1997). In the domain partition approach, the phenomenon of spurious eigenvalues is due to addition of the dummy link in the interiors of the structure and to use of the real-valued kernel functions. This finding is similar to the result in the Ref. of Chen and Sheu (1996). However, the spurious eigenvalue can be filtered out by moving the position of the partitioning structure, ξ, and examining whether the values of λ make the determinant zero or not for all possible ξ. Especially when the ratio of ξ values is chosen to be an irrational number, spurious eigenvalues will occur at different positions for different ξ values. The results of using the direct search method for the eigenvalues in the three cases are schematically shown in Figs. 2 to 7 for the real-valued UT equation and real-valued LM equation.

In comparison with the conventional MR/BEM by Chen and Wong (1997), although the spurious eigenvalues can be filtered out by examining the corresponding eigenmodes obtained from the UT and LM equations, it is only after the eigenmodes have been determined that the spurious eigenvalues can be filtered out. That is, the conventional MR/BEM needs to use both UT and LM equations to filter out the spurious eigenvalues at the stage when the eigenmode is obtained. However, when the domain par-

Table 3. continued

Eigenvalue				Boundary eigenmode at true eigenvalues			Normalized $u(s)^+$
With domain partition			Without domain partition	With domain partition		Without domain partition	
Complex-valued formulation	Real-valued formulation		Real-valued formulation: Chen and Wong (1997)	Complex-valued formulation	Real-valued formulation	Real-valued formulation: Chen and Wong (1997)	
	True	Spurious					
$\sqrt{\lambda}=\frac{(2n-1)\pi}{2}$	$\sqrt{\lambda}=\frac{(2n-1)\pi}{2}$	$\sqrt{\lambda}=\frac{n\pi}{\xi}$; $\sqrt{\lambda}=\frac{n\pi}{1-\xi}$	$\sqrt{\lambda}=\frac{(2n-1)\pi}{2}$	$\begin{Bmatrix} u(\xi)\\ u(1)\\ t(0)\\ t(\xi)\end{Bmatrix}=\begin{Bmatrix}\frac{\sin(\sqrt{\lambda}\xi)}{\sqrt{\lambda}}\\ \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}\\ 1\\ \cos(\sqrt{\lambda}\xi)\end{Bmatrix}$		$\begin{Bmatrix} u(1)\\ t(0)\end{Bmatrix}=\begin{Bmatrix}\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}\\ 1\end{Bmatrix}$	$\sin(\sqrt{\lambda}s)$
$\sqrt{\lambda}=\frac{(2n-1)\pi}{2}$	$\sqrt{\lambda}=\frac{(2n-1)\pi}{2}$	$\sqrt{\lambda}=\frac{n\pi}{\xi}$; $\sqrt{\lambda}=\frac{n\pi}{1-\xi}$	$\sqrt{\lambda}=\frac{(2n-1)\pi}{2}$	$\begin{Bmatrix} u(\xi)\\ u(1)\\ t(0)\\ t(\xi)\end{Bmatrix}=\begin{Bmatrix}\frac{\sin(\sqrt{\lambda}\xi)}{\sqrt{\lambda}}\\ \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}\\ 1\\ \cos(\sqrt{\lambda}\xi)\end{Bmatrix}$		$\begin{Bmatrix} u(0)\\ t(1)\end{Bmatrix}=\begin{Bmatrix}\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}\\ 1\end{Bmatrix}$	$\sin(\sqrt{\lambda}s)$

tition approach is adopted with the UT or LM equation, the spurious eigenvalue can be filtered out in the stage of eigenvalues searching.

