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Boundary collocation method for acoustic eigenanalysis of three-dimensional cavities using radial basis function

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Abstract In this paper, a theoretical formulation based on the collocation method is presented for the eigenanalysis of arbitrarily shaped acoustic cavities. This article can be seen as the extension of non-dimensional influence function (NDIF) method proposed by Kang et al. (1999, 2000a) extending from two-dimensional to three-dimensional case. Unlike the conventional collocation techniques in the literature, approximate functions used in this paper are two-point functions of which the argument is only the distance between the two points. Based on this radial basis expansion, the acoustic field can be represented more exactly. The field solution is obtained through the linear superposition of radial basis function, and boundary conditions can be applied at the discrete points. The influence matrix is symmetric regardless of the boundary shape of the cavity, and the calculated eigenvalues rapidly converge to the exact values by using only a few boundary nodes. Moreover, the method results in true and spurious boundary modes, which can be obtained from the right and left unitary vectors of singular value decomposition, respectively. By employing the updating term and document of singular value decomposition (SVD), the true and spurious eigensolutions can be sorted out, respectively. The validity of the proposed method are illustrated through several numerical examples.

Keywords Boundary collocation method, Acoustic analysis, Radial basis function, SVD, Imaginary-part kernel

1 Introduction

Mesh generation of a complicated geometry is always time consuming in the stage of model creation for engineers in dealing with engineering problems by using numerical methods, such as the finite difference method (FDM), finite element method (FEM) and boundary element method (BEM). Recently, researchers have paid attention to the meshless method which the element is free. The initial idea of meshless method dates back to the smooth particle hydrodynamics (SPH) method for modeling astrophysical phenomena (Gingold, 1977). Several meshless methods have also been reported in the literature, for examples, the element-free Galerkin method (Belystcho et al., 1994) and the reproducing kernel method (Liu et al., 1995).

On the other hand, some mathematicians presented radial basis functions (RBFs) to improve the original basis function, $1 + r$ of the dual reciprocity method (DRM) (Nardini and Brebbia, 1982; Chen et al., 1999) or $C - r$ of the method of particular integral (Banerjee et al., 1988). In fact, the method of particular integral is mathematically equivalent to the DRM. The proof can be found in the reference (Polyzos et al., 1994). The radial basis functions could be classified to globally-supported RBF and compactly-supported RBF. The main idea of globally-supported RBF is to discretize the whole domain by using the scattering nodes. Some mathematicians presented several globally-supported RBFs such as multiquadrics (MQ), reciprocal multiquadrics (RMQ), Gaussian, and thin-plate splines (TPS) to solve partial differential equations (Zhang et al., 2000). But the system matrices resulted from the globally-supported RBF are full and ill-conditioned such that the calculation is difficult. To overcome the drawbacks of globally-supported RBF, compactly-supported RBF was proposed (Zhang et al., 2000). The main idea of compactly-supported RBF is that the calculation depends on the influence domain only (Chen et al., 1999; Golberg et al., 1999). Zhang et al. (2000) has employed RBF for the meshless method.

For acoustics, the integral equations have been utilized to solve the interior and exterior problems for a long time. Several approaches, e.g., complex-valued boundary element method (Yeih et al., 1998), multiple reciprocity method (MRM) (Chen and Wong, 1997, 1998) and the real-part boundary element method (Liou et al., 1999; Chen et al., 1999; Kuo et al., 2000a), have been developed for acoustic problems. To solve acoustic problems by using the complex-valued BEM, the influence coefficient matrix would be complex arithematics (Kamiya et al., 1996). Therefore, Tai and Shaw (1974) employed only the real-part kernel to solve the eigen problems free of the complex-valued computation. The computation of the

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real-part kernel method, the MRM and DRM have some advantages, but it still faces to the singular and hypersingular integrals. In order to avoid the singular and hypersingular integrals, De Mey (1977) used the imaginary-part kernel to solve the eigen problems. At the same time, De Mey also found the spurious eigensolutions but he did not study them analytically. Acoustic eigenproblems have been of great interest in room acoustics. A detailed review for BEM applications on dynamics can be found (Beskos, 1987, 1997). For simple shape problems, e.g., sphere cavity or rectangular room, exact eigensolutions were well established. For problems with a general geometry, numerical methods, e.g., finite element and boundary element methods, have been employed to determine the acoustic eigen frequencies and acoustic modes. Both BEM and FEM need mesh generation in the preparation of data. Kang et al. (1999, 2000a) have employed the so-called non-dimensional influence function (NDIF) method to solve the vibration problem of 2-D membranes. They also termed it "method of point matching" for 2-D cavity eigenproblems, see Kang et al. (2000b). Kang et al. (2001) also extended the NDIF method to solve plate vibration problems. Since only boundary node is necessary, the approach is meshless. Later, Chen et al. (2001) commented that Kang's method is a special case of the imaginary-part dual BEM with concentrated strength. Also, Chen et al. (1999) and Kuo et al. (2000a) employed the theory of circulants to prove that spurious eigensolution and ill-posed problems may occur in case of circular membrane. For general shape problems, the two drawbacks are also inherent. To overcome the problem of spurious eigensolutions, a net approach was proposed by Kang and Lee (2000b). Another alternative to avoid the occurrence of spurious eigensolution was also proposed by Chen et al. (2001) and Chen et al. (2002) using the double-layer approach. This singularity-free method also results in the ill-posed behavior when the number of degrees of freedom becomes large. Kuo et al. (2000b) in our group employed the generalized singular value decomposition (GSVD) method in conjunction with the Tikhonov regularization to deal with the ill-posed problem for the regular formulation. Until now, research of ill-posed problem is an active area as mentioned by Hutchinson (1991). This method is attractive for engineers due to its meshless properties although some other difficulties (e.g. spurious eigenvalues and ill-posed problems) may occur.

By choosing the imaginary-part kernel of the complex-valued fundamental solution as the radial basis function in this paper, the acoustic field can be represented by linear superposition through a finite number of undetermined coefficients. The similar methods, e.g., the edge function method, the method of fundamental solution, the point of matching method, the point collocation method, the Trefftz method, and the boundary collocation method (BCM), represent the solution in terms of fundamental solution or simple solution satisfying the governing equations. The eigenvalue will be detected by directly searching the minimum singular value for the influence matrix through the singular value decomposition (SVD) technique. By plotting the minimum singular value against the wave number k, the dip in this figure indicates the position where the eigenvalue occurs. The nontrivial solution, i.e., true boundary mode, will be sorted out through the vector in the right unitary matrix of SVD. The relation between spurious boundary modes and the left unitary vector will be examined. The examples of a spherical cavity, subject to the Dirichlet and Neumann boundary conditions, will be demonstrated to see the validity of the proposed method.

2 Meshless formulation using the radial basis function of the imaginary-part kernel

The governing equation for an interior acoustic eigenproblem is the Helmholtz equation as follows:

$$(\nabla^2 + k^2)u(x) = 0, \quad x \in D \ , \tag{1}$$

where ∇^2 is the Laplacian operator, D is the domain of the cavity and k is the wave number which is the angular frequency over the speed of sound. The boundary conditions can be either the Neumann or Dirichlet type.

The radial basis function is expressed by

$$G(x_i, s_j) = \varphi(|s_j - x_i|) \ , \tag{2}$$

where φ is a two-point function, x_i and s_j are the ith collocation and jth source points, respectively, the Euclidean norm $|s_j - x_i|$ is referred to as the radial distance between the collocation and source points. The two-point function $(\varphi(|s_j - x_i|))$ is called a radial basis function since it depends on the radial distance only between x_i and s_j.
By considering the imaginary-part of complex-valued fundamental solution for the Helmholtz equation $(U(s_j, x_i) = \mathrm{Im}(e^{ikr}/r) = \sin(k|s_j - x_i|)/(|s_j - x_i|)$, we can choose the four kernels in the dual formulation (Chen and Hong, 1999),

$$U(s,x) = \frac{\sin(k|s-x|)}{|s-x|} = \frac{\sin(kr)}{r} \ , \tag{3}$$

$$T(s,x) = \frac{\partial U(s,x)}{\partial n_s} = \frac{kr\cos(kr) - \sin(kr)}{r^3} y_i n_i \ , \tag{4}$$

$$L(s,x) = \frac{\partial U(s,x)}{\partial n_x} = -\frac{kr\cos(kr) - \sin(kr)}{r^3} y_j \bar{n}_j \ , \tag{5}$$

$$\begin{aligned} M(s,x) &= \frac{\partial^2 U(s,x)}{\partial n_s \partial n_x} \\ &= \frac{3kr\cos(kr) + (k^2r^2 - 3)\sin(kr)}{r^5} y_i y_j n_i \bar{n}_j \\ &\quad + \frac{kr\cos(kr) - \sin(kr)}{r^3} n_i \bar{n}_i \ , \end{aligned} \tag{6}$$

where r is the distance between the source and collocation points; n_i is the ith component of the outnormal vector at the source point s; $\bar{n}_i$ is the ith component of the outnormal vector at the field point x, and $y_i \equiv s_i - x_i$. The radial basis function $U(s,x)$ in Eq. (3) is globally-supported instead of compactly-supported. For the later use of deriving the SVD updating term or document, the dual relationships for the four kernels are shown below (Chen and Hong, 1999):

$$U(s,x) = U(x,s) \;, \tag{7}$$

$$T(s,x) = L(x,s) \;, \tag{8}$$

$$M(s,x) = M(x,s) \;. \tag{9}$$

Based on the dual formulation and the potential theory (Chen et al., 2000), we can represent the acoustic solution by

Single-layer potential approach

$$u(x_i) = \sum_j U(s_j, x_i)\phi_j \xrightarrow{\text{matrix form}} \{u_i\} = [U_{ij}]\{\phi_j\} \;, \tag{10}$$

$$t(x_i) = \sum_j L(s_j, x_i)\phi_j \xrightarrow{\text{matrix form}} \{t_i\} = [L_{ij}]\{\phi_j\} \;, \tag{11}$$

