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Mathematical analysis and numerical study of true and spurious eigenequations for free vibration of plates using real-part BEM

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Abstract In this paper, a real-part BEM for solving the eigenfrequencies of plates is proposed for saving half effort in computation instead of using the complex-valued BEM. By employing the real-part fundamental solution, the spurious eigenequations in conjunction with the true eigenequation are obtained for free vibration of plate. To verify this finding, the circulant is adopted to analytically derive the true and spurious eigenequations in the discrete system of a circular plate. In order to obtain the eigenvalues and boundary modes at the same time, the singular value decomposition (SVD) technique is utilized. For the continuous system, mathematical analysis for the spurious eigenequation was done by using the degenerate kernel and Fourier series. Good agreement of the analytical solutions (continuous and discrete systems) is made. Three cases, clamped, simply-supported and free circular plates, are demonstrated analytically and numerically to see the validity of the present method. SVD updating technique is adopted to suppress the ocurrence of the spurious eigenvalues, and a clamped plate is demonstrated analytically for the discrete system in this paper.

Keywords Real-part BEM, Plate vibration, Spurious eigenvalue, Circulant, Degenerate kernel, Fourier series, SVD updating technique

1 Introduction

For the simply-connected problems of interior acoustics or membrane, either the real-part or imaginary-part BEM results in spurious eigenequations. Tai and Shaw [33] first employed BEM to solve membrane vibration using a complex-valued kernel. De Mey [15, 16], Hutchinson and Wong [21] employed only the real-part kernel to solve the membrane and plate vibrations free of the complex-valued computation in sacrifice of occurrence of spurious eigenequations. Kamiya et al. [25, 26] and Yeih et al. [35] linked the relation of MRM and real-part BEM independently. Wong and Hutchinson [23] have presented a direct BEM for plate vibration involving displacement, slope, moment and shear force. They were able to obtain numerical results for the clamped plates by employing only the real-part BEM with obvious computational gains. However, this saving leads to the spurious eigenvalues in addition to the true ones in free vibration analysis. One has to investigate the mode shapes in order to identify and reject the spurious ones. Shaw [32] commented that only the real-part approach was incorrect since the eigenequation must satisfy the real-part and imaginary-part equations at the same time. Hutchinson [22] replied that the claim of incorrectness was perhaps a little strong since the real-part BEM does not miss any true eigenvalue although the solution is contaminated by spurious ones according to his numerical experience. If we need to look for the eigenmode as well as eigenvalue as usual, the sorting for the spurious eigenequations pay a small price by identifying the mode shapes. Chen et al. [10] commented that the spurious modes can be reasonable which may mislead the judgement of the true and spurious ones, since the true and spurious modes may have the same nodal line in case of different eigenvalues. This is the reason why Chen et al. [10] have developed many systematic techniques, e.g., dual formulation [10], domain partition [7], SVD updating technique [9], CHEEF method [8], for sorting out the true and the spurious eigenvalues. Niwa et al. [31] also stated that "One must take care to use the complete Green's function for outgoing waves, as attempts to use just the real (singular) or imaginary (regular) part separately will not provide the complete spectrum". As quoted from the reply of Hutchinson [22], this comment is not correct since the real-part BEM does not lose any true eigenvalue. The reason is that the real and imaginary-part kernels satisfy the Hilbert transform pair. Complete eigenspectrum is imbedded in either one, real or imaginary-part kernel. The Hilbert transform is the constraint in the frequency domain corresponding to the casual effect in the time-domain fundamental solution. The physical meaning of the real-part kernel is the standing wave [17]. Tai and Shaw [33] claimed that spurious eigenvalues are not present if the complex-valued kernel is employed for the eigenproblem. However, it is true only for the case of problem with a simply-connected domain [12, 13]. For multiply-connected

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problems, spurious eigenequation occur even though the complex-valued BEM is utilized.

In this paper, the spurious eigenequation for the plate eigenproblem will be studied in the real-part BEM. First of all, the true and spurious eigenvalues will be examined for the simply-connected plate using the real-part BEM. Since any two boundary integral equations in the plate formulation (4 equations) can be chosen, 6 (C_2^4) options can be considered. The occurring mechanism for the spurious eigenequation in the plate eigenproblem in each formulation will be studied analytically in the continuous and discrete systems. For the continuous system, degenerate kernels for the fundamental solution and the Fourier series expansion for boundary densities will be employed to derive the true and spurious eigenequations analytically for a circular plate. For the discrete system, the degenerate kernels for the fundamental solution and circulants resulting from the circular boundary will be employed to determine the spurious eigenequation. Three types of plates subject to clamped, simply-supported and free boundary conditions will be illustrated to check the validity of the present formulations. Also, the SVD updating technique is adopted to suppress the ocurrence of the spurious eigenvalues for the free vibration of plate problem, and a clamped plate is demonstrated analytically for the discrete system in this paper.

2
Boundary integral equations for plate eigenproblems

The governing equation for free flexural vibration of a uniform thin plate is written as follows:

$$\nabla^4 u(x) = \lambda^4 u(x), \quad x \in \Omega \ , \tag{1}$$

where u is the lateral displacement, $\lambda^4 = \omega^2 \rho_0 h/D$, λ is the frequency parameter, ω is the circular frequency, ρ_0 is the surface density, D is the flexural rigidity expressed as $D = Eh^3/12(1-\nu^2)$ in terms of Young's modulus E, the Poisson ratio ν, the plate thickness h, and Ω is the domain of the thin plate.

The integral equations for the domain point can be derived from the Rayleigh-Green identity [27] as follows:

$$\begin{aligned} u(x) = &-\int_B U(s,x)v(s)\mathrm{d}B(s) + \int_B \Theta(s,x)m(s)\mathrm{d}B(s) \\ &-\int_B M(s,x)\theta(s)\mathrm{d}B(s) \\ &+\int_B V(s,x)u(s)\mathrm{d}B(s), \quad x \in \Omega \ , \end{aligned} \tag{2}$$

$$\begin{aligned} \theta(x) = &-\int_B U_\theta(s,x)v(s)\mathrm{d}B(s) + \int_B \Theta_\theta(s,x)m(s)\mathrm{d}B(s) \\ &-\int_B M_\theta(s,x)\theta(s)\mathrm{d}B(s) \\ &+\int_B V_\theta(s,x)u(s)\mathrm{d}B(s), \quad x \in \Omega \ , \end{aligned} \tag{3}$$

$$\begin{aligned} m(x) = &-\int_B U_m(s,x)v(s)\mathrm{d}B(s) + \int_B \Theta_m(s,x)m(s)\mathrm{d}B(s) \\ &-\int_B M_m(s,x)\theta(s)\mathrm{d}B(s) \\ &+\int_B V_m(s,x)u(s)\mathrm{d}B(s), \quad x \in \Omega \ , \end{aligned} \tag{4}$$

$$\begin{aligned} v(x) = &-\int_B U_v(s,x)v(s)\mathrm{d}B(s) + \int_B \Theta_v(s,x)m(s)\mathrm{d}B(s) \\ &-\int_B M_v(s,x)\theta(s)\mathrm{d}B(s) \\ &+\int_B V_v(s,x)u(s)\mathrm{d}B(s), \quad x \in \Omega \ , \end{aligned} \tag{5}$$

where B is the boundary, u, θ, m and v mean the displacement, slope, normal moment, effective shear force, s and x are the source and field points, respectively; U, Θ, M and V kernel functions will be elaborated on later.

By moving the field point to the boundary, Eqs. (2)–(5) reduce to

$$\begin{aligned} \alpha u(x) = &-P.V.\int_B U(s,x)v(s)\mathrm{d}B(s) \\ &+P.V.\int_B \Theta(s,x)m(s)\mathrm{d}B(s) \\ &-P.V.\int_B M(s,x)\theta(s)\mathrm{d}B(s) \\ &+P.V.\int_B V(s,x)u(s)\mathrm{d}B(s), \quad x \in B \ , \end{aligned} \tag{6}$$

$$\begin{aligned} \alpha \theta(x) = &-P.V.\int_B U_\theta(s,x)v(s)\mathrm{d}B(s) \\ &+P.V.\int_B \Theta_\theta(s,x)m(s)\mathrm{d}B(s), \\ &-P.V.\int_B M_\theta(s,x)\theta(s)\mathrm{d}B(s) \\ &+P.V.\int_B V_\theta(s,x)u(s)\mathrm{d}B(s), \quad x \in B \ , \end{aligned} \tag{7}$$

$$\begin{aligned} \alpha m(x) = &-P.V.\int_B U_m(s,x)v(s)\mathrm{d}B(s) \\ &+P.V.\int_B \Theta_m(s,x)m(s)\mathrm{d}B(s) \\ &-P.V.\int_B M_m(s,x)\theta(s)\mathrm{d}B(s) \\ &+P.V.\int_B V_m(s,x)u(s)\mathrm{d}B(s), \quad x \in B \ , \end{aligned} \tag{8}$$

$$\alpha v(x) = -P.V.\int_B U_v(s,x)v(s)\mathrm{d}B(s) + P.V.\int_B \Theta_v(s,x)m(s)\mathrm{d}B(s) - P.V.\int_B M_v(s,x)\theta(s)\mathrm{d}B(s) + P.V.\int_B V_v(s,x)u(s)\mathrm{d}B(s), \quad x \in B \ , \tag{9}$$

where $P.V.$ denotes the principal value, and $\alpha = \frac{1}{2}$ for a smooth boundary. The kernel function $U(s,x)$ is the real-part of the fundamental solution $U_c(s,x)$ which satisfies

$$\nabla^4 U_c(s,x) - \lambda^4 U_c(s,x) = \delta(x-s) \ . \tag{10}$$

where $\delta(x-s)$ is the Dirac-Delta function. Considering the two singular solutions ($Y_0(\lambda r)$ and $K_0(\lambda r)$, which are the zeroth-order of second kind Bessel and modified Bessel functions, respectively) [23] and two regular solutions ($J_0(\lambda r)$ and $I_0(\lambda r)$, which are the zeroth-order of first kind Bessel and modified Bessel functions, respectively) in the fundamental solution, we have

$$U_c(s,x) = \frac{1}{8\lambda^2}[(Y_0(\lambda r) + iJ_0(\lambda r)) - \frac{2}{\pi}(K_0(\lambda r) + iI_0(\lambda r))] \tag{11}$$

where $r \equiv |s-x|$ and $i^2 = -1$. The other three kernels, $\Theta(s,x)$, $M(s,x)$ and $V(s,x)$, are defined as follows:

$$\Theta(s,x) = \mathscr{K}_\theta(U(s,x)) \ , \tag{12}$$

$$M(s,x) = \mathscr{K}_m(U(s,x)) \ , \tag{13}$$

$$V(s,x) = \mathscr{K}_v(U(s,x)) \ , \tag{14}$$

where $\mathscr{K}_\theta(\cdot)$, $\mathscr{K}_m(\cdot)$ and $\mathscr{K}_v(\cdot)$ mean the operators defined by

$$\mathscr{K}_\theta(\cdot) = \frac{\partial(\cdot)}{\partial n} \ , \tag{15}$$

$$K_m(\cdot) = v\nabla^2(\cdot) + (1-v)\frac{\partial^2(\cdot)}{\partial n^2} \ , \tag{16}$$

$$\mathscr{K}_v(\cdot) = \frac{\partial\nabla^2(\cdot)}{\partial n} + (1-v)\frac{\partial}{\partial t}[(\frac{\partial^2(\cdot)}{\partial n\partial t})] \ , \tag{17}$$

where n and t are the normal vector and tangential vector, respectively. The operators $\mathscr{K}_\theta$, $\mathscr{K}_m$ and $\mathscr{K}_v$ can be applied to Θ, M and V kernels. The kernel functions can be expressed as:

$$U(s,x) = Re[U_c(s,x)] \ , \tag{18}$$

$$\Theta(s,x) = \mathscr{K}_\theta(U(s,x)) = \frac{\partial U(s,x)}{\partial n_s} \ , \tag{19}$$

$$M(s,x) = \mathscr{K}_m(U(s,x)) = v\nabla_s^2 \ U(s,x) + (1-v)\frac{\partial^2 U(s,x)}{\partial n_s^2} \ , \tag{20}$$

$$V(s,x) = \mathscr{K}_v(U(s,x)) = \frac{\partial\nabla_s^2 U(s,x)}{\partial n_s} + (1-v)\frac{\partial}{\partial t_s}[(\frac{\partial^2 U(s,x)}{\partial n_s\partial t_s})] \ . \tag{21}$$