Another useful approach, the singular value decomposition (SVD) method, has also been proposed to deal with this problem by Chen (1998). It also combines the UT and LM equations to filter out spurious eigenvalues because for the spurious eigenvalues one can find enough constraint equations within the UT and LM equations such that the corresponding eigenmodes are only null vectors. In the SVD approach, when the minimum singular value of the leading coefficient matrix approaches zero, this means that the rank of the matrix is reduced by one, and the eigenvalue is thus known to be the true eigenvalue. Although this approach can filter out spurious eigenvalues in the stage of root searching, it is still necessary for the UT and LM equations to be combined in order to solve the problem.

For the Helmholtz equation in the exterior domain, the fictitious frequencies have also been encountered numerically by Chen and Hong (1992). They are unexpected numerical resonant phenomena rather than real existing

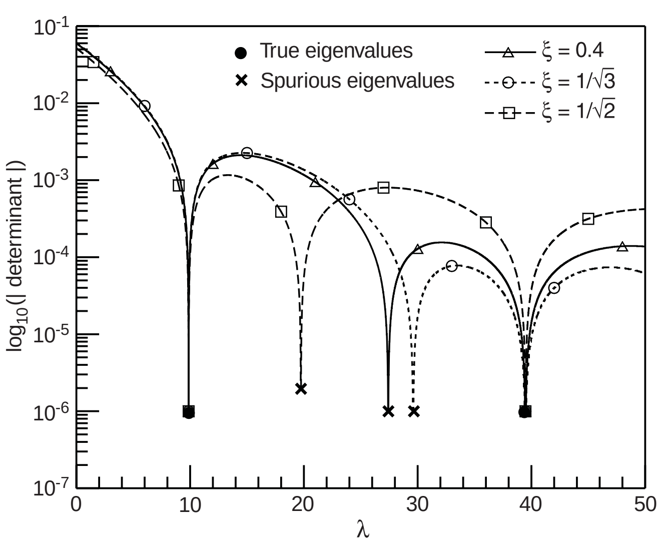


Fig. 2. The spurious eigenvalues are filtered out using the domain partition technique from the UT equation for case I

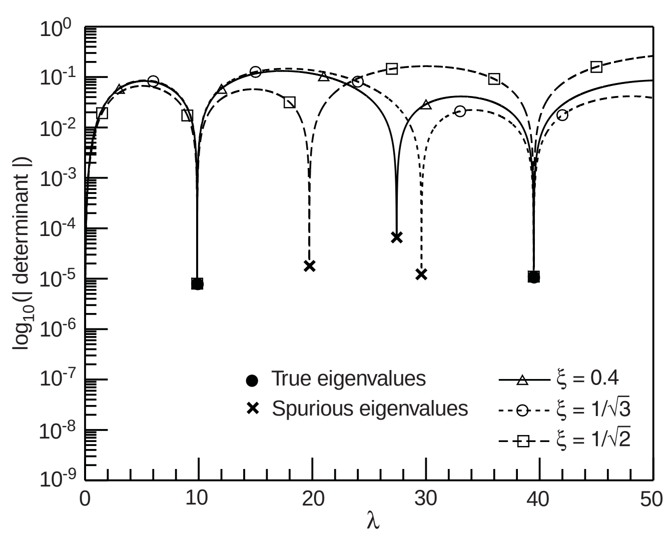


Fig. 3. The spurious eigenvalues are filtered out using the domain partition technique from the LM equation for case I

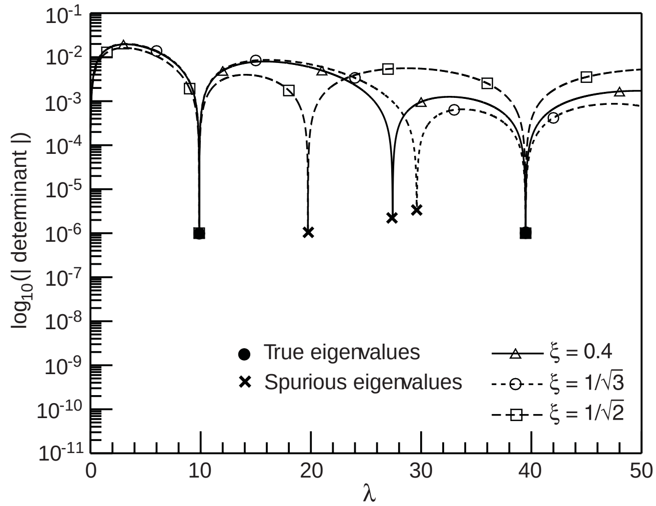


Fig. 4. The spurious eigenvalues are filtered out using the domain partition technique from the UT equation for case II