Double-layer potential approach

$$u(x_i) = \sum_j T(s_j, x_i)\psi_j \xrightarrow{\text{matrix form}} \{u_i\} = [T_{ij}]\{\psi_j\} \;, \tag{12}$$

$$t(x_i) = \sum_j M(s_j, x_i)\psi_j \xrightarrow{\text{matrix form}} \{t_i\} = [M_{ij}]\{\psi_j\} \;, \tag{13}$$

where ϕ_j and ψ_j are the generalized unknowns by using the single-layer potential and double-layer potential approaches, respectively. By adopting the three bases, $j_n(k\rho)$, $j'_n(k\rho)$, and $P_n^m(\cos\theta)$, we can decompose the four kernels into degenerate forms,

$$U(s,x) = \begin{cases} U^I(s,x) = k\sum_{n=0}^{\infty}(2n+1)\sum_{m=0}^{n}\varepsilon_m\dfrac{(n-m)!}{(n+m)!} \\ \quad \cos[m(\phi-\bar{\phi})]\,P_n^m(\cos\theta)\,P_n^m(\cos\bar{\theta}) \\ \quad j_n(k\bar{\rho})j_n(k\rho), \quad \bar{\rho} > \rho \\ U^E(s,x) = k\sum_{n=0}^{\infty}(2n+1)\sum_{m=0}^{n}\varepsilon_m\dfrac{(n-m)!}{(n+m)!} \\ \quad \cos[m(\phi-\bar{\phi})]\,P_n^m(\cos\theta)\,P_n^m(\cos\bar{\theta}) \\ \quad j_n(k\rho)j_n(k\bar{\rho}), \quad \bar{\rho} < \rho \end{cases} \tag{14}$$

$$T(s,x) = \begin{cases} T^I(s,x) = k^2\sum_{n=0}^{\infty}(2n+1)\sum_{m=0}^{n}\varepsilon_m\dfrac{(n-m)!}{(n+m)!} \\ \quad \cos[m(\phi-\bar{\phi})]\,P_n^m(\cos\theta)\,P_n^m(\cos\bar{\theta}) \\ \quad j'_n(k\bar{\rho})j_n(k\rho), \quad \bar{\rho} > \rho \\ T^E(s,x) = k^2\sum_{n=0}^{\infty}(2n+1)\sum_{m=0}^{n}\varepsilon_m\dfrac{(n-m)!}{(n+m)!} \\ \quad \cos[m(\phi-\bar{\phi})]\,P_n^m(\cos\;\theta)\,P_n^m(\cos\bar{\theta}) \\ \quad j_n(k\rho)j'_n(k\bar{\rho}), \quad \bar{\rho} < \rho \end{cases} \tag{15}$$

$$L(s,x) = \begin{cases} L^I(s,x) = k^2\sum_{n=0}^{\infty}(2n+1)\sum_{m=0}^{n}\varepsilon_m\dfrac{(n-m)!}{(n+m)!} \\ \quad \cos[m(\phi-\bar{\phi})]\,P_n^m(\cos\;\theta)\,P_n^m(\cos\bar{\theta}) \\ \quad j_n(k\bar{\rho})j'_n(k\rho), \quad \bar{\rho} > \rho \\ L^E(s,x) = k^2\sum_{n=0}^{\infty}(2n+1)\sum_{m=0}^{n}\varepsilon_m\dfrac{(n-m)!}{(n+m)!} \\ \quad \cos[m(\phi-\bar{\phi})]\,P_n^m(\cos\;\theta)\,P_n^m(\cos\bar{\theta}) \\ \quad j'_n(k\rho)j_n(k\bar{\rho}), \quad \bar{\rho} < \rho \end{cases} \tag{16}$$

$$M(s,x) = \begin{cases} M^I(s,x) = k^3\sum_{n=0}^{\infty}(2n+1)\sum_{m=0}^{n}\varepsilon_m\dfrac{(n-m)!}{(n+m)!} \\ \quad \cos[m(\phi-\bar{\phi})]\,P_n^m(\cos\;\theta)\,P_n^m(\cos\bar{\theta}) \\ \quad j'_n(k\bar{\rho})j'_n(k\rho), \quad \bar{\rho} > \rho \\ M^E(s,x) = k^3\sum_{n=0}^{\infty}(2n+1)\sum_{m=0}^{n}\varepsilon_m\dfrac{(n-m)!}{(n+m)!} \\ \quad \cos[m(\phi-\bar{\phi})]\,P_n^m(\cos\;\theta)\,P_n^m(\cos\bar{\theta}) \\ \quad j'_n(k\rho)j'_n(k\bar{\rho}), \quad \bar{\rho} < \rho \end{cases} \tag{17}$$

where the superscripts "I" and "E" denote the interior and exterior kernels; j_n and j'_n are the nth order spherical Bessel functions of the first kind and its derivative, respectively; (ρ, ϕ, θ) and $(\bar{\rho}, \bar{\phi}, \bar{\theta})$ are the spherical coordinates for x and s, respectively, as shown in Fig. 1; P_n^m is the associated Legendre polynomial function; ε_m is the Neumann factor ($\varepsilon_m = 1$, when $m = 0$ or $\varepsilon_m = 2$, when $m > 0$). By superimposing $2N$ concentrated strength along the boundary, we have the four influence matrices,

$$[U_{ij}] = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,2N-1} & a_{1,2N} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,2N-1} & a_{2,2N} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,2N-1} & a_{3,2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{2N,1} & a_{2N,2} & a_{2N,3} & \cdots & a_{2N,2N-1} & a_{2N,2N} \end{bmatrix} , \tag{18}$$

$$[T_{ij}] = \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} & \cdots & b_{1,2N-1} & b_{1,2N} \\ b_{2,1} & b_{2,2} & b_{2,3} & \cdots & b_{2,2N-1} & b_{2,2N} \\ b_{3,1} & b_{3,2} & b_{3,3} & \cdots & b_{3,2N-1} & b_{3,2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{2N,1} & b_{2N,2} & b_{2N,3} & \cdots & b_{2N,2N-1} & b_{2N,2N} \end{bmatrix} , \tag{19}$$

$$[L_{ij}] = \begin{bmatrix} c_{1,1} & c_{1,2} & c_{1,3} & \cdots & c_{1,2N-1} & c_{1,2N} \\ c_{2,1} & c_{2,2} & c_{2,3} & \cdots & c_{2,2N-1} & c_{2,2N} \\ c_{3,1} & c_{3,2} & c_{3,3} & \cdots & c_{3,2N-1} & c_{3,2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{2N,1} & c_{2N,2} & c_{2N,3} & \cdots & c_{2N,2N-1} & c_{2N,2N} \end{bmatrix} , \tag{20}$$

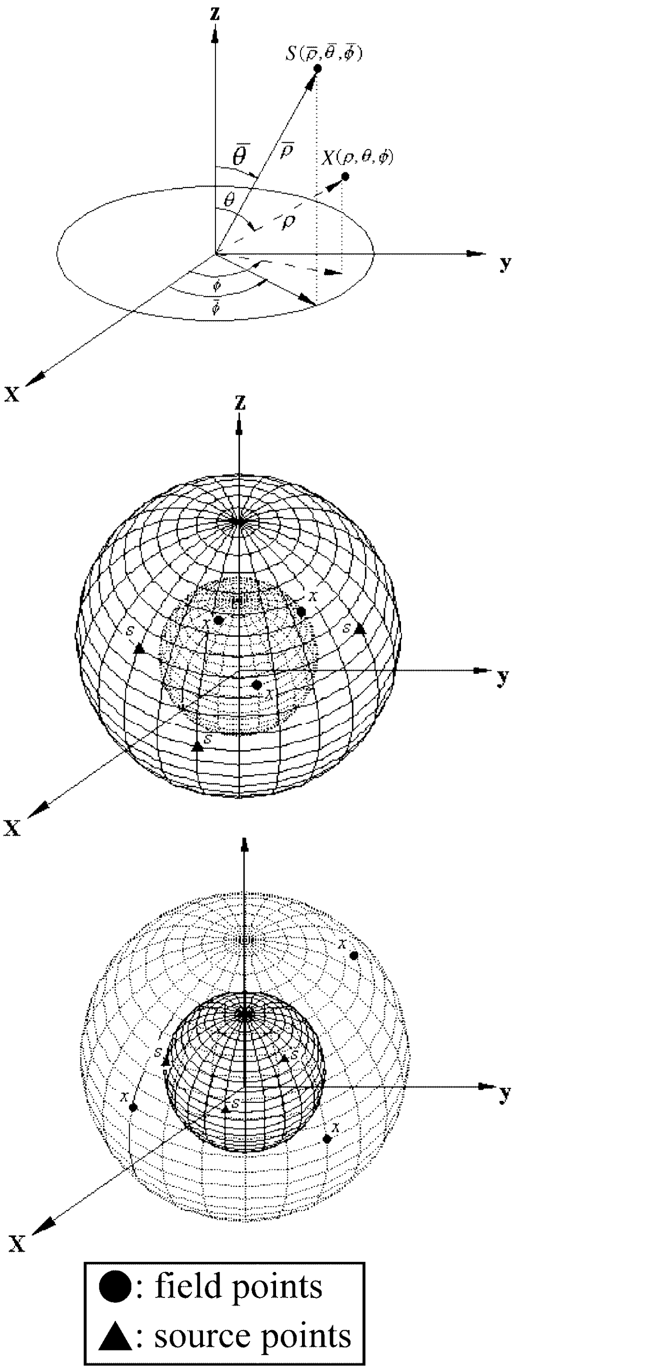

Fig. 1. The definitions of the spherical coordinate and source and collocation points

$$[M_{ij}] = \begin{bmatrix} d_{1,1} & d_{1,2} & d_{1,3} & \cdots & d_{1,2N-1} & d_{1,2N} \\ d_{2,1} & d_{2,2} & d_{2,3} & \cdots & d_{2,2N-1} & d_{2,2N} \\ d_{3,1} & d_{3,2} & d_{3,3} & \cdots & d_{3,2N-1} & d_{3,2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ d_{2N,1} & d_{2N,2} & d_{2N,3} & \cdots & d_{2N,2N-1} & d_{2N,2N} \end{bmatrix} , \tag{21}$$

where the elements can be determined by

$$a_{i,j} = U(s_j, x_i) \ , \tag{22}$$

$$b_{i,j} = T(s_j, x_i) \ , \tag{23}$$

$$c_{i,j} = L(s_j, x_i) \ , \tag{24}$$

$$d_{i,j} = M(s_j, x_i) \ . \tag{25}$$

Then, we can solve ϕ_j and ψ_j by matching the boundary conditions on the collocation points. According to Eqs. (14)–(17), we find that only half effort is required in constructing the four real-valued influence matrices in comparison with the complex-valued BEM.

The diagonal elements in the influence matrices can be solved by using the L'Hôpital's rule since the radial distance is zero ($r = 0$ when $i = j$). Considering the asymptotic behavior for *sine* and *cosine* functions, we can obtain the diagonal elements as follows:

$$\lim_{s\to x} U(s,x) = \lim_{r\to 0} \frac{\sin(kr)}{r} = k \ , \tag{26}$$

$$\lim_{s\to x} T(s,x) = \lim_{r\to 0} \frac{kr\cos(kr) - \sin(kr)}{r^3} y_i n_i = 0 \ , \tag{27}$$

$$\lim_{s\to x} L(s,x) = \lim_{r\to 0} -\frac{kr\cos(kr) - sin(kr)}{r^3} y_j \bar{n}_j = 0 \ , \tag{28}$$

$$\lim_{s\to x} M(s,x) = \lim_{r\to 0} \frac{3kr\cos(kr) + (k^2r^2 - 3)\sin(kr)}{r^5} y_i y_j n_i \bar{n}_j + \lim_{r\to 0} \frac{kr\cos(kr) - \sin(kr)}{r^3} n_i \bar{n}_i = \frac{k^3}{3} \ . \tag{29}$$

3 Relations of kernels and influence matrices in the direct and indirect BEMs

In solving the boundary value problems, the direct method and indirect method can be employed.

The direct method is derived by using the Green's identity in terms of the unknowns which are the actual physical quantities on the boundary. In the direct method, we have the null-field dual equations,

$$0 = \int_B T^E(s,x) u(s) \, dB(s) - \int_B U^E(s,x) t(s) \, dB(s) \ , \tag{30}$$

$$0 = \int_B M^E(s,x) u(s) \, dB(s) - \int_B L^E(s,x) t(s) \, dB(s) \ , \tag{31}$$

where x is outside the domain, $u(s)$ and $t(s)$ are the potential and its normal derivative on the boundary B. By discretizing Eqs. (30) and (31), we have the discrete systems for the direct method,

$$\left[T^E_{ij}\right]\{u_j\} = \left[U^E_{ij}\right]\{t_j\} \ , \tag{32}$$

$$\left[M_{ij}^E\right]\{u_j\} = \left[L_{ij}^E\right]\{t_j\} \ , \tag{33}$$

where $\{u_j\}$ and $\{t_j\}$ are the boundary potential and its normal derivative.