The slope, normal moment and effective shear force are derived by

$$\theta(x) = \mathscr{K}_\theta(u(x)) \ , \tag{22}$$

$$m(x) = \mathscr{K}_m(u(x)) \ , \tag{23}$$

$$v(x) = \mathscr{K}_v(u(x)) \ . \tag{24}$$

Once the field point x locates outside the domain, the null-field BIEs of the direct method in Eqs.(6)–(9) yield

$$0 = -\int_B U(s,x)v(s)\mathrm{d}B(s) + \int_B \Theta(s,x)m(s)\mathrm{d}B(s) - \int_B M(s,x)\theta(s)\mathrm{d}B(s) + \int_B V(s,x)u(s)\mathrm{d}B(s), \quad x \in \Omega^e \ , \tag{25}$$

$$0 = -\int_B U_\theta(s,x)v(s)\mathrm{d}B(s) + \int_B \Theta_\theta(s,x)m(s)\mathrm{d}B(s) - \int_B M_\theta(s,x)\theta(s)\mathrm{d}B(s) + \int_B V_\theta(s,x)u(s)\mathrm{d}B(s), \quad x \in \Omega^e \ , \tag{26}$$

$$0 = -\int_B U_m(s,x)v(s)\mathrm{d}B(s) + \int_B \Theta_m(s,x)m(s)\mathrm{d}B(s) - \int_B M_m(s,x)\theta(s)\mathrm{d}B(s) + \int_B V_m(s,x)u(s)\mathrm{d}B(s), \quad x \in \Omega^e \ , \tag{27}$$

$$0 = -\int_B U_v(s,x)v(s)\mathrm{d}B(s) + \int_B \Theta_v(s,x)m(s)\mathrm{d}B(s) - \int_B M_v(s,x)\theta(s)\mathrm{d}B(s) + \int_B V_v(s,x)u(s)\mathrm{d}B(s), \quad x \in \Omega^e \ , \tag{28}$$

where Ω^e is the complementary domain. Note that the null-field BIEs are not singular, since x and s never coincide.

When the boundary is discretized into $2N$ constant elements, the linear algebraic equations of Eqs. (6)–(9) can be obtained as follows:

$$0 = [U]\{v\} + [\Theta]\{m\} + [M]\{\theta\} + [V]\{u\} \ , \quad (29)$$

$$0 = [U_\theta]\{v\} + [\Theta_\theta]\{m\} + [M_\theta]\{\theta\} + [V_\theta]\{u\} \ , \quad (30)$$

$$0 = [U_m]\{v\} + [\Theta_m]\{m\} + [M_m]\{\theta\} + [V_m]\{u\} \ , \quad (31)$$

$$0 = [U_v]\{v\} + [\Theta_v]\{m\} + [M_v]\{\theta\} + [V_v]\{u\} \ , \quad (32)$$

where $[U]$, $[\Theta]$, $[M]$, $[V]$, $[U_\theta]$, $[\Theta_\theta]$, $[M_\theta]$, $[V_\theta]$, $[U_m]$, $[\Theta_m]$, $[M_m]$, $[V_m]$, $[U_v]$, $[\Theta_v]$, $[M_v]$ and $[V_v]$ are the sixteen influence matrices with a dimension of $2N$ by $2N$, $\{u\}$, $\{\theta\}$, $\{m\}$ and $\{v\}$ are the vectors of boundary data with a dimension of $2N$ by one.

3 Mathematical analysis for the true and spurious eigenequations for free vibration of plate by using the real-part BEM

In order to obtain the true and spurious eigenequations for plate vibration by using the real-part BEM, the degenerate kernel is adopted to analytically derive the true and spurious eigenequations in the continuous and discrete systems of a circular plate. For the continuous system, the spurious eigenequation is derived by using the degenerate kernel and Fourier series. For the discrete system, mathematical analysis for the spurious eigenequation is done by using the degenerate kernel and circulants. Three cases (clamped, simply-supported and free plate) are demonstrated analytically in the continuous and the discrete systems, respectively, in the following subsections.

3.1 Mathematical analysis in the continuous system by using the degenerate kernel and Fourier series

Case 1. Clamped circular plate

For the clamped circular plate ($u = 0$ and $\theta = 0$) with a radius a, we can obtain the eigenequation in the continuous formulation. The moment and shear force, $m(s)$ and $v(s)$, can be expanded into Fourier series by

$$m(s) = p_0^c + \sum_{n=1}^{\infty}(p_n^c \cos(n\overline{\phi}) + q_n^c \sin(n\overline{\phi})), \quad s \in B \ , \quad (33)$$

$$v(s) = a_0^c + \sum_{n=1}^{\infty}(a_n^c \cos(n\overline{\phi}) + b_n^c \sin(n\overline{\phi})), \quad s \in B \ , \quad (34)$$

where the superscript "c" denotes the clamped case, $\overline{\phi}$ is the angle on the circular boundary, a_n^c, b_n^c, p_n^c and q_n^c are the undetermined Fourier coefficients. Substituting Eqs.(33) and (34) into Eqs.(25) and (26), we have

$$0 = -\int_0^{2\pi} U(s,x)[a_0^c + \sum_{n=1}^{\infty}(a_n^c \cos(n\overline{\phi}) + b_n^c \sin(n\overline{\phi}))]\mathrm{d}B(s) + \int_0^{2\pi} \Theta(s,x)[p_0^c + \sum_{n=1}^{\infty}(p_n^c \cos(n\overline{\phi}) + q_n^c \sin(n\overline{\phi}))]\mathrm{d}B, \quad x \in B \ , \quad (35)$$

$$0 = -\int_0^{2\pi} U_\theta(s,x)[a_0^c + \sum_{n=1}^{\infty}(a_n^c \cos(n\overline{\phi}) + b_n^c \sin(n\overline{\phi}))]\mathrm{d}B(s) + \int_0^{2\pi} \Theta_\theta(s,x)[p_0^c + \sum_{n=1}^{\infty}(p_n^c \cos(n\overline{\phi}) + q_n^c \sin(n\overline{\phi}))]\mathrm{d}B, \quad x \in B \ . \quad (36)$$

The kernel functions, $U(s,x)$, $\Theta(s,x)$, $U_\theta(s,x)$ and $\Theta_\theta(s,x)$, can be expanded by using the expansion formulae

$$Y_0(\lambda r) = \begin{cases} Y_0^i(\lambda r) = \sum_{m=-\infty}^{\infty} Y_m(\lambda\overline{\rho}) J_m(\lambda\rho)\cos(m(\overline{\phi}-\phi)) \ , & \overline{\rho} > \rho \ , \\ Y_0^e(\lambda r) = \sum_{m=-\infty}^{\infty} Y_m(\lambda\rho) J_m(\lambda\overline{\rho})\cos(m(\overline{\phi}-\phi)) \ , & \rho > \overline{\rho} \ , \end{cases} \quad (37)$$

$$K_0(\lambda r) = \begin{cases} K_0^i(\lambda r) = \sum_{m=-\infty}^{\infty} K_m(\lambda\overline{\rho}) I_m(\lambda\rho)\cos(m(\overline{\phi}-\phi)) \ , & \overline{\rho} > \rho \ , \\ K_0^e(\lambda r) = \sum_{m=-\infty}^{\infty} K_m(\lambda\rho) I_m(\lambda\overline{\rho})\cos(m(\overline{\phi}-\phi)) \ , & \rho > \overline{\rho} \ , \end{cases} \quad (38)$$

where J_m and I_m denote the first kind of the mth-order Bessel and modified Bessel functions, Y_m and K_m denote the second kind of the mth-order Bessel and modified Bessel functions. The superscripts "i" and "e" denote the interior point ($\overline{\rho} > \rho$) and the exterior point ($\overline{\rho} < \rho$), $s = (\overline{\rho}, \overline{\phi})$ and $x = (\rho, \phi)$ are the polar coordinates of s and x, respectively. In this case, $\overline{\rho} = \rho = a$ and $\mathrm{d}B(s) = a\mathrm{d}\overline{\phi}$ for the circular plate with a radius a. Similarly, the other kernels can also be expanded into degenerate forms. By using the degenerate kernels into Eq. (35) and by employing the orthogonality condition of the Fourier series, the Fourier coefficients a_n^c, b_n^c, p_n^c and q_n^c satisfy

$$p_n^c = \frac{1}{\lambda}\frac{[Y_n(\lambda a)J_n(\lambda a) - \frac{2}{\pi}K_n(\lambda a)I_n(\lambda a)]}{[Y_n(\lambda a)J_n'(\lambda a) - \frac{2}{\pi}K_n(\lambda a)I_n'(\lambda a)]}a_n^c \ , \quad n = 0, 1, 2, \ldots, \quad (39)$$

$$q_n^c = \frac{1}{\lambda}\frac{[Y_n(\lambda a)J_n(\lambda a) - \frac{2}{\pi}K_n(\lambda a)I_n(\lambda a)]}{[Y_n(\lambda a)J_n'(\lambda a) - \frac{2}{\pi}K_n(\lambda a)I_n'(\lambda a)]}b_n^c \ , \quad n = 0, 1, 2, \ldots. \quad (40)$$