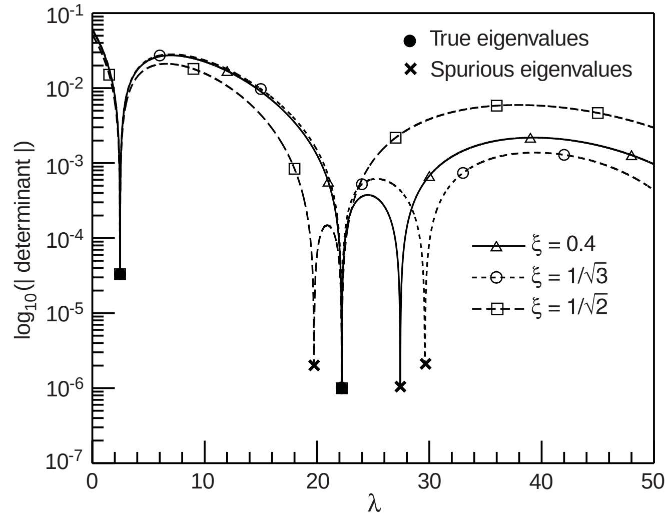


Fig. 6. The spurious eigenvalues are filtered out using the domain partition technique from the UT equation for case III

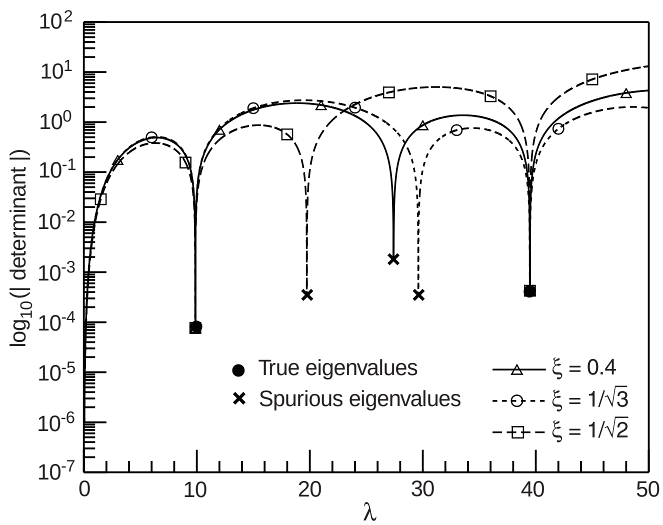


Fig. 5. The spurious eigenvalues are filtered out using the domain partition technique from the LM equation for case II

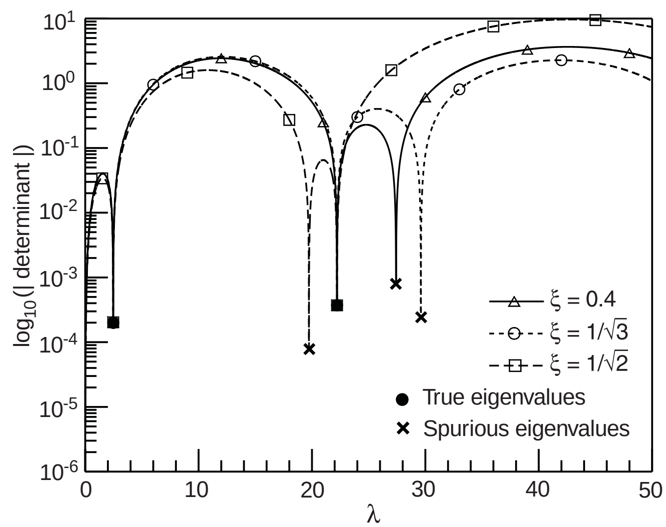
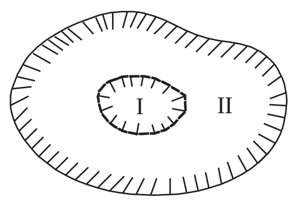
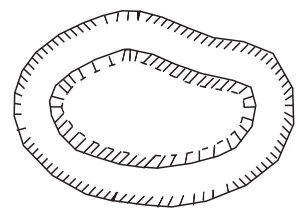