A key point of the indirect method is assuming a solution which can satisfy the governing equation and boundary conditions. By considering the superposition principle for the potential, we have

Single-layer potential approach

$$u(x) = \int_B U^I(s,x)\phi(s)\,\mathrm{d}B(s) \ ,$$

$$t(x) = \int_B L^I(s,x)\phi(s)\,\mathrm{d}B(s) \ . \tag{34}$$

Double-layer potential approach

$$\begin{aligned} u(x) &= \int_B T^I(s,x)\psi(s)\,\mathrm{d}B(s), \\ t(x) &= \int_B M^I(s,x)\psi(s)\,\mathrm{d}B(s) \ , \end{aligned} \tag{35}$$

where $\phi(s)$ and $\psi(s)$ are the single-layer and double-layer densities. By discretizing Eq. (34) and Eq. (35), we have the discrete system for the indirect method,

Single-layer potential approach

$$\{u(x_j)\} = \left[U_{ij}^I\right]\{\phi_i\}, \quad \{t(x_j)\} = \left[L_{ij}^I\right]\{\phi_i\} \ . \tag{36}$$

Double-layer potential approach

$$\{u(x_j)\} = \left[T_{ij}^I\right]\{\psi_i\}, \quad \{t(x_j)\} = \left[M_{ij}^I\right]\{\psi_i\} \ , \tag{37}$$

where $\{\phi_j\}$ and $\{\psi_j\}$ are the column vectors of single and double-layer boundary densities, respectively. By considering the degenerate kernels of Eqs. (14)–(17) and comparing Eqs. (32) and (33) with Eqs. (36) and (37), we have the dual relations between the interior and exterior kernels, i.e.,

$$U^E(s,x) = U^I(x,s) \quad \text{or} \quad [U_{ij}^E] = [U_{ij}^I] \ , \tag{38}$$

$$T^E(s,x) = L^I(x,s) \quad \text{or} \quad [T_{ij}^E] = [L_{ij}^I] \ , \tag{39}$$

$$L^E(s,x) = T^I(x,s) \quad \text{or} \quad [L_{ij}^E] = [T_{ij}^I] \ , \tag{40}$$

$$M^E(s,x) = M^I(x,s) \quad \text{or} \quad [M_{ij}^E] = [M_{ij}^I] \ . \tag{41}$$

4 The derivation of true and spurious eigensolutions for the spherical cavity using the degenerate kernels and Fourier series

For the spherical cavity, we have the degenerate kernel functions in Eqs. (14)–(17). By considering the spherical harmonics for boundary density functions, we can assume the single and double-layer density functions on the spherical boundary as

$$\phi(s) = \sum_{v=0}^{\infty}\sum_{w=0}^{v} A_{vw}\,P_v^w(\cos\bar{\theta})\cos(w\bar{\phi}) \ , \tag{42}$$

$$\psi(s) = \sum_{v=0}^{\infty}\sum_{w=0}^{v} B_{vw}\,P_v^w(\cos\bar{\theta})\cos(w\bar{\phi}) \ , \tag{43}$$

where A_{vw} and B_{vw} are the unknown coefficients. By substituting Eq. (42) into Eq. (34) for the Dirichlet problem, we have

$$\begin{aligned} 0 &= \int_B U^I(s,x)\phi(s)\,\mathrm{d}B(s) \\ &= \int_0^{2\pi}\int_0^{\pi} k\sum_{n=0}^{\infty}(2n+1) \\ &\quad \sum_{m=0}^{n}\varepsilon_m\frac{(n-m)!}{(n+m)!}\cos[m(\phi-\bar{\phi})]\,P_n^m(\cos\theta) \\ &\quad P_n^m(\cos\ \bar{\theta})j_n(k\bar{\rho})j_n(k\rho) \\ &\quad \sum_{v=0}^{\infty}\sum_{w=0}^{v} A_{vw}P_v^w(\cos\bar{\theta})\cos(w\bar{\phi})\bar{\rho}^2\ \sin(\bar{\theta})\,\mathrm{d}\bar{\theta}\,\mathrm{d}\bar{\phi} \ , \end{aligned} \tag{44}$$

where $\bar{\rho}$ is the radius of the sphere in Eq. (44). By considering the orthogonality of the associated Legendre polynomial function, we have

$$\begin{aligned} &\int_0^{\pi} P_n^m(\cos\bar{\theta})P_v^w(\cos\bar{\theta})\sin(\bar{\theta})\mathrm{d}\bar{\theta} \\ &= \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}\delta_{nv}\delta_{mw} \ , \end{aligned} \tag{45}$$

$$\int_0^{2\pi}\cos[m(\phi-\bar{\phi})]\cos(w\bar{\phi})\mathrm{d}\bar{\phi} = \frac{2\pi}{\varepsilon_w}\delta_{mw}\cos(m\phi), \tag{46}$$

where δ is the Kronecker delta symbol. According to Eqs. (45) and (46), Eq. (44) is simplified to

$$\begin{aligned} 0 &= k\int_0^{\pi}\sum_{n=0}^{\infty}\sum_{v=0}^{\infty}\varepsilon_m(2n+1)j_n(k\bar{\rho})j_n(k\rho)\frac{(n-m)!}{(n+m)!} \\ &\quad P_n^m(\cos\bar{\theta})P_v^w(\cos\ \bar{\theta})\sin(\bar{\theta})\bar{\rho}^2\,\mathrm{d}\bar{\theta}\,P_n^m(\cos\theta) \\ &\quad \int_0^{2\pi}\sum_{m=0}^{n}\sum_{w=0}^{v} A_{vw}\cos[m(\phi-\bar{\phi})]\cos(w\bar{\phi})\,\mathrm{d}\bar{\phi} \\ &= 4\pi k\sum_{n=0}^{\infty}\sum_{m=0}^{n} A_{nm}\,j_n(k\bar{\rho})j_n(k\rho) \\ &\quad P_n^m(\cos\theta)\cos(m\phi)\bar{\rho}^2 \ . \end{aligned} \tag{47}$$

Thus, we obtain the eigenequation, $j_n(k\rho)j_n(k\bar{\rho}) = 0$, by using the single-layer potential method for the interior Dirichlet problem.

Similarly, we obtain

$$0=\int_B T^I(s,x)\psi(s)\,\mathrm{d}B(s)$$

$$=k^2\int_0^{2\pi}\int_0^{\pi}\sum_{n=0}^{\infty}(2n+1)\sum_{m=0}^{n}\varepsilon_m\frac{(n-m)!}{(n+m)!}\cos[m(\phi-\bar{\phi})]$$

$$P_n^m(\cos\,\theta)\,P_n^m(\cos\,\bar{\theta})j_n(k\bar{\rho})j'_n(k\rho)$$

$$\sum_{v=0}^{\infty}\sum_{w=0}^{v}B_{vw}P_v^w(\cos\,\bar{\theta})\cos(w\bar{\phi})\bar{\rho}^2\sin\,(\bar{\theta})\,\mathrm{d}\bar{\theta}\,\mathrm{d}\bar{\phi}\quad, \tag{48}$$

by using the double-layer potential approach. Considering the orthogonality of the associated Legendre polynomial function, Eq. (48) reduces to

$$0=k^2\int_0^{\pi}\sum_{n=0}^{\infty}\sum_{v=0}^{\infty}\varepsilon_m(2n+1)j_n(k\bar{\rho})j'_n(k\rho)\frac{(n-m)!}{(n+m)!}$$

$$P_n^m(\cos\bar{\theta})P_v^w(\cos\bar{\theta})\bar{\rho}^2\sin(\bar{\theta})\,\mathrm{d}\bar{\theta}\,P_n^m(\cos\theta)$$

$$\int_0^{2\pi}\sum_{m=0}^{n}\sum_{w=0}^{v}B_{vw}\cos[m(\phi-\bar{\phi})]\cos(w\bar{\phi})\,\mathrm{d}\bar{\phi}$$

$$=4\pi k^2\sum_{n=0}^{\infty}\sum_{m=0}^{n}B_{nm}\,j_n(k\bar{\rho})j'_n(k\rho)\,P_n^m(\cos\theta)\cos(m\phi)\bar{\rho}^2\;. \tag{49}$$

Thus, we obtain the eigenequation, $j'_n(k\rho)j_n(k\bar{\rho})=0$, by using the double-layer potential method for the interior Dirichlet problem.

For the Neumann problem, we have

$$0=\int_B L^I(s,x)\phi(s)\,\mathrm{d}B(s)$$

$$=\int_0^{2\pi}\int_0^{\pi}k^2\sum_{n=0}^{\infty}(2n+1)\sum_{m=0}^{n}\varepsilon_m\frac{(n-m)!}{(n+m)!}\cos[m(\phi-\bar{\phi})]$$

$$P_n^m(\cos\,\theta)\,P_n^m(\cos\,\bar{\theta})j'_n(k\bar{\rho})j_n(k\rho)$$

$$\sum_{v=0}^{\infty}\sum_{w=0}^{v}A_{vw}\,P_v^w(\cos\,\bar{\theta})\cos(w\bar{\phi})\sin(\bar{\theta})\bar{\rho}^2\,\mathrm{d}\bar{\theta}\,\mathrm{d}\bar{\phi}\quad, \tag{50}$$

by substituting Eqs. (16) and (42) into Eq. (35). Considering the orthogonality of the associated Legendre polynomial function Eqs. (45) and (46), Eq. (50) is simplified to

$$0=k^2\int_0^{\pi}\sum_{n=0}^{\infty}\sum_{v=0}^{\infty}\varepsilon_m(2n+1)j'_n(k\bar{\rho})j_n(k\rho)\frac{(n-m)!}{(n+m)!}$$

$$P_n^m(\cos\,\bar{\theta})P_v^w(\cos\,\bar{\theta})\bar{\rho}^2\sin(\bar{\theta})\,\mathrm{d}\bar{\theta}P_n^m(\cos\,\theta)$$

$$\int_0^{2\pi}\sum_{m=0}^{n}\sum_{w=0}^{v}A_{vw}\cos[m(\phi-\bar{\phi})]\cos(w\bar{\phi})\mathrm{d}\bar{\phi}$$

$$=4\pi k^2\sum_{n=0}^{\infty}\sum_{m=0}^{n}A_{nm}\,j'_n(k\bar{\rho})\;\;j_n(k\rho)$$

$$P_n^m(\cos\,\theta)\cos(m\phi)\bar{\rho}^2\quad. \tag{51}$$

Therefore, we obtain the eigenequation, $j'_n(k\bar{\rho})j_n(k\rho)=0$, by using the single-layer potential method for the interior Neumann problem.