Similarly, Eq. (36) yields,

$$p_n^c = \frac{1}{\lambda}\frac{[Y_n'(\lambda a)J_n(\lambda a) - \frac{2}{\pi}K_n'(\lambda a)I_n(\lambda a)]}{[Y_n'(\lambda a)J_n'(\lambda a) - \frac{2}{\pi}K_n'(\lambda a)I_n'(\lambda a)]}a_n^c \ , \quad n = 0, 1, 2, \ldots, \tag{41}$$

$$q_n^c = \frac{1}{\lambda}\frac{[Y_n'(\lambda a)J_n(\lambda a) - \frac{2}{\pi}K_n'(\lambda a)I_n(\lambda a)]}{[Y_n'(\lambda a)J_n'(\lambda a) - \frac{2}{\pi}K_n'(\lambda a)I_n'(\lambda a)]}b_n^c \ , \quad n = 0, 1, 2, \ldots. \tag{42}$$

To seek nontrivial data for the generalized coefficients of a_n^c, p_n^c, b_n^c and q_n^c, we can obtain the eigenequation by using either Eqs. (39) and (41) or Eqs. (40) and (42)

$$\frac{Y_n(\lambda a)J_n(\lambda a) - \frac{2}{\pi}K_n(\lambda a)I_n(\lambda a)}{Y_n(\lambda a)J_n'(\lambda a) - \frac{2}{\pi}K_n(\lambda a)I_n'(\lambda a)} = \frac{Y_n'(\lambda a)J_n(\lambda a) - \frac{2}{\pi}K_n'(\lambda a)I_n(\lambda a)}{Y_n'(\lambda a)J_n'(\lambda a) - \frac{2}{\pi}K_n'(\lambda a)I_n'(\lambda a)} \ . \tag{43}$$

After recollecting the terms, Eq. (43) can be simplified to

$$[K_{n+1}(\lambda a)Y_n(\lambda a) - Y_{n+1}(\lambda a)K_n(\lambda a)]\{I_{n+1}(\lambda a)J_n(\lambda a) + J_{n+1}(\lambda a)I_n(\lambda a)\} = 0 \tag{44}$$

The former part in Eq. (44) inside the middle bracket is the spurious eigenequation while the latter part inside the big bracket is found to be the true eigenequation after comparing with the exact eigenequation [24, 30].

Case 2. Simply-supported circular plate

For the simply-supported circular plate ($u = 0$ and $m = 0$) with a radius a, we can obtain the eigenequation in the continuous formulation. Similarly, the moment and shear force, $\theta(s)$ and $v(s)$, can be expanded into Fourier series by

$$\theta(s) = p_0^s + \sum_{n=1}^{\infty}(p_n^s\cos(n\overline{\phi}) + q_n^s\sin(n\overline{\phi})), \quad s \in B \ , \tag{45}$$

$$v(s) = a_0^s + \sum_{n=1}^{\infty}(a_n^s\cos(n\overline{\phi}) + b_n^s\sin(n\overline{\phi})), \quad s \in B \ , \tag{46}$$

where the superscript "s" denotes the simply-supported case, $\overline{\phi}$ is the angle on the circular boundary, a_n^s, b_n^s, p_n^s and q_n^s are the undetermined Fourier coefficients. Substituting Eqs. (45) and (46) and using the degenerate kernels of $U(s,x)$, $M(s,x)$, $U_\theta(s,x)$ and $M_\theta(s,x)$ into Eq. (25), we have

$$p_n^s = -\frac{[Y_n(\lambda a)J_n(\lambda a) - \frac{2}{\pi}K_n(\lambda a)I_n(\lambda a)]}{[Y_n(\lambda a)\alpha_n^J(\lambda a) - \frac{2}{\pi}K_n(\lambda a)\alpha_n^I(\lambda a)]}a_n^s, \quad n = 0, 1, 2, \ldots, \tag{47}$$

$$q_n^s = -\frac{[Y_n(\lambda a)J_n(\lambda a) - \frac{2}{\pi}K_n(\lambda a)I_n(\lambda a)]}{[Y_n(\lambda a)\alpha_n^J(\lambda a) - \frac{2}{\pi}K_n(\lambda a)\alpha_n^I(\lambda a)]}b_n^s, \quad n = 0, 1, 2, \ldots. \tag{48}$$

where

$$\alpha_n^J(\lambda a) = \lambda^2 J_n''(\lambda a) + v[\frac{1}{a}\lambda J_n'(\lambda a) - (\frac{n}{a})^2 J_n(\lambda a)] \ , \tag{49}$$

$$\alpha_n^I(\lambda a) = \lambda^2 I_n''(\lambda a) + v[\frac{1}{a}\lambda I_n'(\lambda a) - (\frac{n}{a})^2 I_n(\lambda a)] \ . \tag{50}$$

Similarly, Eq. (26) yields,

$$p_n^s = -\frac{[Y_n'(\lambda a)J_n(\lambda a) - \frac{2}{\pi}K_n'(\lambda a)I_n(\lambda a)]}{[Y_n'(\lambda a)\alpha_n^J(\lambda a) - \frac{2}{\pi}K_n'(\lambda a)\alpha_n^I(\lambda a)]}a_n^s, \quad n = 0, 1, 2, \ldots, \tag{51}$$

$$q_n^s = -\frac{[Y_n'(\lambda a)J_n(\lambda a) - \frac{2}{\pi}K_n'(\lambda a)I_n(\lambda a)]}{[Y_n'(\lambda a)\alpha_n^J(\lambda a) - \frac{2}{\pi}K_n'(\lambda a)\alpha_n^I(\lambda a)]}b_n^s, \quad n = 0, 1, 2, \ldots. \tag{52}$$

To seek nontrivial data for the generalized coefficients of a_n^s, p_n^s, b_n^s and q_n^s, we can obtain the eigenequations as

$$\frac{Y_n(\lambda a)J_n(\lambda a) - \frac{2}{\pi}K_n(\lambda a)I_n(\lambda a)}{Y_n(\lambda a)\alpha_n^J(\lambda a) - \frac{2}{\pi}K_n(\lambda a)\alpha_n^I(\lambda a)} = \frac{Y_n'(\lambda a)J_n(\lambda a) - \frac{2}{\pi}K_n'(\lambda a)I_n(\lambda a)}{Y_n'(\lambda a)\alpha_n^J(\lambda a) - \frac{2}{\pi}K_n'(\lambda a)\alpha_n^I(\lambda a)} \tag{53}$$

After recollecting the terms, Eq. (53) can be simplified to

Table 1. True eigenequations for a circular plate ($a = 1$)

True eigenequations for circular plate		
Clamped	Leissa [27]	$I_{\ell+1}J_\ell + I_\ell J_{\ell+1} = 0$
	Real-part BEM	$I_{\ell+1}J_\ell + I_\ell J_{\ell+1} = 0$
Simply-supported	Leissa [27]	$\frac{J_{\ell+1}}{J_\ell} + \frac{I_{\ell+1}}{I_\ell} = \frac{2\lambda}{(1-v)}$
	Real-part BEM	$(1-v)(I_\ell J_{\ell+1} + I_{\ell+1}J_\ell) - 2\lambda I_\ell J_\ell = 0$
Free	Leissa [27]	$\downarrow$ $\frac{\lambda^2 J_\ell + (1-v)[\lambda J_\ell' - \ell^2 J_\ell]}{\lambda^2 I_\ell - (1-v)[\lambda I_\ell' - \ell^2 I_\ell]} = \frac{\lambda^3 J_\ell' + (1-v)\ell^2[\lambda J_\ell' - J_\ell]}{\lambda^3 I_\ell' - (1-v)\ell^2[\lambda I_\ell' - I_\ell]}$ $\downarrow$
		$\frac{\lambda^2 J_\ell + (1-v)[\lambda J_\ell' - \ell^2 J_\ell]}{\lambda^2 I_\ell - (1-v)[\lambda I_\ell' - \ell^2 I_\ell]} = \frac{\lambda^3 I_\ell' + (1-v)\ell^2[\lambda J_\ell' - J_\ell]}{\lambda^3 I_\ell' - (1-v)\ell^2[\lambda I_\ell' - I_\ell]}$
	Real-part BEM	or $\lambda(1-v)[-4\ell^2(\ell-1)I_\ell J_\ell - 2\lambda^2 I_{\ell+1}J_{\ell+1}] + 2\ell\lambda^2(1-v)(1-\ell)(I_{\ell+1}J_\ell - I_\ell J_{\ell+1}) + [\ell^2(1-v)^2(\ell^2 - -1) + \lambda^4](I_{\ell+1}J_\ell + I_\ell J_{\ell+1}) = 0$

$\ell = 0, \pm 1, \pm 2, \ldots$

$$[K_{n+1}(\lambda a)Y_n(\lambda a) - Y_{n+1}(\lambda a)K_n(\lambda a)] \times \{(1-\nu)I_n(\lambda a)J_{n+1}(\lambda a) + I_{n+1}(\lambda a)J_n(\lambda a) - 2\lambda a I_n(\lambda a)J_n(\lambda a)\} = 0 \quad (54)$$

The former part in Eq. (54) inside the middle bracket is the spurious eigenequation while the latter part inside the big bracket is found to be the true eigenequation after comparing with the exact eigenequation [24, 30].

Case 3. Free circular plate

For the free circular plate ($m = 0$ and $\nu = 0$) with a radius a, we can obtain the eigenequation in the continuous formulation. Similarly, the displacement and slope, $u(s)$ and $\theta(s)$, can be expanded into Fourier series by

$$u(s) = p_0^f + \sum_{n=1}^{\infty}(p_n^f \cos(n\overline{\phi}) + q_n^f \sin(n\overline{\phi})), \quad s \in B, \quad (55)$$

$$\theta(s) = a_0^f + \sum_{n=1}^{\infty}(a_n^f \cos(n\overline{\phi}) + b_n^f \sin(n\overline{\phi})), \quad s \in B, \quad (56)$$

where the superscript "f" denotes the free case, $\overline{\phi}$ is the angle on the circular boundary, a_n^f, b_n^f, p_n^f and q_n^f are the undetermined Fourier coefficients. Substituting Eqs. (55) and (56) and using the degenerate kernels of $M(s,x)$, $V(s,x)$, $M_\theta(s,x)$ and $V_\theta(s,x)$ into Eq. (25), we have

$$p_n^f = -\frac{[Y_n(\lambda a)\alpha_n^J(\lambda a) - \frac{2}{\pi}K_n(\lambda a)\alpha_n^I(\lambda a)]}{[Y_n(\lambda a)\beta_n^J(\lambda a) - \frac{2}{\pi}K_n(\lambda a)\beta_n^I(\lambda a) + \frac{1-\nu}{a}[Y_n(\lambda a)\gamma_n^J(\lambda a) - \frac{2}{\pi}K_n(\lambda a)\gamma_n^I(\lambda a)]]}a_n^f, \quad n = 0,1,2,\ldots, \quad (57)$$

$$q_n^f = -\frac{[Y_n(\lambda a)\alpha_n^J(\lambda a) - \frac{2}{\pi}K_n(\lambda a)\alpha_n^I(\lambda a)]}{[Y_n(\lambda a)\beta_n^J(\lambda a) - \frac{2}{\pi}K_n(\lambda a)\beta_n^I(\lambda a) + \frac{1-\nu}{a}[Y_n(\lambda a)\gamma_n^J(\lambda a) - \frac{2}{\pi}K_n(\lambda a)\gamma_n^I(\lambda a)]]}b_n^f, \quad n = 0,1,2,\ldots. \quad (58)$$

where

$$\beta_n^J(\lambda a) = \lambda^3 J_n'''(\lambda a) + \nu[\frac{1}{a}\lambda^2 J_n''(\lambda a) - (\frac{n}{a})^2 \lambda J_n'(\lambda a) - \frac{1}{a^2}\lambda J_n'(\lambda a) + (\frac{2n^2}{a^3})J_n(\lambda a)], \quad (59)$$

$$\beta_n^I(\lambda a) = \lambda^3 I_n'''(\lambda a) + \nu[\frac{1}{a}\lambda^2 I_n''(\lambda a) - (\frac{n}{a})^2 \lambda I_n'(\lambda a) - \frac{1}{a^2}\lambda I_n'(\lambda a) + (\frac{2n^2}{a^3})I_n(\lambda a)], \quad (60)$$

$$\gamma_n^J(\lambda a) = -n^2[\frac{1}{a^2}J_n(\lambda a) + \frac{\lambda}{a}J'(\lambda a)], \quad (61)$$

$$\gamma_n^I(\lambda a) = -n^2[\frac{1}{a^2}I_n(\lambda a) + \frac{\lambda}{a}I'(\lambda a)]. \quad (62)$$