Fig. 7. The spurious eigenvalues are filtered out using the domain partition technique from the LM equation for case III

physical resonant frequencies since there are radiation phenomena but no resonant phenomena in the exterior domain. In the generalized indirect BEM, one creates an artificial boundary outside the domain of interest, and fictitious frequencies will occur depending on the position of the artificial boundary. Furthermore, it is known that by moving the position of the artificial boundary, fictitious frequencies can be filtered out. For the Helmholtz equation in an interior problem, the spurious resonant frequencies (called "spurious" eigenvalues should be more accurate and in contrast of "fictitious") happen when the domain partition technique is used in conjunction with the real-valued UT or LM equation as mentioned above. The mechanism of such spurious resonant frequencies mainly results from adding the internal links into the structure. To filter out such spurious resonant frequencies, one can move the position of the internal links, which is apparently very similar to the way in which we filter out fictitious numerical resonant frequencies in cases involving the exterior domain for the exterior problem. A comparison of these interesting phenomena in solving the Helmholtz equation using the domain partition technique for interior domain and using the integral formulation for exterior domain is tabulated in Table 4.

After filtering out the spurious eigenvalues, the real-valued MR/BEM results in indeterminacy of the eigenmodes in case I with the UT equation and case II with the LM equation because the leading coefficient matrix becomes a null one. Nevertheless, this problem disappears when the domain partition technique is introduced. The cause is the added dummy links, which produce constraints. That is to say, the true eigenvalue may cause the leading coefficient matrix to be a null one. However, when the domain partition concept is introduced, the local elements in the leading coefficient matrix corresponding to the position of the internal links will not be zero, so the

Table 4. A comparison of the phenomena of spurious eigenvalues (for the interior problem using the domain partition technique) and fictitious eigenvalues (for the exterior problem using the integral formulation)

	Interior problem using the domain partition technique	Exterior problem using the integral equation
Spurious (fictitious) eigenvalues	Yes	Yes
Mechanism	The spurious eigenvalues are embedded in the interface of the substructures	The fictitious eigenvalues are embedded in the representation for the solution
Objectives	Filter out spurious physical resonance frequencies	Filter out fictitious numerical resonance frequencies
Methods used to filter out spurious (fictitious) eigenvalues	Move the position of internal dummy links	Move fictitious boundary in generalized indirect BEM
Auxiliary boundary		

leading coefficient matrix cannot be a null one. This concept is very similar to that proposed by Lin, Shiau, Huang (1992), in which external dummy links are added to solve the singular matrix problem.

Once the boundary eigenvectors are determined, the solved boundary eigenvector and boundary conditions can be substituted into the interior field point equation, Eq. (7), for the complex-valued formulation or into Eq. (9) for the real-valued formulation to determine the corresponding mode shapes. It should be mentioned here that the mode shapes corresponding to each element ought to be determined separately when the domain partition technique is adopted. All the determined analytical eigenmodes are shown in Tables 1 to 3. By using the following expansions:

$$\sin x = \sum_{j=0}^{\infty} (-1)^j \frac{x^{(2j+1)}}{(2j+1)!} \;; \tag{30}$$

$$\cos x = \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j}}{(2j)!} \;, \tag{31}$$

the real-valued formulation can then be reduced to the MR/BEM. The numerical results for the first two normalized eigenmodes obtained by means of series expansion (MR/BEM) in cases I, II, and III are shown in Figs. 8 to 10. Since the eigenmodes obtained using all the above mentioned formulations are almost the same, only the results