Similarly, we use the double-layer potential method to obtain

$$0=\int_B M^I(s,x)\psi(s)\,\mathrm{d}B(s)$$

$$=\int_0^{2\pi}\int_0^{\pi}k^3\sum_{n=0}^{\infty}(2n+1)\sum_{m=0}^{n}\varepsilon_m\frac{(n-m)!}{(n+m)!}\cos[m(\phi-\bar{\phi})]$$

$$P_n^m(\cos\,\theta)\,P_n^m(\cos\,\bar{\theta})j'_n(k\bar{\rho})j'_n(k\rho)$$

$$\sum_{v=0}^{\infty}\sum_{w=0}^{v}B_{vw}P_v^w(\cos\,\bar{\theta})\cos(w\bar{\phi})\bar{\rho}^2\;\sin(\bar{\theta})\mathrm{d}\bar{\theta}\,\mathrm{d}\bar{\phi}\quad. \tag{52}$$

By considering the orthogonality of the associated Legendre polynomial function, Eq. (52) reduces to

$$0=k^3\int_0^{\pi}\sum_{n=0}^{\infty}\sum_{v=0}^{\infty}\varepsilon_m(2n+1)j'_n(k\bar{\rho})j'_n(k\rho)\frac{(n-m)!}{(n+m)!}$$

$$P_n^m(\cos\,\bar{\theta})P_v^w(\cos\,\bar{\theta})\bar{\rho}^2\sin(\bar{\theta})\,\mathrm{d}\bar{\theta}\,P_n^m(\cos\,\theta)$$

$$\int_0^{2\pi}\sum_{m=0}^{n}\sum_{w=0}^{v}B_{vw}\cos[m(\phi-\bar{\phi})]\cos(w\bar{\phi})\,\mathrm{d}\bar{\phi}$$

$$=4\pi k^3\sum_{n=0}^{\infty}\sum_{m=0}^{n}B_{nm}\,j'_n(k\bar{\rho})j'_n(k\rho)$$

$$P_n^m(\cos\theta)\cos(m\phi)\bar{\rho}^2\quad. \tag{53}$$

Therefore, we obtain the eigenequation, $j'_n(k\rho)j'_n(k\bar{\rho})=0$, by using the double-layer potential method for the interior Neumann problem. All the relations of true and spurious eigenvalues by using the single and double-layer potential approaches are shown in Table 1 where $\bar{\rho}=\rho$. It is found that the intersection set of the solutions using different methods is a true solution. Otherwise, it is a spurious solution for different problems once the numerical method is chosen.

Table 1. SVD updating technique to filter out the true and spurious eigensolutions of a spherical cavity

Boundary value problem	Eigensolutions	Density function		True and spurious eigenvalues	
		Single layer potential	Double layer potential	Direct method	Indirect method
Dirichlet problem	True eigensolution	$j_m(k\rho)=0$	$j_m(k\rho)=0$	SVD updating term $\begin{bmatrix} U^E \\ L^E \end{bmatrix}$	SVD updating term $\begin{bmatrix} U^I \\ T^I \end{bmatrix}$
	Spurious eigensolution	$j_m(k\rho)=0$	$j'_m(k\rho)=0$	SVD updating document $\begin{bmatrix} L^E M^E \end{bmatrix}$	SVD updating document $\begin{bmatrix} T^I M^I \end{bmatrix}$
Neumann problem	True eigensolution	$j'_m(k\rho)=0$	$j'_m(k\rho)=0$	SVD Updating term $\begin{bmatrix} T^E \\ M^E \end{bmatrix}$	SVD updating term $\begin{bmatrix} L^I \\ M^I \end{bmatrix}$
	Spurious eigensolution	$j_m(k\rho)=0$	$j'_m(k\rho)=0$	SVD updating document $\begin{bmatrix} U^E T^E \end{bmatrix}$	SVD updating document $\begin{bmatrix} U^I L^I \end{bmatrix}$

$U^I = U^E, L^I = T^E,\ T^I = L^E$ and $M^I = M^E$

5 Method to extract the true eigensolutions

The SVD technique is an important tool in linear algebra. The matrix A with dimension M by N can be decomposed into a product of the orthogonal matrix Φ (M by M), the diagonal matrix Σ (M by N) with positive or zero elements, and the orthogonal matrix Ψ (N by N),

$$[A]_{M\times N} = [\Phi]_{M\times M}[\Sigma]_{M\times N}[\Psi]^{\mathrm{T}}_{N\times N} \quad , \tag{54}$$

where the superscript "T" is the transpose, Φ and Ψ are both orthogonal in the sense that their column vectors are orthogonal,

$$\phi_i \cdot \phi_j = \delta_{ij} \quad , \tag{55}$$

$$\psi_i \cdot \psi_j = \delta_{ij} \quad , \tag{56}$$

in which $\Phi^{\mathrm{T}}\Phi = \Psi^{\mathrm{T}}\Psi = I$. For the eigenproblem, we can obtain a nontrivial solution for the homogeneous system from a column vector $\{\psi_i\}$ of Ψ when the singular value (σ_i) is zero. Equations (32) and (33) of the direct method can be represented by

Singular equation (UT method)

$$\left[T^E\right]\{u\} = \left[U^E\right]\{t\} = \{0\} \quad , \tag{57}$$

Hypersingular equation (LM method)

$$\left[M^E\right]\{u\} = \left[L^E\right]\{t\} = \{0\} \quad , \tag{58}$$

for the Dirichlet problem. Equations (57) and (58) can be combined to

$$\begin{bmatrix} [U^E] \\ [L^E] \end{bmatrix}\{t\} = \{0\} \quad . \tag{59}$$

By using the SVD technique, the two submatrices in Eq. (59) can be decomposed into

$$\begin{aligned} \left[U^E\right] &= \left[\Phi^{(U)}\right]\left[\Sigma^{(U)}\right]\left[\Psi^{(U)}\right]^{\mathrm{T}} \text{ or} \\ \{U^E\} &= \sum_j \sigma_j^{(U)}\{\phi_j^{(U)}\}\{\psi_j^{(U)}\}^{\mathrm{T}}, \\ \left[L^E\right] &= \left[\Phi^{(L)}\right]\left[\Sigma^{(L)}\right]\left[\Psi^{(L)}\right]^{\mathrm{T}} \text{ or} \\ \{L^E\} &= \sum_j \sigma_j^{(L)}\{\phi_j^{(L)}\}\{\psi_j^{(L)}\}^{\mathrm{T}} \quad , \end{aligned} \tag{60}$$

where the superscripts, (U) and (L), denote the corresponding matrices. For the linear algebraic system, $\{t\}$ is a column vector $\{\psi_i\}$ in the $[\Psi]$ matrix corresponding to the zero singular value ($\sigma_i = 0$). By setting $\{t\}$ as a $\{\psi_i\}$ vector in the right unitary matrix $[\Psi]$ for the true eigenvalue k_t, Eq. (59) reduces to

$$\begin{aligned} \left[U^E(k_t)\right]\{\psi_i\} &= \{0\} \quad , \\ \left[L^E(k_t)\right]\{\psi_i\} &= \{0\} \quad . \end{aligned} \tag{61}$$

Table 2. Zeros of the spherical Bessel functions, $j_n(k)$ $j_n(x) = \sqrt{\frac{\pi}{2x}}J_{n+\frac{1}{2}}(x)$

Problem	Eigenvalues	1	2	3	4	5
Dirichlet problem	$j_0(k)$	3.1416	6.2832	9.4248	12.5664	15.708
	$j_1(k)$	4.4934	7.7253	10.9041	14.0662	17.2207
	$j_2(k)$	5.7635	9.0950	12.3229	15.5146	18.689
	$j_3(k)$	6.9879	10.4171	13.698	16.9236	20.1218
	$j_4(k)$	8.1826	11.7049	15.0397	18.3013	21.5254
	$j_5(k)$	9.3558	12.9665	16.3547	19.6531	22.9045

Substituting Eq. (60) into Eq. (61), we have

$$\begin{aligned}&\sum_j \sigma_j^{(U)}\{\phi_j^{(U)}\}\{\psi_j^{(U)}\}^{\mathrm{T}}\{\psi_i\} = \{0\}\\ &\xrightarrow{\psi_j^{(U)}\cdot\psi_i=\delta_{ij}} \sigma_i^{(U)}\{\phi_i^{(U)}\} = \{0\},\ (i \text{ no sum})\\ &\sum_j \sigma_j^{(L)}\{\phi_j^{(L)}\}\{\psi_j^{(L)}\}^{\mathrm{T}}\{\psi_i\} = \{0\}\\ &\xrightarrow{\psi_j^{(L)}\cdot\psi_i=\delta_{ij}} \sigma_i^{(L)}\{\phi_i^{(L)}\} = \{0\},\ (i \text{ no sum})\end{aligned} \tag{62}$$

where $\{\phi_i\}$ and $\{\psi_i\}$ are the orthonormal bases, $\sigma_j^{(U)}$ and $\sigma_j^{(L)}$ are the singular values of $[U^E]$ and $[L^E]$ matrices, respectively. We can easily extract out the true eigensolutions ($\sigma_i^{(U)} = \sigma_i^{(L)} = 0$) since there exists the same eigensolution ($\{t\} = \{\psi_i\}$) for the Dirichlet problem using Eqs. (59) and (62) together. In a similar way, Eqs. (57) and (58) can be combined to

$$\begin{bmatrix}[T^E(k_t)]\\ [M^E(k_t)]\end{bmatrix}\{u\} = \{0\}\ , \tag{63}$$

for the Neumann problem. We can easily extract the true eigensolutions,

$$\begin{aligned}&\sum_j \sigma_j^{(T)}\{\phi_j^{(T)}\}\{\psi_j^{(T)}\}^{\mathrm{T}}\{\psi_i\} = \{0\}\\ &\xrightarrow{\psi_j^{(T)}\cdot\psi_i=\delta_{ij}} \sigma_i^{(T)}\{\phi_i^{(T)}\} = \{0\},\ (i \text{ no sum})\\ &\sum_j \sigma_j^{(M)}\{\phi_j^{(M)}\}\{\psi_j^{(M)}\}^{\mathrm{T}}\{\psi_i\} = \{0\}\\ &\xrightarrow{\psi_j^{(M)}\cdot\psi_j=\delta_{ij}} \sigma_i^{(M)}\{\phi_i^{(M)}\} = \{0\},\ (i \text{ no sum})\end{aligned} \tag{64}$$

where the superscripts, (T) and (M), denote the corresponding matrices. Since there exists the same eigensolution ($\{u\} = \{\psi_i\}$) corresponding to the zero singular values ($\sigma_i^{(T)} = \sigma_i^{(M)} = 0$) by using Eqs. (63) and (64).

According to the relations of the direct method and indirect method (Eqs. (38)–(41)), we can extend to extract out the true eigensolutions in the indirect method. By employing the indirect method to derive the influence matrices for the Dirichlet problem, Eq. (61) transforms to

$$\begin{bmatrix}[U^E(k_t)]\\ [L^E(k_t)]\end{bmatrix}\{t\} = \{0\} \xrightarrow[L^E=T^I]{U^E=U^I} \begin{bmatrix}[U^I(k_t)]\\ [T^I(k_t)]\end{bmatrix}\{t\} = \{0\}\ , \tag{65}$$

after using Eqs. (38) and (40). By using the SVD technique and Eq. (65), we have

$$\begin{aligned}&\sum_j \sigma_j^{(U)}\{\phi_j^{(U)}\}\{\psi_j^{(U)}\}^{\mathrm{T}}\{\psi_i\} = \{0\}\\ &\xrightarrow{\psi_j^{(U)}\cdot\psi_i=\delta_{ij}} \sigma_i^{(U)}\{\phi_i^{(U)}\} = \{0\},\ (i \text{ no sum})\\ &\sum_j \sigma_j^{(T)}\{\phi_j^{(T)}\}\{\psi_j^{(T)}\}^{\mathrm{T}}\{\psi_i\} = \{0\}\\ &\xrightarrow{\psi_j^{(T)}\cdot\psi_i=\delta_{ij}} \sigma_i^{(T)}\{\phi_i^{(T)}\} = \{0\},\ (i \text{ no sum})\end{aligned} \tag{66}$$

where $\{\psi_i\}$ is the orthonormal basis; $\sigma_j^{(U)}$ and $\sigma_j^{(T)}$ are the singular values of $[U^I]$ and $[T^I]$ matrices, respectively. We can easily extract out the true eigensolutions for the Dirichlet problem by using Eq. (66) with respect to the zero singular values of $\sigma_i^{(U)} = \sigma_i^{(T)} = 0$.