Similarly, Eq. (26) yields,

$$p_n^f = -\frac{[Y_n'(\lambda a)J_n(\lambda a) - \frac{2}{\pi}K_n'(\lambda a)I_n(\lambda a)]}{[Y_n'(\lambda a)\beta_n^J(\lambda a) - \frac{2}{\pi}K_n'(\lambda a)\beta_n^I(\lambda a) + \frac{1-\nu}{a}[Y_n'(\lambda a)\gamma_n^J(\lambda a) - \frac{2}{\pi}K_n'(\lambda a)\gamma_n^I(\lambda a)]]}a_n^f, \quad n = 0,1,2,\ldots, \quad (63)$$

$$q_n^f = -\frac{[Y_n'(\lambda a)J_n(\lambda a) - \frac{2}{\pi}K_n'(\lambda a)I_n(\lambda a)]}{[Y_n'(\lambda a)\beta_n^J(\lambda a) - \frac{2}{\pi}K_n'(\lambda a)\beta_n^I(\lambda a) + \frac{1-\nu}{a}[Y_n'(\lambda a)\gamma_n^J(\lambda a) - \frac{2}{\pi}K_n'(\lambda a)\gamma_n^I(\lambda a)]]}b_n^f, \quad n = 0,1,2,\ldots. \quad (64)$$

To seek nontrivial data for the generalized coefficients of a_n^f, p_n^f, b_n^f and q_n^f, we can obtain the eigenequation

$$\frac{[Y_n(\lambda a)\alpha_n^J(\lambda a) - \frac{2}{\pi}K_n(\lambda a)\alpha_n^I(\lambda a)]}{[Y_n(\lambda a)\beta_n^J(\lambda a) - \frac{2}{\pi}K_n(\lambda a)\beta_n^I(\lambda a) + \frac{1-\nu}{a}[Y_n(\lambda a)\gamma_n^J(\lambda a) - \frac{2}{\pi}K_n(\lambda a)\gamma_n^I(\lambda a)]]} = \frac{[Y_n'(\lambda a)J_n(\lambda a) - \frac{2}{\pi}K_n'(\lambda a)I_n(\lambda a)]}{[Y_n'(\lambda a)\beta_n^J(\lambda a) - \frac{2}{\pi}K_n'(\lambda a)\beta_n^I(\lambda a) + \frac{1-\nu}{a}[Y_n'(\lambda a)\gamma_n^J(\lambda a) - \frac{2}{\pi}K_n'(\lambda a)\gamma_n^I(\lambda a)]]}. \quad (65)$$

After recollecting the terms, Eq. (65) can be simplified to

$$[K_{n+1}(\lambda a)Y_n(\lambda a) - Y_{n+1}(\lambda a)K_n(\lambda a)]\{\lambda a(1-\nu) \times [-4n^2(n-1)I_n(\lambda a)J_n(\lambda a) - 2\lambda^2 a^2 I_{n+1}(\lambda a)J_{n+1}(\lambda a)] + 2n\lambda^2 a^2(1-\nu)(1-n)(I_{n+1}(\lambda a)J_n(\lambda a) - I_n(\lambda a)J_{n+1}(\lambda a)) + [n^2(1-\nu)^2(n^2-1) + \lambda^4 a^4](I_{n+1}(\lambda a)J_n(\lambda a) + I_n(\lambda a)J_{n+1}(\lambda a))\} = 0 \quad (66)$$

The former part in Eq. (66) inside the middle bracket is the spurious eigenequation which also appears in the clamped and simply-supported cases. It indicates that the spurious eigenequations of Eqs. (44), (54) and (66) are the same since the same formulation (null-field integral formulation of Eqs. (25) and (26)) is used. This reconfirms that spurious eigenequation depends on the formulation instead of the specified boundary condition. It is noted that the true eigenequation of free plate does not agree with that of the Leissa result [30] as shown in Table 1. An original paper can also be consulted [24]. However, the same true eigenvalues are obtained numerically between the present and Leissa's results. After finding the eigenvalues according to the Leissa's eigenequation, the eigenvalues are not consistent in his book. The possible explanation is that the eigenequation in the Leissa's book for the free case was wrongly typed [19].

3.2 Mathematical analysis in the discrete system by using the degenerate kernel and circulants

Case 1. Clamped circular plate

For the clamped circular plate ($u = 0$ and $\theta = 0$) with a radius a, Eqs. (29) and (30) can be rewritten as

$$\{0\} = [U]\{v\} + [\Theta]\{m\} \ , \tag{67}$$

$$\{0\} = [U_\theta]\{v\} + [\Theta_\theta]\{m\} \ , \tag{68}$$

By assembling Eqs. (67) and (68) together, we have

$$[SM^c]\begin{Bmatrix} v \\ m \end{Bmatrix} = \{0\} \ , \tag{69}$$

where the superscript "c" denotes the clamped case and

$$[SM^c] = \begin{bmatrix} U & \Theta \\ U_\theta & \Theta_\theta \end{bmatrix}_{4N\times 4N} . \tag{70}$$

For the existence of nontrivial solution of $\begin{Bmatrix} v \\ m \end{Bmatrix}$, the determinant of the matrix versus eigenvalue must be zero, i.e.,

$$\det[SM^c] = 0. \tag{71}$$

Since the rotation symmetry is preserved for a circular boundary, the influence matrices for the discrete system are found to be circulants with the following forms into Eq. (67), we have

$$[U] = \begin{bmatrix} z_0 & z_1 & z_2 & \cdots & z_{2N-1} \\ z_{2N-1} & z_0 & z_1 & \cdots & z_{2N-2} \\ z_{2N-2} & z_{2N-1} & z_0 & \cdots & z_{2N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_1 & z_2 & z_3 & \cdots & z_0 \end{bmatrix}_{2N\times 2N} \tag{72}$$

The coefficients of each element can be obtained by using degenerate kernel

$$\begin{aligned} z_m &= \int_{(m-\frac{1}{2})\Delta\overline{\phi}}^{(m+\frac{1}{2})\Delta\overline{\phi}} [-U(a,\overline{\phi};a,\phi)]ad\overline{\phi} \\ &\approx [-U(a,\overline{\phi}_m;a,\phi)]a\Delta\overline{\phi}, \\ & m = 0,1,2,\ldots,2N-1 \ . \end{aligned} \tag{73}$$

where $\Delta\overline{\phi} = \frac{2\pi}{2N}$, $\overline{\phi}_m = m\Delta\overline{\phi}$. By introducing the following bases for circulants, $[I]$, $([C_{2N}])^1$, $([C_{2N}])^2$, $\cdots$, $([C_{2N}])^{2N-1}$, we can expand matrix $[U]$ into

$$\begin{aligned} [U] &= z_0[I] + z_1([C_{2N}])^1 + z_2([C_{2N}])^2 \\ &\quad + \cdots + z_{2N-1}([C_{2N}])^{2N-1}, \end{aligned} \tag{74}$$

where

$$[C_{2N}] = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}_{2N\times 2N} . \tag{75}$$

Based on the similar properties for the matrices of $[U]$ and $[C_{2N}]$, we have

$$\begin{aligned} \mu_\ell^{[U]} &= z_0 + z_1\alpha_\ell + z_2\alpha_\ell^2 + \cdots + z_{2N-1}\alpha_\ell^{2N-1}, \\ & \ell = 0,1,2,\cdots,2N-1 \ . \end{aligned} \tag{76}$$

where $\mu_\ell^{[U]}$ and α_ℓ are the eigenvalues for $[U]$ and $[C_{2N}]$, respectively. It is easily found that the eigenvalues for the circulants $[C_{2N}]$, are the roots for $\alpha^{2N} = 1$ as shown below:

$$\begin{aligned} \alpha_\ell &= e^{i\frac{2\pi\ell}{2N}}, \quad \ell = 0,\pm1,\pm2,\cdots,\pm(N-1),N \text{ or} \\ & \ell = 0,1,2,\cdots,2N-1 \ . \end{aligned} \tag{77}$$

The eigenvector for the circulant $[C_{2N}]$ is

$$\{\phi_\ell\} = \begin{Bmatrix} 1 \\ \alpha_\ell \\ \alpha_\ell^2 \\ \vdots \\ \alpha_\ell^{2N-1} \end{Bmatrix}_{2N\times 1} . \tag{78}$$

Substituting Eq. (77) into Eq. (76), we have

$$\begin{aligned} \mu_\ell^{[U]} &= \sum_{m=0}^{2N-1} z_m\alpha_\ell^m = \sum_{m=0}^{2N-1} z_m e^{i\frac{2\pi}{2N}m\ell}, \\ & \ell = 0,\pm1,\pm2,\cdots,\pm(N-1),N. \end{aligned} \tag{79}$$

According to the definition for z_m in Eq. (73), we have

$$z_m = z_{2N-m}, \quad m = 0,1,2,\cdots,2N-1 \ . \tag{80}$$

Substitution of Eq. (80) into Eq. (79) yields

$$\begin{aligned} \mu_\ell^{[U]} &= z_0 + (-1)^\ell z_N + \sum_{m=1}^{N-1}(\alpha_\ell^m + \alpha_\ell^{2N-m})z_m \\ &= \sum_{m=0}^{2N-1} \cos(m\ell\Delta\overline{\phi})z_m \ . \end{aligned} \tag{81}$$

Substituting Eq. (73) into Eq. (81) for $\phi = 0$ without loss of generality, the Reimann sum of infinite terms reduces to the following integral

$$\mu_\ell^{[U]} = \lim_{N\to\infty} \sum_{m=0}^{2N-1} \cos(m\ell\Delta\overline{\phi})[-U(a,\overline{\phi}_m;a,0)]\, a\Delta\overline{\phi}$$
$$\approx \int_0^{2\pi} \cos(\ell\overline{\phi})[-U(a,\overline{\phi}_m;a,0)]ad\overline{\phi} \ , \qquad (82)$$