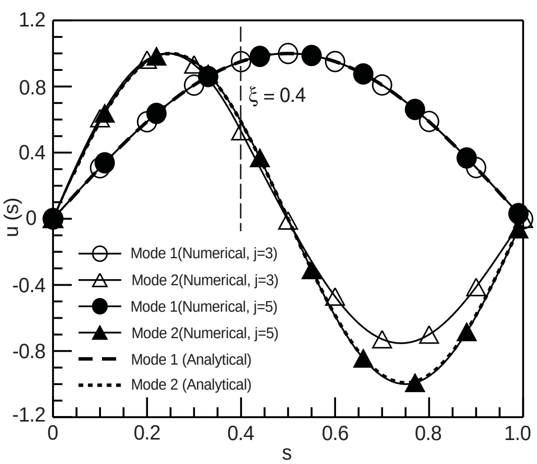


Fig. 8. The first two normalized eigenmodes for case I

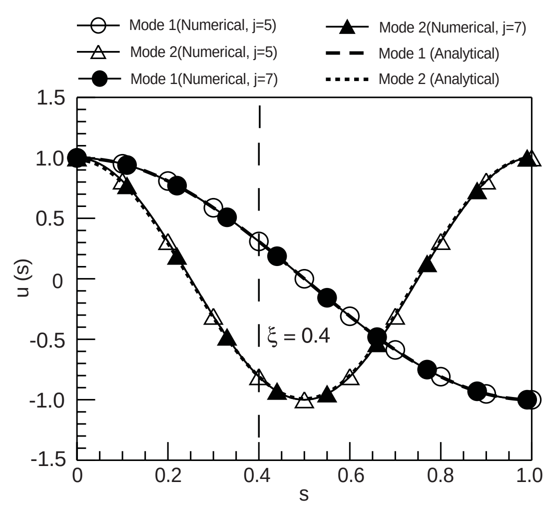


Fig. 9. The first two normalized eigenmodes for case II

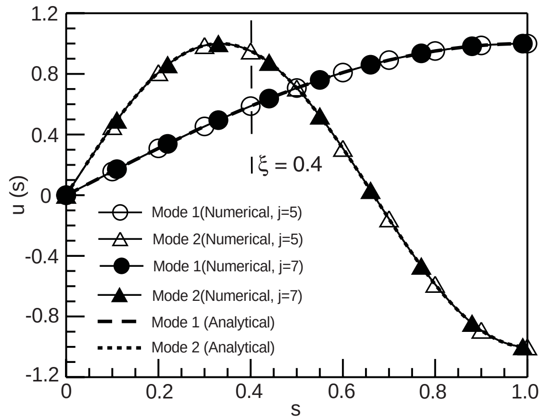


Fig. 10. The first two normalized eigenmodes for case III

obtained using the real-valued singular equation (the real-valued UT equation) in conjunction with the domain partition technique are illustrated. The figures show that numerical result matches the analytical solution well when the number of series terms increases.

4 Conclusions

In this paper, we have combined the dual BEM with the domain partition technique to find the natural frequencies and natural modes of a rod analytically. It has been found that this new approach not only eliminates the possible indeterminacy of eigenmodes in the conventional real-valued formulation, which does not use the domain partition concept, but also filters out spirious eigenvales in the stage of root searching without introducing the hypersingular equation. In addition, the domain partition approach has been used to determine the true eigenvalues and eigenmodes whatever the singular or hypersingular equation is chosen. Three examples with different boundary conditions have been presented to show the validity of the current approach.

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