In a similar way, Eq. (63) transforms to

$$\begin{bmatrix}[T^E(k_t)]\\ [M^E(k_t)]\end{bmatrix}\{u\} = \{0\} \xrightarrow[M^E=M^I]{T^E=L^I} \begin{bmatrix}[L^I(k_t)]\\ [M^I(k_t)]\end{bmatrix}\{u\} = \{0\}\ , \tag{67}$$

after using Eqs. (39) and (41). By employing the SVD technique and Eq. (67), we have

$$\begin{aligned}&\sum_j \sigma_j^{(L)}\{\phi_j^{(L)}\}\{\psi_j^{(L)}\}^{\mathrm{T}}\{\psi_i\} = \{0\}\\ &\xrightarrow{\psi_j^{(L)}\cdot\psi_i=\delta_{ij}} \sigma_i^{(L)}\{\phi_i^{(L)}\} = \{0\},\ (i \text{ no sum})\\ &\sum_j \sigma_j^{(M)}\{\phi_j^{(M)}\}\{\psi_j^{(M)}\}^{\mathrm{T}}\{\psi_i\} = \{0\}\\ &\xrightarrow{\psi_j^{(M)}\cdot\psi_i=\delta_{ij}} \sigma_i^{(M)}\{\phi_i^{(M)}\} = \{0\},\ (i \text{ no sum})\end{aligned} \tag{68}$$

where $\{\psi_i\}$ is the orthonormal basis; $\sigma_j^{(L)}$ and $\sigma_j^{(M)}$ are the singular values of $[L^I]$ and $[M^I]$ matrices, respectively. We can easily extract out the true eigensolutions for the Neumann problem by using Eq.(68) with respect to the zero singular values of $\sigma_i^{(L)} = \sigma_i^{(M)} = 0$.

6 Method to filter out the spurious eigensolutions

Fredholm's alternative theorem

(1) Nonsingular system

The equation, $[H]\{g\} = \{p\}$, has a unique solution $\{g\} = [H]^{-1}\{p\}$ with $\det|H| \neq 0$. Besides, the equation has a solution $\{g\} = \{0\}$ if and only if the only continuous solution to the homogeneous equation

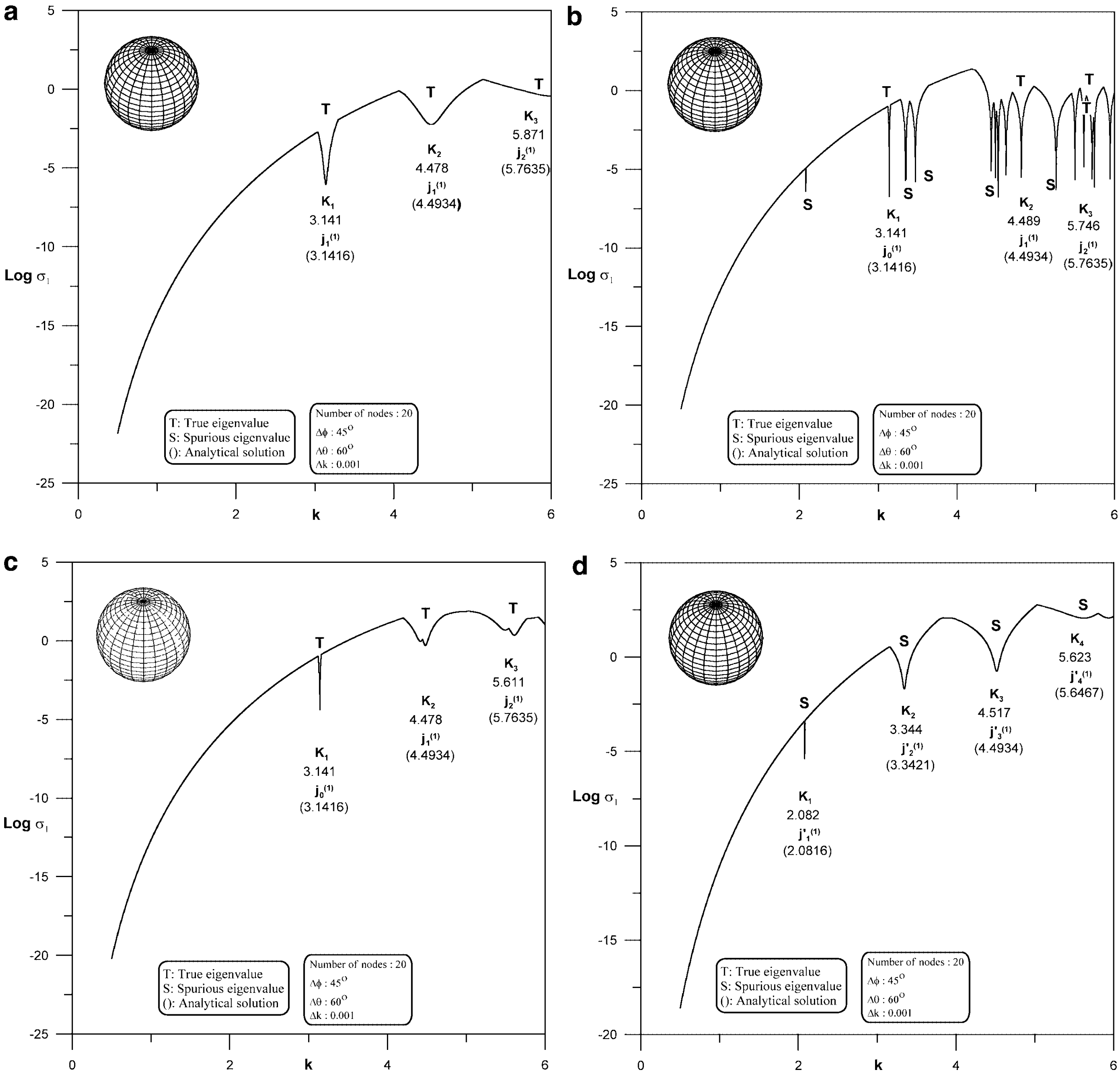

Fig. 2. The first minimum singular value for different wave numbers by using the **a** single-layer potential approach for the Dirichlet problem; **b** double-layer potential approach for the Dirichlet problem, **c** SVD updating term $\begin{bmatrix} U \\ T \end{bmatrix}$ for the Dirichlet problem, **d** SVD updating document $[T \quad M]$ for the Dirichlet problem

$$[H]\{g\} = \{0\} \ . \tag{69}$$

(2) Singular system
Alternatively, the equation has at least one solution if the homogeneous adjoint equation has at least one solution $\{\phi_i\}$ such that,

$$[H]^{\dagger}\{\phi_i\} = \{0\} \ , \tag{70}$$

where $[H]^{\dagger}$ is the transpose conjugate matrix of H (Davis and Thomsom, 2000). If the matrix $[H]$ is real, the transpose conjugate is equal to its transpose only, i.e., $[H]^{\dagger} = [H]^{T}$. Moreover, a necessary and sufficient *solvability condition* (Stakgold, 1979) is that the constraint, $\{p\}^{\dagger}\{\phi_i\} = 0$, must be satisfied. Then, the general solution can be written as

$$\{g\} = \{\bar{g}\} + \sum_{i=1}^{N_n} c_i\{\phi_i\} \ , \tag{71}$$

where $\{\bar{g}\}$ is any particular solution, N_n is the nullity of the matrix $[H]$, c_i is an arbitrary constant and $\{\phi_i\}$ are the bases obtained by Eq. (70).

By employing the *LM* formulation in the direct method, we have

$$\left[M^E\right]\{u\} = \left[L^E\right]\{t\} = \{p\} \ , \tag{72}$$

Table 3. Zeros of the derivative for the spherical Bessel functions, $j'_n(k)$
$j'_n(x)=\frac{1}{2}\sqrt{\frac{\pi}{2x}}(\frac{-1}{x}J_{n+\frac{1}{2}}(x)+J_{n-\frac{1}{2}}(x)+J_{n+\frac{3}{2}}(x))$

Problem	Eigenvalues	1	2	3	4	5
Neumann problem	$j'_0(k)$	0	4.4934	7.7253	10.9041	14.0662
	$j'_1(k)$	2.0816	5.9404	9.2058	12.4044	15.5792
	$j'_2(k)$	3.3421	7.2899	10.6139	13.8461	17.0429
	$j'_3(k)$	4.5141	8.5838	11.9727	15.2445	18.4681
	$j'_4(k)$	5.6467	9.8405	13.2956	16.6093	19.8624
	$j'_5(k)$	6.7565	11.0702	15.5906	17.9472	21.2311

where $\{u\}$ and $\{t\}$ are the boundary excitations. Since spurious eigenvalues k_s are embedded in both the Dirichlet and Neumann problems, we have

$$\{p\}^{\dagger}\{\phi_i\}=0 \ , \tag{73}$$

where $\{\phi_i\}$ satisfies

$$\begin{aligned}&\left[L^E(k_s)\right]^{\mathrm{T}}\{\phi_i\}=\{0\}, \quad \text{for the Dirichlet problem,}\\ &\left[M^E(k_s)\right]^{\mathrm{T}}\{\phi_i\}=\{0\}, \quad \text{for the Neumann problem}\ ,\end{aligned} \tag{74}$$

according to the Fredholm alternative theorem. By substituting Eq. (72) into Eq. (73), we have

$$\begin{aligned}&\{u\}^{\mathrm{T}}\left[M^E\right]^{\mathrm{T}}\{\phi_i\}=0, \quad \text{for the Dirichlet problem,}\\ &\{t\}^{\mathrm{T}}\left[L^E\right]^{\mathrm{T}}\{\phi_i\}=0, \quad \text{for the Neumann problem}\ .\end{aligned} \tag{75}$$

Since $\{u\}$ and $\{t\}$ can be arbitrary boundary excitation for the Dirichlet problem and Neumann problems, respectively, we have

$$\begin{aligned}&\left[M^E\right]^{\mathrm{T}}\{\phi_i\}=\{0\}, \quad \text{for the Dirichlet problem,}\\ &\left[L^E\right]^{\mathrm{T}}\{\phi_i\}=\{0\}, \quad \text{for the Neumann problem}\ .\end{aligned} \tag{76}$$

By combining Eq. (74) with Eq. (76) for the Dirichlet problem, we have

$$\begin{bmatrix}[L^E]^{\mathrm{T}}\\ [M^E]^{\mathrm{T}}\end{bmatrix}\{\phi_i\}=\{0\} \ \text{ or } \ \{\phi_i\}^{\mathrm{T}}\left[[L^E] \quad [M^E]\right]=\{0\}\ . \tag{77}$$

It indicates that the two matrices have the same spurious boundary mode $\{\phi_i\}$ corresponding to the same zero singular values. By using the SVD technique, the two matrices in Eq. (77) can be decomposed into

$$\begin{aligned}&\left[L^E\right]^{\mathrm{T}}=\left[\Psi^{(L)}\right]\left[\Sigma^{(L)}\right]\left[\Phi^{(L)}\right]^{\mathrm{T}} \ \text{or}\\ &\quad\{L^E_\lambda\}=\sum_j \sigma_j^{(L)}\{\psi_j^{(L)}\}\{\phi_j^{(L)}\}^{\mathrm{T}}\ ,\\ &\left[M^E\right]^{\mathrm{T}}=\left[\Psi^{(M)}\right]\left[\Sigma^{(M)}\right]\left[\Phi^{(M)}\right]^{\mathrm{T}} \ \text{or}\\ &\quad\{M^E_\lambda\}=\sum_j \sigma_j^{(M)}\{\psi_j^{(M)}\}\{\phi_j^{(M)}\}^{\mathrm{T}}\ .\end{aligned} \tag{78}$$