By using the degenerate kernel for $U(s,x)$ and the orthogonal conditions, Eq. (82) reduces to

$$\mu_\ell^{[U]} = -\frac{\pi a}{4\lambda^2}[Y_\ell(\lambda a)J_\ell(\lambda a) - \frac{2}{\pi}K_\ell(\lambda a)I_\ell(\lambda a)],$$
$$\ell = 0, \pm1, \pm2, \cdots, \pm(N-1), N \ . \qquad (83)$$

Similarly, we have

$$\mu_\ell^{[\Theta]} = \frac{\pi a}{4\lambda}[Y_\ell(\lambda a)J'_\ell(\lambda a) - \frac{2}{\pi}K_\ell(\lambda a)I'_\ell(\lambda a)],$$
$$\ell = 0, \pm1, \pm2, \cdots, \pm(N-1), N \ . \qquad (84)$$

$$\kappa_\ell^{[U]} = -\frac{\pi a}{4\lambda}[Y'_\ell(\lambda a)J_\ell(\lambda a) - \frac{2}{\pi}K'_\ell(\lambda a)I_\ell(\lambda a)],$$
$$\ell = 0, \pm1, \pm2, \cdots, \pm(N-1), N \ . \qquad (85)$$

$$\kappa_\ell^{[\Theta]} = \frac{\pi a}{4}[Y'_\ell(\lambda a)J'_\ell(\lambda a) - \frac{2}{\pi}K'_\ell(\lambda a)I'_\ell(\lambda a)],$$
$$\ell = 0, \pm1, \pm2, \cdots, \pm(N-1), N \ . \qquad (86)$$

where $\mu_\ell^{[\Theta]}$, $\kappa_\ell^{[U]}$ and $\kappa_\ell^{[\Theta]}$ are the eigenvalues of $[\Theta]$, $[U_\theta]$ and $[\Theta_\theta]$ matrices, respectively. Since the four matrices $[U]$, $[\Theta]$, $[U_\theta]$ and $[\Theta_\theta]$ are all symmetric circulants, they can be expressed by

$$[U] = \Phi\Sigma_U\Phi^{-1}$$
$$= \Phi \begin{bmatrix} \mu_0^{[U]} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \mu_1^{[U]} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \mu_{-1}^{[U]} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mu_{(N-1)}^{[U]} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \mu_{-(N-1)}^{[U]} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \mu_N^{[U]} \end{bmatrix}_{2N\times2N} \Phi^{-1} \ , \qquad (87)$$

$$[\Theta] = \Phi\Sigma_\Theta\Phi^{-1}$$
$$= \Phi \begin{bmatrix} \mu_0^{[\Theta]} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \mu_1^{[\Theta]} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \mu_{-1}^{[\Theta]} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mu_{(N-1)}^{[\Theta]} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \mu_{-(N-1)}^{[\Theta]} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \mu_N^{[\Theta]} \end{bmatrix}_{2N\times2N} \Phi^{-1} \ , \qquad (88)$$

$$[U_\theta] = \Phi\Sigma_{U_\theta}\Phi^{-1}$$
$$= \Phi \begin{bmatrix} \kappa_0^{[U]} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \kappa_1^{[U]} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \kappa_{-1}^{[U]} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \kappa_{(N-1)}^{[U]} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \kappa_{-(N-1)}^{[U]} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \kappa_N^{[U]} \end{bmatrix}_{2N\times2N} \Phi^{-1} \ , \qquad (89)$$

$$[\Theta_\theta] = \Phi\Sigma_{\Theta_\theta}\Phi^{-1}$$
$$= \Phi \begin{bmatrix} \kappa_0^{[\Theta]} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \kappa_1^{[\Theta]} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \kappa_{-1}^{[\Theta]} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \kappa_{(N-1)}^{[\Theta]} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \kappa_{-(N-1)}^{[\Theta]} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \kappa_N^{[\Theta]} \end{bmatrix}_{2N\times2N} \Phi^{-1} \ , \qquad (90)$$

where
By employing Eqs. (87)–(90) for Eq. (70), we have

$$\Phi = \frac{1}{\sqrt{2N}} \begin{bmatrix} 1 & 1 & 0 & \cdots & 1 & 0 & 1 \\ 1 & \cos(\frac{2\pi}{2N}) & \sin(\frac{2\pi}{2N}) & \cdots & \cos(\frac{2\pi(N-1)}{2N}) & \sin(\frac{2\pi(N-1)}{2N}) & \cos(\frac{2\pi N}{2N}) \\ 1 & \cos(\frac{4\pi}{2N}) & \sin(\frac{4\pi}{2N}) & \cdots & \cos(\frac{4\pi(N-1)}{2N}) & \sin(\frac{4\pi(N-1)}{2N}) & \cos(\frac{4\pi N}{2N}) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & \cos(\frac{2\pi(2N-2)}{2N}) & \sin(\frac{2\pi(2N-2)}{2N}) & \cdots & \cos(\frac{\pi(4N-4)(N-1)}{2N}) & \sin(\frac{\pi(4N-4)(N-1)}{2N}) & \cos(\frac{\pi(4N-4)(N)}{2N}) \\ 1 & \cos(\frac{2\pi(2N-1)}{2N}) & \sin(\frac{2\pi(2N-1)}{2N}) & \cdots & \cos(\frac{\pi(4N-2)(N-1)}{2N}) & \sin(\frac{\pi(4N-2)(N-1)}{2N}) & \cos(\frac{\pi(4N-2)(N)}{2N}) \end{bmatrix}_{2N\times 2N} \quad (91) \;.$$

$$[SM^c] = \begin{bmatrix} \Phi\Sigma_U\Phi^{-1} & \Phi\Sigma_\Theta\Phi^{-1} \\ \Phi\Sigma_{U_\theta}\Phi^{-1} & \Phi\Sigma_{\Theta_\theta}\Phi^{-1} \end{bmatrix}_{4N\times 4N} , \quad (92)$$

Eq. (92) can be reformulated into

$$[SM^c] = \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix} \begin{bmatrix} \Sigma_U & \Sigma_\Theta \\ \Sigma_{U_\theta} & \Sigma_{\Theta_\theta} \end{bmatrix} \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix}^{-1} . \quad (93)$$

Since Φ is orthogonal ($\det[\Phi] = \det[\Phi^{-1}] = 1$), the determinant of $[SM^c]_{4N\times 4N}$ is

$$\det[SM^c] = \det \begin{bmatrix} \Sigma_U & \Sigma_\Theta \\ \Sigma_{U_\theta} & \Sigma_{\Theta_\theta} \end{bmatrix} = \prod_{\ell=-(N-1)}^{N} (\mu_\ell^{[U]}\kappa_\ell^{[\Theta]} - \mu_\ell^{[\Theta]}\kappa_\ell^{[U]}) \;, \quad (94)$$

By employing Eqs. (83)–(86) for Eq. (94), we have

$$\begin{aligned} \det[SM^c] = \prod_{\ell=-(N-1)}^{N} \frac{\pi^2 a^2}{16\lambda^2} \{ & [Y_\ell(\lambda a) J_\ell(\lambda a) \\ & - \frac{2}{\pi} K_\ell(\lambda a) I_\ell(\lambda a)][Y_\ell'(\lambda a) J_\ell'(\lambda a) \\ & - \frac{2}{\pi} K_\ell'(\lambda a) I_\ell'(\lambda a)] - [Y_\ell(\lambda a) J_\ell'(\lambda a) \\ & - \frac{2}{\pi} K_\ell(\lambda a) I_\ell'(\lambda a)][Y_\ell'(\lambda a) J_\ell(\lambda a) \\ & - \frac{2}{\pi} K_\ell'(\lambda a) I_\ell(\lambda a)] \} \end{aligned} \quad (95)$$

Eq. (95) can be simplified into

$$\begin{aligned} & \det[SM^c] \\ & = \prod_{\ell=-(N-1)}^{N} \frac{\pi a^2}{8\lambda^2} [K_{\ell+1}(\lambda a) Y_\ell(\lambda a) - Y_{\ell+1}(\lambda a) K_\ell(\lambda a)] \\ & \times \{ I_{\ell+1}(\lambda a) J_\ell(\lambda a) + J_{\ell+1}(\lambda a) I_\ell(\lambda a) \} = 0 \end{aligned} \quad (96)$$

Zero determinant in Eq. (96) implies that the eigenequation is

$$\begin{aligned} & [K_{\ell+1}(\lambda a) Y_\ell(\lambda a) - Y_{\ell+1}(\lambda a) K_\ell(\lambda a)] \\ & \quad \times \{ I_{\ell+1}(\lambda a) J_\ell(\lambda a) + J_{\ell+1}(\lambda a) I_\ell(\lambda a) \} = 0, \\ & \quad \ell = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N \;. \end{aligned} \quad (97)$$

After comparing with the analytical solution for the clamped circular plate [30], the true eigenequation for a continuous system can be obtained by approaching N in the the discrete system to infinity. The former part in Eq. (97) inside the middle bracket is the spurious eigenequation while the latter part inside the big bracket is found to be the true eigenequation. The result of Eq. (97) in the discrete system matches well with Eq. (44) in the continuous system.

Case 2. Simply-supported circular plate

For the simply-supported circular plate ($u = 0$ and $m = 0$) with a radius a, we have

$$[SM^s] = \begin{bmatrix} U & M \\ U_\theta & M_\theta \end{bmatrix}_{4N\times 4N} , \quad (98)$$

where the superscript "s" denotes the simply-supported case. Since the rotation symmetry is preserved for a circular boundary, the eigenvalues of the influence matrices for the discrete system can be found by using circulants, we have

$$\begin{aligned} \mu_\ell^{[M]} &= -\frac{\pi a}{4\lambda^2} [Y_\ell(\lambda a) \alpha_\ell^J(\lambda a) - \frac{2}{\pi} K_\ell(\lambda a) \alpha_\ell^I(\lambda a)], \\ & \quad \ell = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N \;. \end{aligned} \quad (99)$$

$$\begin{aligned} \kappa_\ell^{[M]} &= -\frac{\pi a}{4\lambda} [Y_\ell'(\lambda a) \alpha_\ell^J(\lambda a) - \frac{2}{\pi} K_\ell'(\lambda a) \alpha_\ell^I(\lambda a)], \\ & \quad \ell = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N \;. \end{aligned} \quad (100)$$

where $\mu_\ell^{[M]}$ and $\kappa_\ell^{[M]}$ are the eigenvalues of $[M]$ and $[M_\theta]$ matrices, respectively. Since the two matrices $[M]$ and $[M_\theta]$ are all symmetric circulants, they can be expressed by

$$[M] = \Phi \Sigma_M \Phi^T \;, \quad (101)$$

$$[M_\theta] = \Phi \Sigma_{M_\theta} \Phi^T \;. \quad (102)$$

By employing Eqs. (87), (89), (101) and (102) for Eq. (98), we have

$$[SM^s] = \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix} \begin{bmatrix} \Sigma_U & \Sigma_M \\ \Sigma_{U_\theta} & \Sigma_{M_\theta} \end{bmatrix} \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix}^T . \quad (103)$$

Since Φ is orthogonal, the determinant of $[SM^s]_{4N\times 4N}$ is

$$\det[SM^s] = \det \begin{bmatrix} \Sigma_U & \Sigma_M \\ \Sigma_{U_\theta} & \Sigma_{M_\theta} \end{bmatrix} = \prod_{\ell=-(N-1)}^{N} (\mu_\ell^{[U]}\kappa_\ell^{[M]} - \mu_\ell^{[M]}\kappa_\ell^{[U]}) \;, \quad (104)$$

By employing Eqs. (83), (85), (99) and (100) for Eq. (104), we have

$$\det[SM^s] = \prod_{\ell=-(N-1)}^{N} \frac{\pi^2 a^2}{16\lambda^3}$$
$$\times \{[Y_\ell(\lambda a)J_\ell(\lambda a) - \frac{2}{\pi}K_\ell(\lambda a)I_\ell(\lambda a)]$$
$$\times [Y'_\ell(\lambda a)\alpha^J_\ell(\lambda a) - \frac{2}{\pi}K'_\ell(\lambda a)\alpha^I_\ell(\lambda a)]$$
$$- [Y_\ell(\lambda a)\alpha^J_\ell(\lambda a) - \frac{2}{\pi}K_\ell(\lambda a)\alpha^I_\ell(\lambda a)]$$
$$\times [Y'_\ell(\lambda a)J_\ell(\lambda a) - \frac{2}{\pi}K'_\ell(\lambda a)I_\ell(\lambda a)]\} \quad (105)$$

Eq. (144) can be simplified into

$$\det[SM^s] = \prod_{\ell=-(N-1)}^{N} \frac{\pi a}{8\lambda^2}$$
$$\times [K_{\ell+1}(\lambda a)Y_\ell(\lambda a) - Y_{\ell+1}(\lambda a)K_\ell(\lambda a)]$$
$$\times \{(1-\nu)I_\ell(\lambda a)J_{n+1}(\lambda a) + I_{n+1}(\lambda a)J_\ell(\lambda a)$$
$$- 2\lambda a I_\ell(\lambda a)J_\ell(\lambda a)\} = 0 \ . \quad (106)$$

Zero determinant in Eq. (106) implies that the eigenequation is

$$[K_{\ell+1}(\lambda a)Y_\ell(\lambda a) - Y_{\ell+1}(\lambda a)K_\ell(\lambda a)]$$
$$\times \{(1-\nu)I_\ell(\lambda a)J_{n+1}(\lambda a) + I_{n+1}(\lambda a)J_\ell(\lambda a)$$
$$- 2\lambda a I_\ell(\lambda a)J_\ell(\lambda a)\} = 0,$$
$$\ell = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N \ . \quad (107)$$

After comparing with the analytical solution for the simply-supported circular plate [30], the true eigenequation for a continuous system can be obtained by approaching N in the discrete system to infinity. The former part in Eq. (107) inside the middle bracket is the spurious eigenequation while the latter part inside the big bracket is found to be the true eigenequation. The result of Eq. (107) in the discrete system match well with Eq. (54) in the continuous system.

Case 3. Free circular plate

For the free circular plate ($m = 0$ and $v = 0$) with a radius a, we have

$$[SM^f] = \begin{bmatrix} M & V \\ M_\theta & V_\theta \end{bmatrix}_{4N\times 4N} , \quad (108)$$

where the superscript "f" denotes the free case. Since the rotation symmetry is preserved for a circular boundary, the eigenvalues of the influence matrices for the discrete system can be found by using circulants, we have

$$\mu_\ell^{[V]} = -\frac{\pi a}{4\lambda^2}[Y_\ell(\lambda a)\beta^J_\ell(\lambda a) - \frac{2}{\pi}K_\ell(\lambda a)\beta^I_\ell(\lambda a)$$
$$+ \frac{1-\nu}{a}[Y_\ell(\lambda a)\gamma^J_\ell(\lambda a) - \frac{2}{\pi}K_\ell(\lambda a)\gamma^I_n(\lambda a)],$$
$$\ell = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N \ . \quad (109)$$

$$\kappa_\ell^{[V]} = -\frac{\pi a}{4\lambda}[Y'_\ell(\lambda a)\beta^J_\ell(\lambda a) - \frac{2}{\pi}K'_\ell(\lambda a)\beta^I_\ell(\lambda a)$$
$$+ \frac{1-\nu}{a}[Y'_\ell(\lambda a)\gamma^J_\ell(\lambda a) - \frac{2}{\pi}K'_\ell(\lambda a)\gamma^I_n(\lambda a)],$$
$$\ell = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N \ . \quad (110)$$

where $\mu_\ell^{[V]}$ and $\kappa_\ell^{[V]}$ are the eigenvalues of $[V]$ and $[V_\theta]$ matrices, respectively. Since the two matrices $[V]$ and $[V_\theta]$ are all symmetric circulants, they can be expressed by

$$[V] = \Phi\Sigma_V\Phi^T \ , \quad (111)$$
$$[V_\theta] = \Phi\Sigma_{V_\theta}\Phi^T \ , \quad (112)$$

By employing Eqs. (101), (102), (111) and (112) for Eq. (108), we have

$$[SM^f] = \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix}\begin{bmatrix} \Sigma_M & \Sigma_V \\ \Sigma_{M_\theta} & \Sigma_{V_\theta} \end{bmatrix}\begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix}^T . \quad (113)$$

Since Φ is orthogonal, the determinant of $[SM^f]_{4N\times 4N}$ is

$$\det[SM^f] = \det\begin{bmatrix} \Sigma_U & \Sigma_M \\ \Sigma_{U_\theta} & \Sigma_{M_\theta} \end{bmatrix}$$
$$= \prod_{\ell=-(N-1)}^{N} (\mu_\ell^{[M]}\kappa_\ell^{[V]} - \mu_\ell^{[V]}\kappa_\ell^{[M]}) \ , \quad (114)$$

Table 2. Spurious eigenequations in the six formulations by using the real-part BEM

	Spurious eigenequations for the real-part BEM
u, θ formulation Eq. (2) and Eq. (3)	$K_{\ell+1}Y_\ell - K_\ell Y_{\ell+1} = 0$
u, m formulation Eq. (2) and Eq. (4)	$(1-\nu)(K_\ell Y_{\ell+1} - K_{\ell+1}Y_\ell) - 2\lambda a K_\ell Y_\ell = 0$
u, v formulation Eq. (2) and Eq. (5)	$\ell^2(1-\nu)(K_\ell Y_{\ell+1} - K_{\ell+1}Y_\ell) - 2\lambda a\ell K_\ell Y_\ell$ $+ \lambda^2 a^2(K_{\ell+1}Y_\ell + K_\ell Y_{\ell+1}) = 0$
θ, m formulation Eq. (3) and Eq. (4)	$\ell^2(1-\nu)(K_\ell Y_{\ell+1} - K_{\ell+1}Y_\ell) - 2\lambda a\,\ell\, K_\ell Y_\ell$ $+ \lambda^2 a^2(K_{\ell+1}Y_\ell + K_\ell Y_{\ell+1}) = 0$
θ, v formulation Eq. (3) and Eq. (5)	$2\lambda a(\ell^2 K_\ell Y_\ell + \lambda^2\rho^2 K_{\ell+1}Y_{\ell+1}) - 2\lambda^2 a^2\,\ell\,(K_{\ell+1}Y_\ell + K_\ell Y_{\ell+1})$ $+\ \ell^2\,(1-\nu)(K_{\ell+1}Y_\ell - K_\ell Y_{\ell+1}) = 0$
m, v formulation Eq. (4) and Eq. (5)	$\lambda a(1-\nu)[4\ell^2(\ell-1)K_\ell Y_\ell - 2\lambda^2 a^2 K_{\ell+1}Y_{\ell+1}]$ $+ 2\ell\lambda^2 a^2(1-\nu)(1-\ell)\,(K_{\ell+1}Y_\ell + K_\ell Y_{\ell+1})$ $+\ [\ell^2\,(1-\nu)^2(\ell^2-1) + \lambda^4 a^4](K_{\ell+1}Y_\ell - K_\ell Y_{\ell+1}) = 0$

where $\ell = 0, \pm 1, \pm 2, \pm 3, \ldots$

By employing Eqs. (99), (100), (109) and (110) for Eq. (114), we have

$$\det[SM^f] = \prod_{\ell=-(N-1)}^{N} \frac{\pi^2 a^2}{16\lambda^3} \times \{[Y_\ell(\lambda a)J_\ell(\lambda a) - \frac{2}{\pi}K_\ell(\lambda a)I_\ell(\lambda a)] \times [Y'_\ell(\lambda a)\alpha^J_\ell(\lambda a) - \frac{2}{\pi}K'_\ell(\lambda a)\alpha^I_\ell(\lambda a)] - [Y_\ell(\lambda a)\alpha^J_\ell(\lambda a) - \frac{2}{\pi}K_\ell(\lambda a)\alpha^I_\ell(\lambda a)] \times [Y'_\ell(\lambda a)J_\ell(\lambda a) - \frac{2}{\pi}K'_\ell(\lambda a)I_\ell(\lambda a)]\} \quad . \quad (115)$$

Eq. (115) can be simplified into

$$\det[SM^f] = \prod_{\ell=-(N-1)}^{N} \frac{\pi a}{8\lambda^2} \times [K_{\ell+1}(\lambda a)Y_\ell(\lambda a) - Y_{\ell+1}(\lambda a)K_\ell(\lambda a)] \times \{\lambda a(1-\nu)[-4\ell^2(\ell-1)]I_\ell(\lambda a)J_\ell(\lambda a) - 2\lambda^2a^2 I_{\ell+1}(\lambda a)J_{\ell+1}(\lambda a)] + 2\ell\lambda^2a^2(1-\nu)(1-\ell)(I_{\ell+1}(\lambda a)J_\ell(\lambda a) - I_\ell(\lambda a)J_{\ell+1}(\lambda a)) + [\ell^2(1-\nu)^2(\ell^2-1) + \lambda^4a^4](I_{\ell+1}(\lambda a)J_\ell(\lambda a) + I_\ell(\lambda a)J_{\ell+1}(\lambda a))\} = 0 \quad . \quad (116)$$

Zero determinant in Eq. (116) implies that the eigenequation is

$$[K_{\ell+1}(\lambda a)Y_\ell(\lambda a) - Y_{\ell+1}(\lambda a)K_\ell(\lambda a)] \times \{\lambda a(1-\nu)[-4\ell^2(\ell-1)I_\ell(\lambda a)J_\ell(\lambda a) - 2\lambda^2a^2 I_{\ell+1}(\lambda a)J_{\ell+1}(\lambda a)] + 2\ell\lambda^2a^2(1-\nu)(1-\ell)(I_{\ell+1}(\lambda a)J_\ell(\lambda a) - I_\ell(\lambda a)J_{\ell+1}(\lambda a)) + [\ell^2(1-\nu)^2(\ell^2-1) + \lambda^4a^4] \times (I_{\ell+1}(\lambda a)J_\ell(\lambda a) + I_\ell(\lambda a)J_{\ell+1}(\lambda a))\} = 0 \quad \ell = 0, \pm1, \pm2, \cdots, \pm(N-1), N \quad . \quad (117)$$

After comparing with the analytical solution for the free circular plate [30], the true eigenequation for a continuous system can be obtained by approaching N in the discrete system to infinity. The former part in Eq. (117) inside the middle bracket is the spurious eigenequation while the latter part inside the big bracket is the true eigenequation. The result of Eq. (117) in the discrete system match well with Eq. (66) in the continuous system. After comparing Eq. (97) with Eqs. (107) and (117), the same spurious eigenequation ($[K_{\ell+1}(\lambda a)Y_\ell(\lambda a) - Y_{\ell+1}(\lambda a)K_\ell(\lambda a)] = 0$) is embedded in the u, θ formulation no matter what the boundary condition is.