By substituting Eq. (78) into Eq. (76), we have

$$\begin{aligned}&\sum_j \sigma_j^{(L)}\{\psi_j^{(L)}\}\{\phi_j^{(L)}\}^{\mathrm{T}}\{\phi_i\} \ =\{0\}\\ &\xrightarrow{\phi_j^{(L)}\cdot\phi_i=\delta_{ij}} \sigma_i^{(L)}\{\psi_i^{(L)}\}=\{0\}, \ \ (i \text{ no sum})\\ &\sum_j \sigma_j^{(M)}\{\psi_j^{(M)}\}\{\phi_j^{(M)}\}^{\mathrm{T}}\{\phi_i\} \ =\{0\}\\ &\xrightarrow{\phi_j^{(M)}\cdot\phi_i=\delta_{ij}} \sigma_i^{(M)}\{\psi_i^{(M)}\}=\{0\}, \ \ (i \text{ no sum})\end{aligned} \tag{79}$$

where $\{\phi_i\}$ and $\{\psi_i\}$ are the orthonormal bases, $\sigma_i^{(L)}$ and $\sigma_i^{(M)}$ are the zero singular values of $[L^E]$ and $[M^E]$ matrices, respectively. We can easily extract out the spurious eigenvalues since there exists the same spurious boundary mode $\{\phi_i\}$ corresponding to the same zero singular values $(\sigma_i^{(L)}=\sigma_i^{(M)}=0)$.

Similarly, we can employ the *UT* formulation in the direct method, we have

$$\left[T^E\right]\{u\}=\left[U^E\right]\{t\}=\{p\}\ . \tag{80}$$

Since spurious eigenvalues are embedded in both the Dirichlet and Neumann problems, we have

$$\begin{aligned}&\left[U^E\right]^{\mathrm{T}}\{\phi_i\}=\{0\}, \quad \text{for the Dirichlet problem}\ ,\\ &\left[T^E\right]^{\mathrm{T}}\{\phi_i\}=\{0\}, \quad \text{for the Neumann problem}\ ,\end{aligned} \tag{81}$$

according to the Fredholm alternative theorem. By substituting Eq. (80) into Eq. (73), we have

$$\begin{aligned}&\{u\}^{\mathrm{T}}\left[T^E\right]^{\mathrm{T}}\{\phi_i\}=0, \quad \text{for the Dirichlet problem,}\\ &\{t\}^{\mathrm{T}}\left[U^E\right]^{\mathrm{T}}\{\phi_i\}=0, \quad \text{for the Neumann problem}\ .\end{aligned} \tag{82}$$

Since $\{u\}$ and $\{t\}$ can be arbitrary for the Dirichlet problem and Neumann problems, respectively, we have

$$\begin{aligned}&\left[T^E\right]^{\mathrm{T}}\{\phi_i\}=\{0\}, \quad \text{for the Dirichlet problem,}\\ &\left[U^E\right]^{\mathrm{T}}\{\phi_i\}=\{0\}, \quad \text{for the Neumann problem}\ .\end{aligned} \tag{83}$$

By combining Eq. (81) with Eq. (83) for the Neumann problem, we have

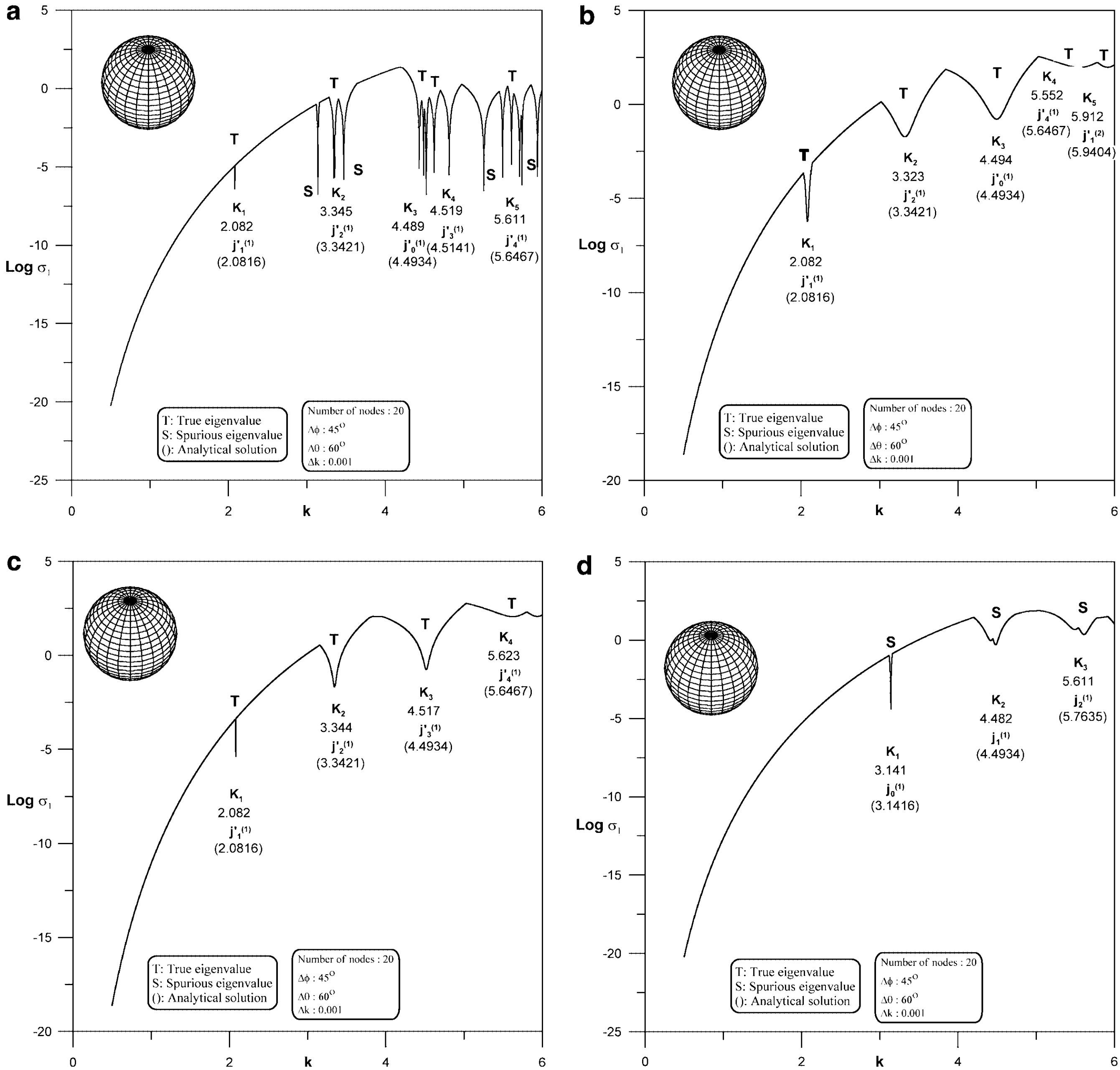

Fig. 3a–k. The first minimum singular value for different wave numbers by using the **a** single-layer potential approach for the Neumann problem, **b** double-layer potential approach for the Neumann problem, **c** SVD updating term $\begin{bmatrix} L \\ M \end{bmatrix}$ for the Dirichlet problem, **d** SVD updating document $[\,U \quad T\,]$ for the Dirichlet problem. **e** The determination of multiplicities for a spherical cavity. **f** Contours of the former three boundary modes by using the single-layer potential approach for the Neumann problem. **g** 3-D plots of the former three boundary modes by using the single-layer potential approach for the Neumann problem. The former three analytical interior modes for the Dirichlet problem. **h** Contours of the former three boundary modes by using the double-layer potential approach for the Neumann problem. **i** 3-D plots of the former three boundary modes by using the double-layer potential approach for the Neumann problem. **j** Contours of the former three analytical boundary modes for the Neumann problem. **k** 3-D plots of the former three analytical boundary modes for the Neumann problem

$$\begin{bmatrix} [U^E(k_s)]^{\mathrm{T}} \\ [T^E(k_s)]^{\mathrm{T}} \end{bmatrix} \{\phi_i\} = \{0\} \;\text{ or }\; \{\phi_i\}^{\mathrm{T}} \big[[U^E] \quad [T^E] \big] = \{0\} \;, \tag{84}$$

where k_s is the spurious eigenvalue. It indicates that the two matrices have the same spurious boundary mode $\{\phi_i\}$ corresponding to the same zero singular values. By using the SVD technique, Eq. (84) can be expressed as

$$\begin{aligned} \left[U^E\right]^{\mathrm{T}} &= \left[\Psi^{(U)}\right]\left[\Sigma^{(U)}\right]\left[\Phi^{(U)}\right]^{\mathrm{T}} \;\text{ or} \\ \{U^E\} &= \sum_j \sigma_j^{(U)} \{\psi_j^{(U)}\}\{\phi_j^{(U)}\}^{\mathrm{T}} \;, \\ \left[T^E\right]^{\mathrm{T}} &= \left[\Psi^{(T)}\right]\left[\Sigma^{(T)}\right]\left[\Phi^{(T)}\right]^{\mathrm{T}} \;\text{ or} \\ \{T^E\} &= \sum_j \sigma_j^{(T)} \{\psi_j^{(T)}\}\{\phi_j^{(T)}\}^{\mathrm{T}} \;. \end{aligned} \tag{85}$$

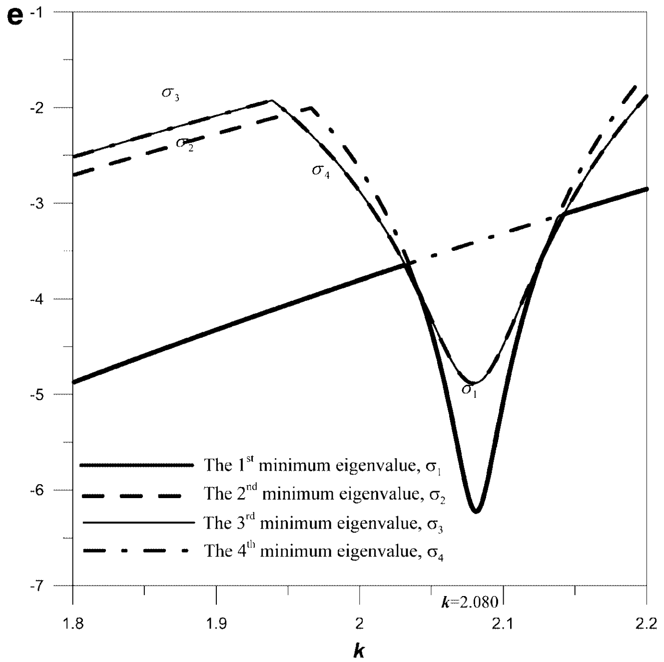

Fig. 3. (Contd.)