Since any two equations in the plate formulation (Eqs. (29)–(32)) can be chosen, $6(C_2^4)$ options of the formulation can be considered. If we choose different formulae for either one of the the clamped, simply-supported or free circular plate cases, we can obtain the same true eigenequation but different spurious eigenequations. At the same time, either clamped or simply-supported circular plate results in the same spurious eigenequation, once we use the same formulation. The occurrence of spurious eigenequation only depends on the formulation instead of the specified boundary condition. True eigenequation depends on the specified boundary condition instead of the formulation. All the resluts are summarized in Table 2.

4 Extraction of the true eigenvalues using SVD updating technique in the discrete system

In this section, SVD updating technique is adopted to suppress the ocurrence of the spurious eigenvalues for the free vibration of plate problem. A clamped case is demonstrated analytically for the discrete system in the following subsection.

4.1 SVD updating technique

A conventional approach to detect the nonunique solution is the criterion of satisfying all Eqs. (29)–(32) at the same time. For the clamped plate ($u = 0$ and $\theta = 0$), the Eqs. (29)–(32) reduce to

$$0 = [U]\{v\} + [\Theta]\{m\} \quad , \quad (118)$$

$$0 = [U_\theta]\{v\} + [\Theta_\theta]\{m\} \quad , \quad (119)$$

$$0 = [U_m]\{v\} + [\Theta_m]\{m\} \quad , \quad (120)$$

$$0 = [U_v]\{v\} + [\Theta_v]\{m\} \quad , \quad (121)$$

After rearranging the terms, Eqs. (118) and (119) can be assembled to

$$[SM_1]\begin{Bmatrix} v \\ m \end{Bmatrix} = 0 \quad , \quad (122)$$

where

$$[SM_1] = \begin{bmatrix} U & \Theta \\ U_\theta & \Theta_\theta \end{bmatrix} \quad . \quad (123)$$

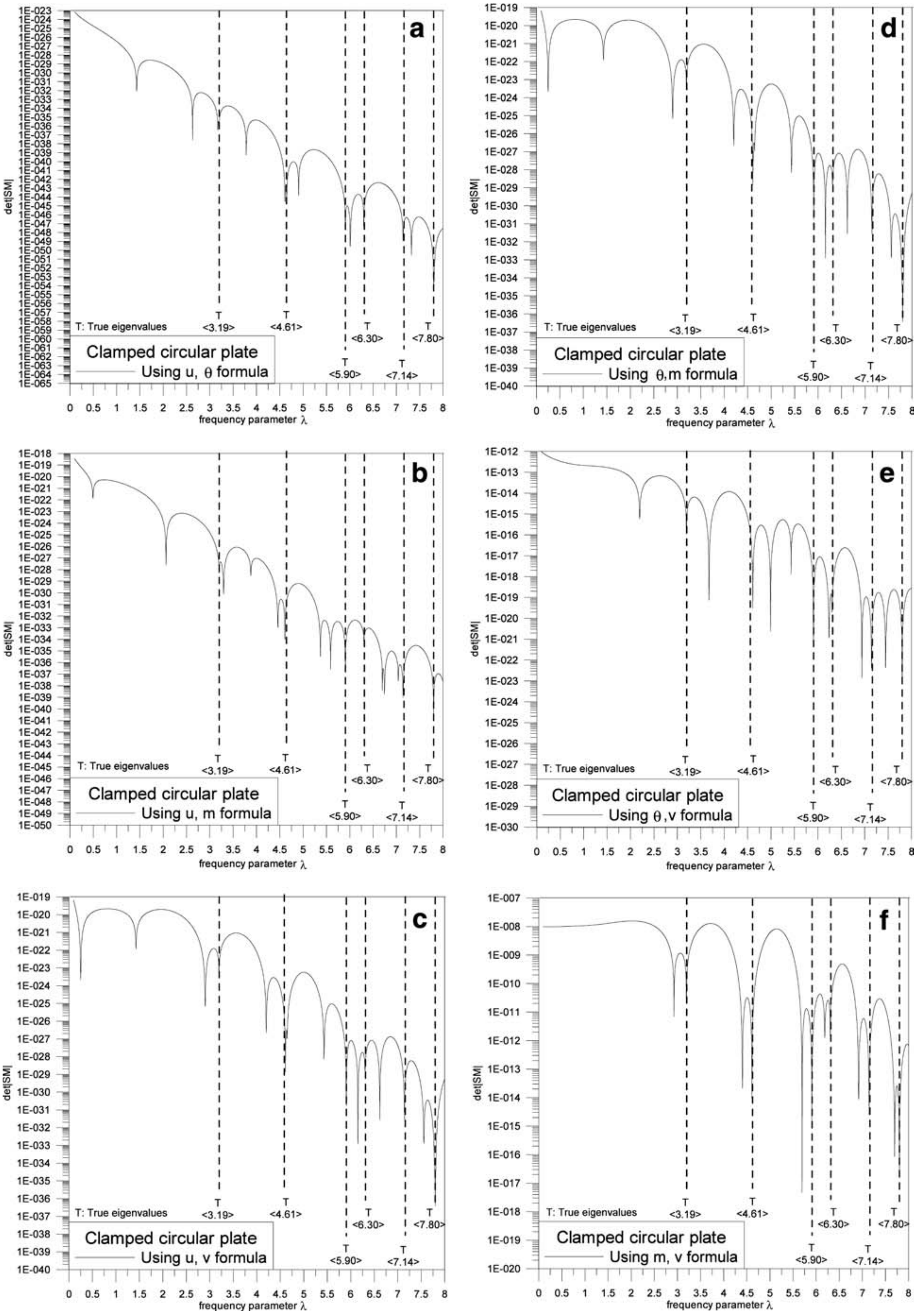
Similarly, Eqs. (120) and (121) yield

$$[SM_2]\begin{Bmatrix} v \\ m \end{Bmatrix} = 0 \quad , \quad (124)$$

where

$$[SM_2] = \begin{bmatrix} U_m & \Theta_m \\ U_v & \Theta_v \end{bmatrix} \quad . \quad (125)$$

Since the real-part BEM misses the imaginary-part information, we can reconstruct the independment equation by differentiation. To obtain an overdetermined system, we can combine Eqs. (122) and (124) by using the updating term,


a
b
c
d
e
f
det[SM]
frequency parameter λ
T: True eigenvalues
<3.19>
<4.61>
<5.90>
<6.30>
<7.14>
<7.80>
Clamped circular plate
Using u, θ formula
Using u, m formula
Using u, v formula
Using θ,m formula
Using θ,v formula
Using m, v formula

◀

Fig. 1. Logarithm curve for the $det[SM]$ versus frequency parameter λ for the clamped circular plate using the six formulations (a) u, θ formulation of Eq. (2) and Eq. (3) for the clamped case. (b) u, m formulation of Eq. (2) and Eq. (4) for the clamped case. (c) u, v formulation of Eq. (2) and Eq. (5) for the clamped case. (d) θ, m formulation of Eq. (3) and Eq. (4) for the clamped case. (e) θ, v formulation of Eq. (3) and Eq. (5) for the clamped case. (f) m, v formulation of Eq. (4) and Eq. (5) for the clamped case

$$[C]\begin{Bmatrix} v \\ m \end{Bmatrix} = 0 \tag{126}$$

where

$$[C] = \begin{bmatrix} SM_1 \\ SM_2 \end{bmatrix} \tag{127}$$

Since the eigenequation is nontrival, the rank of the matrix $[C]$ must be smaller then $4N$, the $4N$ singular values for the matrix $[C]$ must have at least one zero value. The explicit form for the matrix $[C]$ can be decomposed into

$$[C] = \begin{bmatrix} \Phi & 0 & 0 & 0 \\ 0 & \Phi & 0 & 0 \\ 0 & 0 & \Phi & 0 \\ 0 & 0 & 0 & \Phi \end{bmatrix} \begin{bmatrix} \Sigma_U & \Sigma_\Theta \\ \Sigma_{U_\theta} & \Sigma_{\Theta_\theta} \\ \Sigma_{U_m} & \Sigma_{\Theta_m} \\ \Sigma_{U_v} & \Sigma_{\Theta_v} \end{bmatrix} \begin{bmatrix} \Phi^T & 0 \\ 0 & \Phi^T \end{bmatrix} . \tag{128}$$

Based on the equivalence between the SVD technique and the least-squares method in mathematical essence. The least square form leads to

Table 3. The true eigenvalues (λ) for the clamped circular plate ($a = 1$)

m	λ for values of n of					
	0	1	2	3	4	5
1	3.19	4.61	5.90	7.14	8.34	9.52
2	6.30	7.80	9.19	10.53	11.83	13.10
3	9.44	10.95	12.40	13.79	15.15	16.47
4	12.57	14.10	15.58	17.00	18.39	19.75
5	15.71	17.25	18.74	20.19	21.60	22.99

where n refers to the number of nodal diameters and m is the number of nodal circles, not including the boundary circle

Table 4. The true eigenvalues (λ) for the simply-supported circular plate ($a = 1$) $v = 0.33$

m	λ for values of n of					
	0	1	2	3	4	5
1	2.23	3.73	5.06	6.32	7.54	8.73
2	5.45	6.96	8.37	9.72	11.03	12.31
3	8.61	10.14	11.59	12.98	14.34	15.67
4	11.76	13.29	14.77	16.20	17.59	18.96
5	14.90	16.45	17.94	19.39	20.81	22.20

where n refers to the number of nodal diameters and m is the number of nodal circles, not including the boundary circle