By substituting Eq. (85) into Eq. (83), we have

$$\begin{aligned}
&\sum_j \sigma_j^{(U)}\{\psi_j^{(U)}\}\{\phi_j^{(U)}\}^{\mathrm{T}}\{\phi_i\} = \{0\} \\
&\xrightarrow{\phi_i^{(U)}\cdot\phi_j=\delta_{ij}} \sigma_i^{(U)}\{\psi_i^{(U)}\} = \{0\}, \quad (i \text{ no sum}) \\
&\sum_j \sigma_j^{(T)}\{\psi_j^{(T)}\}\{\phi_j^{(T)}\}^{\mathrm{T}}\{\phi_i\} = \{0\} \\
&\xrightarrow{\phi_j^{(T)}\cdot\phi_i=\delta_{ij}} \sigma_i^{(T)}\{\psi_i^{(T)}\} = \{0\}, \quad (i \text{ no sum})
\end{aligned} \tag{86}$$

where $\{\phi_i\}$ and $\{\psi_i\}$ are the orthonormal bases, $\sigma_i^{(U)}$ and $\sigma_i^{(T)}$ are the zero singular values of $[U^E]$ and $[T^E]$ matrices, respectively. We can easily extract out the spurious eigenvalues since there exists the same spurious boundary mode $\{\phi_i\}$ corresponding to the same zero singular values ($\sigma_i^{(U)} = \sigma_i^{(T)} = 0$).

According to the relations of the matrices between the direct method and indirect method (Eqs. (38)–(41)), we can filter out the spurious eigensolutions in the indirect method which is expressed as

$$\begin{bmatrix} [L^E(k_s)]^{\mathrm{T}} \\ [M^E(k_s)]^{\mathrm{T}} \end{bmatrix}\{\phi_i\} = \{0\} \xrightarrow[M^E=M^I]{L^E=T^I} \begin{bmatrix} [T^I(k_s)]^{\mathrm{T}} \\ [M^I(k_s)]^{\mathrm{T}} \end{bmatrix}\{\phi_i\} = \{0\}. \tag{87}$$

By using the SVD technique, we obtain

$$\begin{aligned}
&\sum_j \sigma_j^{(T)}\{\psi_j^{(T)}\}\{\phi_j^{(T)}\}^{\mathrm{T}}\{\phi_i\} = \{0\} \\
&\xrightarrow{\phi_j^{(T)}\cdot\phi_i=\delta_{ij}} \sigma_i^{(T)}\{\psi_i^{(T)}\} = \{0\}, \quad (i \text{ no sum}) \\
&\sum_j \sigma_j^{(M)}\{\psi_j^{(M)}\}\{\phi_j^{(M)}\}^{\mathrm{T}}\{\phi_i\} = \{0\} \\
&\xrightarrow{\phi_i^{(M)}\cdot\phi_j=\delta_{ij}} \sigma_i^{(M)}\{\psi_i^{(M)}\} = \{0\}, \quad (i \text{ no sum})
\end{aligned} \tag{88}$$

where $\{\phi_i\}$ and $\{\psi_i\}$ are the orthonormal bases, $\sigma_i^{(T)}$ and $\sigma_i^{(M)}$ are the zero singular values of $[T^I]$ and $[M^I]$ matrices, respectively. We can easily extract out the spurious eigenvalues since there exists the same spurious boundary mode $\{\phi_i\}$ corresponding to the same zero singular values ($\sigma_i^{(T)} = \sigma_i^{(M)} = 0$).

In a similar way, we have the influence matrices by using the indirect method and Eq. (84),

$$\begin{bmatrix} [U^E(k_s)]^{\mathrm{T}} \\ [T^E(k_s)]^{\mathrm{T}} \end{bmatrix}\{\phi_i\} = \{0\} \xrightarrow[T^E=L^I]{U^E=U^I} \begin{bmatrix} [U^I(k_s)]^{\mathrm{T}} \\ [L^I(k_s)]^{\mathrm{T}} \end{bmatrix}\{\phi_i\} = \{0\}\ . \tag{89}$$

By using the SVD technique, we obtain

$$\begin{aligned}
&\sum_j \sigma_j^{(U)}\{\psi_j^{(U)}\}\{\phi_j^{(U)}\}^{\mathrm{T}}\{\phi_i\} = \{0\} \\
&\xrightarrow{\phi_i^{(U)}\cdot\phi_j=\delta_{ij}} \sigma_i^{(U)}\{\psi_i^{(U)}\} = \{0\}, \quad (i \text{ no sum}) \\
&\sum_j \sigma_j^{(L)}\{\psi_j^{(L)}\}\{\phi_j^{(L)}\}^{\mathrm{T}}\{\phi_i\} = \{0\} \\
&\xrightarrow{\phi_i^{(L)}\cdot\phi_j=\delta_{ij}} \sigma_i^{(L)}\{\psi_i^{(L)}\} = \{0\}, \quad (i \text{ no sum})
\end{aligned} \tag{90}$$

where $\{\phi_i\}$ and $\{\psi_i\}$ are the orthonormal bases, $\sigma_i^{(U)}$ and $\sigma_i^{(L)}$ are the singular values of $[U^I]$ and $[L^I]$ matrices, respectively. We can easily extract out the spurious eigenvalues since there exists the same spurious boundary $\{\phi_i\}$ corresponding to the same zero singular-values ($\sigma_i^{(U)} = \sigma_i^{(L)} = 0$). Therefore, all the SVD updating techniques for extracting out the eigensolutions are summarized in Table 1.

7 Numerical results and discussions

Case 1: Spherical cavity (Dirichlet case)

A spherical cavity with a unit radius subjected to the Dirichlet boundary condition ($u = 0$) is considered. In this case, an analytical solution is available as follows:

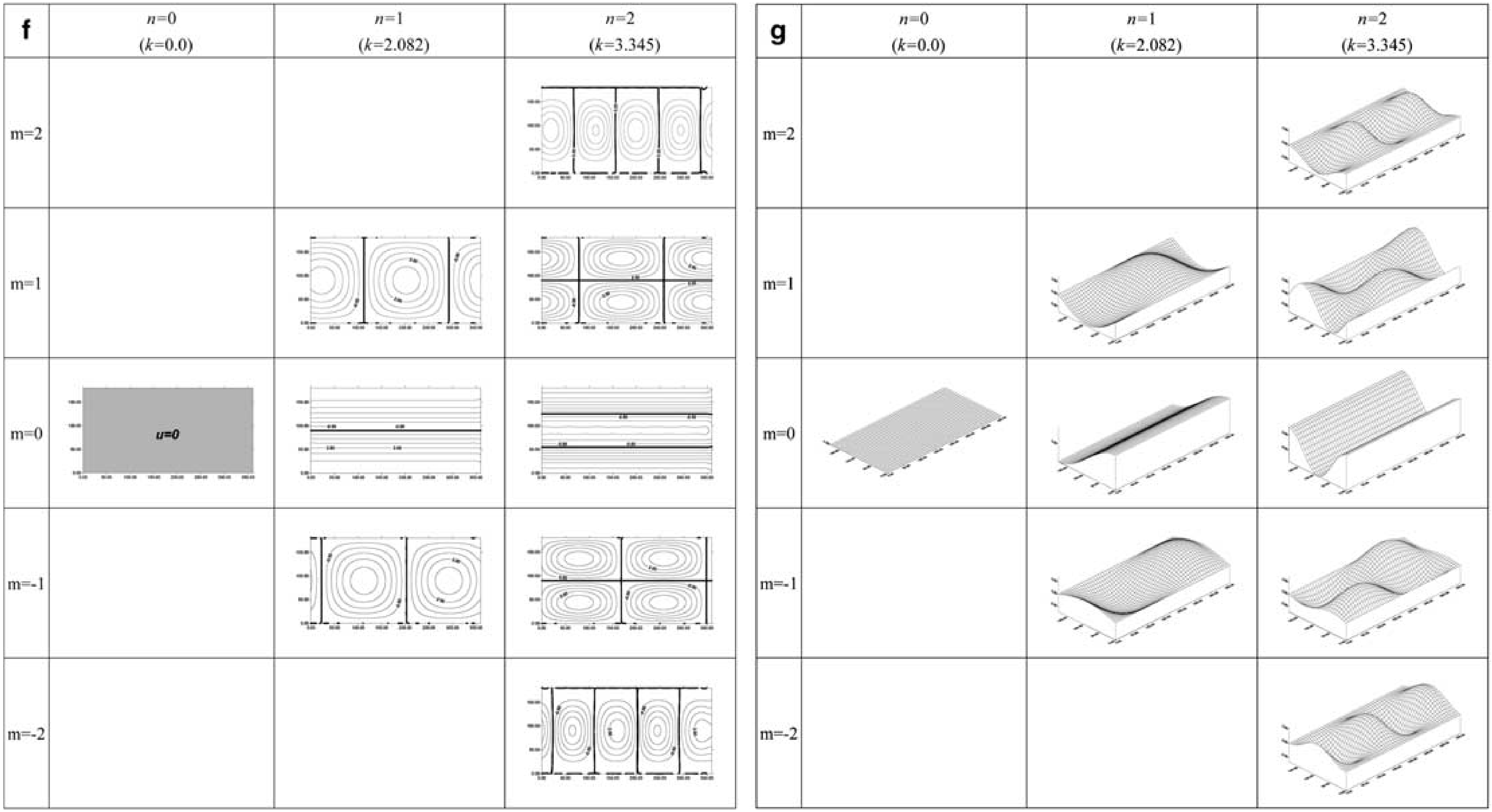
f
n=0 (k=0.0)
n=1 (k=2.082)
n=2 (k=3.345)
m=2
m=1
m=0
m=-1
m=-2
u=0
g
n=0 (k=0.0)
n=1 (k=2.082)
n=2 (k=3.345)
m=2
m=1
m=0
m=-1
m=-2

Fig. 3. (Contd.)

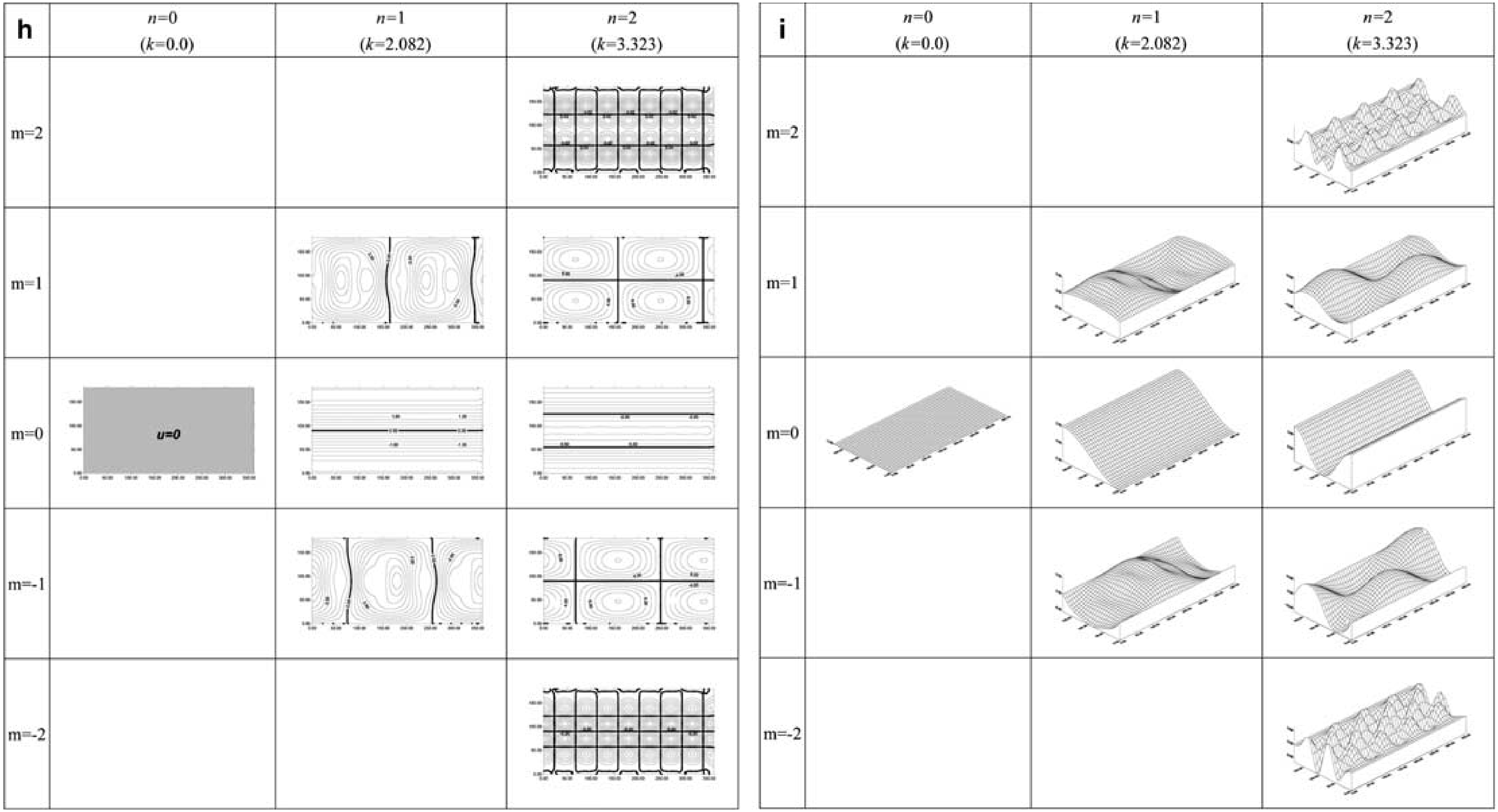
h
n=0
(k=0.0)
n=1
(k=2.082)
n=2
(k=3.323)
m=2
m=1
m=0
m=-1
m=-2
u=0
i
n=0
(k=0.0)
n=1
(k=2.082)
n=2
(k=3.323)
m=2
m=1
m=0
m=-1
m=-2

Fig. 3. (Contd.)

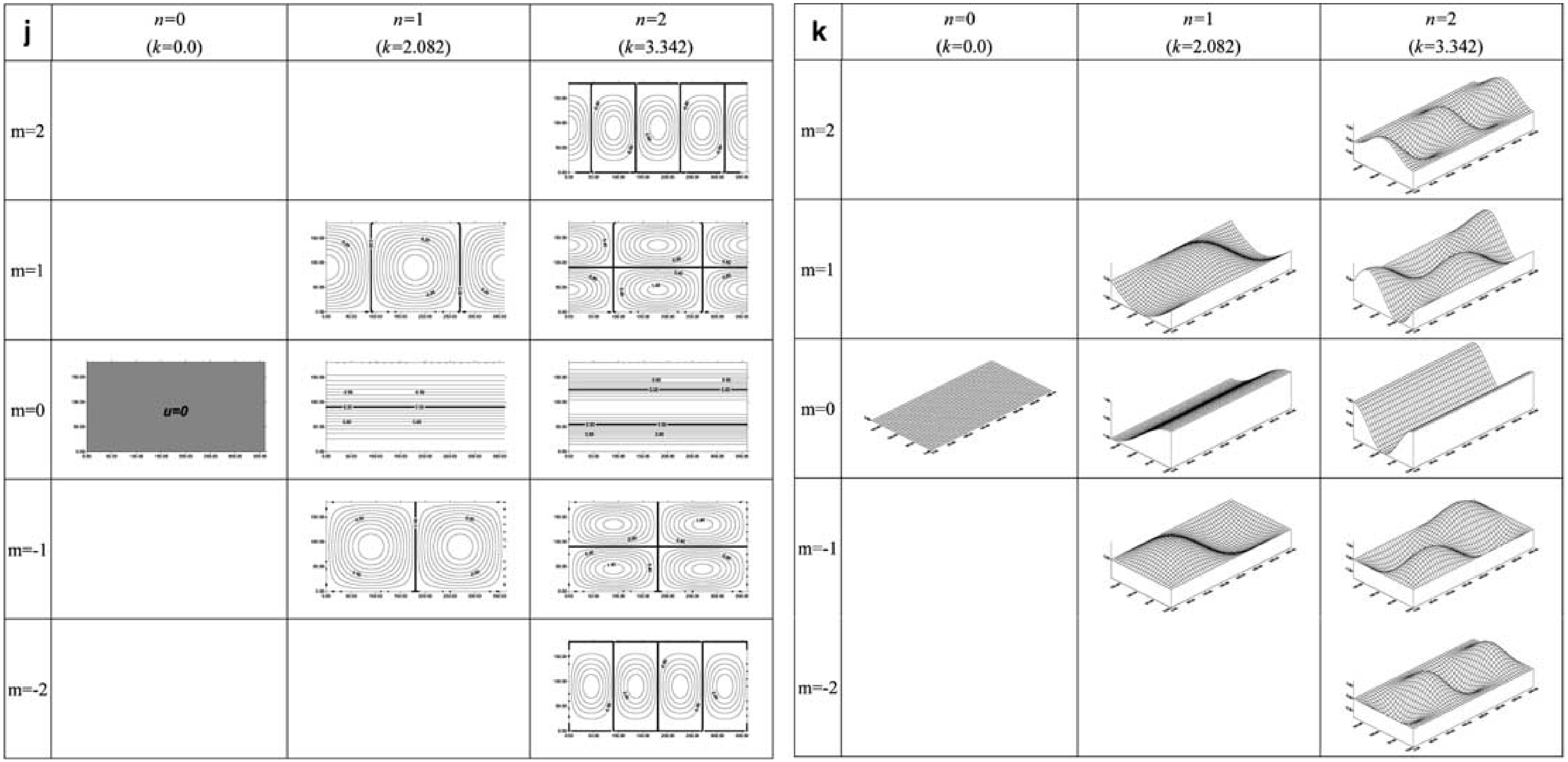
j
n=0
(k=0.0)
n=1
(k=2.082)
n=2
(k=3.342)
m=2
m=1
m=0
m=-1
m=-2
u=0
k
n=0
(k=0.0)
n=1
(k=2.082)
n=2
(k=3.342)
m=2
m=1
m=0
m=-1
m=-2

Fig. 3. (Contd.)

eigenequation:

$j_m(k_n) = 0$,

eigenmode:

$$u_{mn}(\rho, \theta, \phi) = j_m(k_n\rho)P_n^m(\cos\theta)\mathrm{e}^{\mathrm{i}m\phi},$$
$$0 < \rho \le 1, \ \ 0 \le \theta \le 2\pi, \ \ 0 \le \phi \le \pi \ ,$$

where $m = 0, 1, \ldots, n$ and $n = 1, 2, 3, \ldots$.

By discretizing twenty nodes on the spherical boundary, two results by using the single and double-layer potential approaches are obtained. The exact eigenvalues of a spherical cavity subjected to the Dirichlet boundary condition are shown in Table 2. Fig. 2a shows the minimum singular value versus k using the single-layer potential approach. As predicted in Eq. (47), only true eigenvalues occur. The former three true eigenvalues are obtained by considering the near zero minimum singular values as shown in Fig. 2a where no spurious eigenvalues are found if only the single-layer potential method is chosen. Figure 2b shows the minimum singular value versus k using the double-layer potential approach. The true eigenvalues occur at the positions of zeros for $j_m(k_n\rho)$ as predicted in Eq. (49) while the spurious eigenvalues occur at the positions of zeros for $j'_m(k_n\rho)$ if the double-layer potential approach is chosen. In order to extract out the true eigenvalues, we can combine the $[U^I]$ with $[T^I]$ by using the SVD updating term. The minimum singular value versus k is shown in Fig. 2c, it is found that dips occur only at the positions of true eigenvalues. In a similar way, we can combine the $[T^I]$ with $[M^I]$ by using the SVD updating document in order to sort out the spurious eigenvalues. The minimum singular value versus k is shown in Fig. 2d, it is found that the dips occur only at the positions of spurious eigenvalues.

Case 2: Spherical cavity (Neumann case)
A spherical cavity with a unit radius subjected to the Neumann boundary condition ($t = 0$) is considered. In this case, an analytical solution is available as follows:

eigenequation:

$j'_m(k_n) = 0$,

eigenmode:

$$u_{mn}(\rho, \theta, \phi) = j_m(k_n\rho)P_n^m(\cos\ \theta)\mathrm{e}^{\mathrm{i}m\phi},$$
$$0 < \rho \le 1, \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi \ ,$$

where $m = 0, 1, \ldots, n$ and $n = 1, 2, 3, \ldots$.

By collocating the twenty nodes on the spherical boundary, two results by using the single and double-layer potential approaches are obtained. The exact eigenvalues of a spherical cavity subjected to the Neumann boundary condition are shown in Table 3. Figure 3a shows the minimum singular value versus k using the single-layer potential approach. The former five true eigenvalues can be obtained as shown in Fig. 3a by considering the near zero minimum singular values if only the single-layer potential method is chosen. As expected in Eq. (51), spurious eigenvalues are present. The true eigenvalues occur at the positions of zeros for $j'_m(k_n\rho)$ while the spurious eigenvalues occur at the positions of zeros for $j_m(k_n\rho)$ if the double-layer potential approach is chosen. Figure 3b shows the minimum singular value versus k using the double-layer potential approach. No spurious eigenvalues occurs as theoretically predicted in Eq. (53). In order to extract out the true eigenvalues, we can combine the $[L^I]$ with $[M^I]$ by using the SVD updating term. The minimum singular value versus k is shown in Fig. 3c, it is found that the dips occur only at the positions of true eigenvalues. In a similar way, we can combine the $[U^I]$ with $[L^I]$ by using the SVD updating document in order to sort out the spurious eigenvalues. The minimum singular value versus k is shown in Fig. 3d, it is found that the dips occur only at the positions of spurious eigenvalues. Moreover, the multiplicity is found to be three in Fig. 3e obtained by using the three successive zero singular values at the same value of k. The former three boundary modes for the single and double-layer potential approaches are shown in Fig. 3f–i and are compared with the analytical modes in Fig. 3j and k. Good agreement is made.

According to the numerical experiments, the guideline on the number of nodes depends on two criteria, one is large enough for the number of nodes to detect the minimum former eigenvalues, and the other is small enough for the number of nodes to avoid the ill-posed condition. As quoted from the Hutchinson's paper (1991) "The major drawback seems to be with the belief that the BCM produces ill-conditioned. While this is sometimes the cases, it is hoped that future research can alleviate this drawback." This topic is under investigation in our group.

8
Conclusions

We have developed a boundary collocation method by using the imaginary-part kernel for the acoustic eigenproblems of three-dimensional cavities. The imaginary-part of the complex-valued fundamental solution was chosen as the radial basis function to approximate the solution. Although this method is very simple by using only two-point function for the influence matrices, it results in spurious eigensolutions and ill-posed problems. The true and spurious boundary modes were extracted from the right and left unitary vectors of SVD, respectively. Two approaches, the SVD updating terms and updating documents in conjunction with the dual formulation, were proposed to extract out the true eigensolutions and to sort out the spurious eigensolutions, respectively, as shown in Table 1. A spherical cavity, subject to the Dirichlet and the Neumann boundary conditions, was demonstrated to check the validity of the meshless formulation.

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