Table 5. The true eigenvalues (λ) for the free circular plate ($a = 1$) $v = 0.33$

m	λ for values of n of					
	0	1	2	3	4	5
1			2.29	3.50	4.64	5.75
2	3.012	4.53	5.93	7.27	8.56	9.82
3	6.20	7.73	9.18	10.57	11.93	13.25
4	9.37	10.91	12.38	13.80	15.19	16.55
5	12.52	14.06	15.55	17.00	18.42	19.81

where n refers to the number of nodal diameters and m is the number of nodal circles, not including the boundary circle

$$[C]^T[C] = \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix} [D]_{4N\times 4N} \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix}^T \tag{129}$$

where

$$[D] = \begin{bmatrix} \Sigma_U & \Sigma_{U_\theta} & \Sigma_{U_m} & \Sigma_{U_v} \\ \Sigma_\Theta & \Sigma_{\Theta_\theta} & \Sigma_{\Theta_m} & \Sigma_{\Theta_v} \end{bmatrix} \begin{bmatrix} \Sigma_U & \Sigma_\Theta \\ \Sigma_{U_\theta} & \Sigma_{\Theta_\theta} \\ \Sigma_{U_m} & \Sigma_{\Theta_m} \\ \Sigma_{U_v} & \Sigma_{\Theta_v} \end{bmatrix} \tag{130}$$

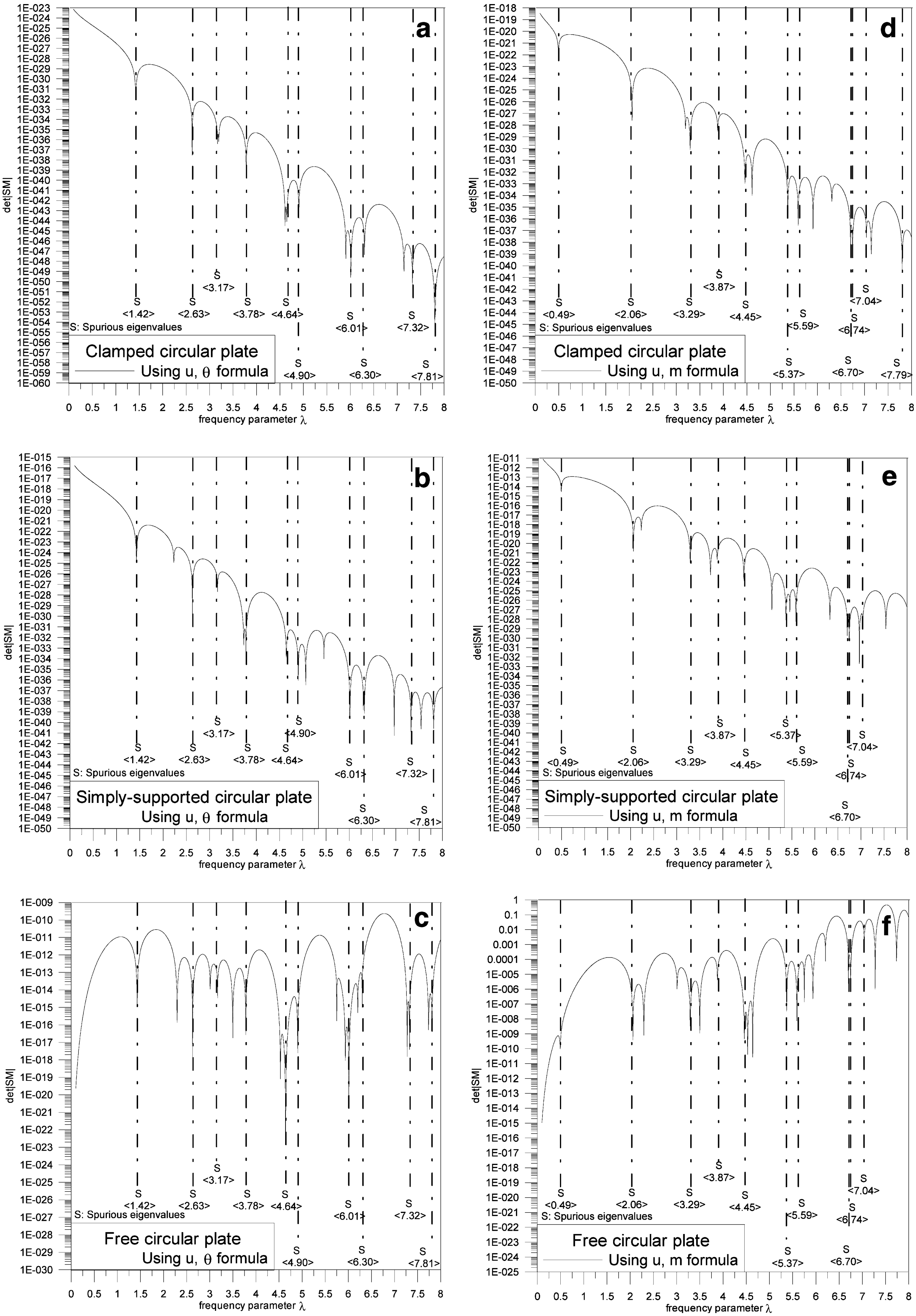
If the determinant of the matrix $[C]^T[C]$ is zero, we can obtain the nontrival solution. Since Φ is orthogonal, the determinant of the matrix $[C]^T[C]$ is equal to the determinant of the matrix $[D]$. By calculating the determinant of the matrix $[D]$, we have

$$\begin{aligned} \det[D] = \prod_{\ell=-(N-1)}^{N} & [(\mu_\ell^{[U]}\kappa_\ell^{[\Theta]} - \mu_\ell^{[\Theta]}\kappa_\ell^{[U]})^2 \\ & + (\mu_\ell^{[U]}\zeta_\ell^{[\Theta]} - \mu_\ell^{[\Theta]}\zeta_\ell^{[U]})^2 + (\mu_\ell^{[U]}\delta_\ell^{[\Theta]} - \mu_\ell^{[\Theta]}\delta_\ell^{[U]})^2 \\ & + (\kappa_\ell^{[U]}\zeta_\ell^{[\Theta]} - \kappa_\ell^{[\Theta]}\zeta_\ell^{[U]})^2 + (\kappa_\ell^{[U]}\delta_\ell^{[\Theta]} - \kappa_\ell^{[\Theta]}\delta_\ell^{[U]})^2 \\ & + (\zeta_\ell^{[U]}\delta_\ell^{[\Theta]} - \zeta_\ell^{[\Theta]}\delta_\ell^{[U]})^2] , \end{aligned} \tag{131}$$

where $\zeta_\ell^{[U]}$, $\zeta_\ell^{[\Theta]}$, $\delta_\ell^{[U]}$ and $\delta_\ell^{[\Theta]}$ are the eigenvalues of the matrics $[U_m]$, $[\Theta_m]$, $[U_v]$ and $[\Theta_v]$, respectively. The only possibility for the zero determinant of the matrix $[D]$ occurs when the six terms $(\mu_\ell^{[U]}\kappa_\ell^{[\Theta]} - \mu_\ell^{[\Theta]}\kappa_\ell^{[U]})$, $(\mu_\ell^{[U]}\zeta_\ell^{[\Theta]} - \mu_\ell^{[\Theta]}\zeta_\ell^{[U]})$, $(\mu_\ell^{[U]}\delta_\ell^{[\Theta]} - \mu_\ell^{[\Theta]}\delta_\ell^{[U]})$, $(\kappa_\ell^{[U]}\zeta_\ell^{[\Theta]} - \kappa_\ell^{[\Theta]}\zeta_\ell^{[U]})$, $(\kappa_\ell^{[U]}\delta_\ell^{[\Theta]} - \kappa_\ell^{[\Theta]}\delta_\ell^{[U]})$ and $(\zeta_\ell^{[U]}\delta_\ell^{[\Theta]} - \zeta_\ell^{[\Theta]}\delta_\ell^{[U]})$ are all zeros at the same time for the same ℓ. Here we can find that the six terms result in the six different spurious eigenequations as shown in Table 2, and the same true eigenequation is commonly imbedded in the six formulations. The only possibility for the zero determinant of the matrix $[D]$ is only the common term (true eigenequation) to be zero, such that

$$\{I_{\ell+1}(\lambda a)J_\ell(\lambda a) + J_{\ell+1}(\lambda a)I_\ell(\lambda a)\} = 0, \\ \ell = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N . \tag{132}$$

This indicates that only the true eigenequation of the clamped circular plate is sorted out in the SVD updating


a
Clamped circular plate
Using u, θ formula
S: Spurious eigenvalues
<1.42> <2.63> <3.17> <3.78> <4.64> <4.90> <6.01> <6.30> <7.32> <7.81>
d
Clamped circular plate
Using u, m formula
<0.49> <2.06> <3.29> <3.87> <4.45> <5.37> <5.59> <6.70> <6.74> <7.04> <7.79>
b
Simply-supported circular plate
Using u, θ formula
<1.42> <2.63> <3.17> <3.78> <4.64> <4.90> <6.01> <6.30> <7.32> <7.81>
e
Simply-supported circular plate
Using u, m formula
<0.49> <2.06> <3.29> <3.87> <4.45> <5.37> <5.59> <6.70> <6.74> <7.04>
c
Free circular plate
Using u, θ formula
<1.42> <2.63> <3.17> <3.78> <4.64> <4.90> <6.01> <6.30> <7.32> <7.81>
f
Free circular plate
Using u, m formula
<0.49> <2.06> <3.29> <3.87> <4.45> <5.37> <5.59> <6.70> <6.74> <7.04>
det|SM|
frequency parameter λ

◄

Fig. 2. Logarithm curve for $det[SM]$ versus frequency parameter λ of the same formulation with different boundary conditions. (a) u, θ formulation of Eq. (2) and Eq. (3) for the clamped case. (b) u, θ formulation of Eq. (2) and Eq. (3) for the simply-supported case. (c) u, θ formulation of Eq. (2) and Eq. (3) for the free case. (d) u, m formulation of Eq. (2) and Eq. (4) for the clamped case. (e) u, m formulation of Eq. (2) and Eq. (4) for the simply-supported case. (f) u, m formulation of Eq. (2) and Eq. (4) for the free case

matrix since the true eigenequation is simultaneously embedded in the six formulations. The result matches well with Eq. (97) and Eq. (44) in the discrete and continuous systems, respectively.

5
Numerical results and discussion

Circular plate (clamped, simply-supported and free boundary conditions $\nu = 0.33$).

A circular plate with a radius ($a = 1$ m) is considered. When the boundary is discretized into ten constant elements, the true eigenvalues by using the real-part BEM are obtained as shown in Tables 3, 4 and 5 for the clamped, simply-supported and free circular plates, respectively. Good agreement is made. Since any two equations in the plate formulation (Eqs. (29)–(32)) can be chosen, $6(C_2^4)$ options of the formulation can be considered. Figure 1.(a)–(f) show the determinant of $[SM^c]$ versus frequency parameter λ for the clamped circular plate using the six formulations. We find that the true eigenvalues depends on the specified boundary condition instead of the formulation. Figures 2.(a)–(c) and (d)–(f) show the determinant of $[SM]$ versus λ using the same formualtion (*e.g.* u, θ and u, m formulae) to solve plates subject to different boundary conditions. Either clamped, simply-supported or free circular case results in the same spurious eigenvalues, once we use the same formulation. The occurrence of spurious eigenvalues only depends on the formulation instead of the specified boundary condition. All the resluts are summarized in Table 2, and the eigenvalues agree well with the data in Leissa [30].

6
Conclusions

A real-part BEM formulation has been derived for the free vibration of plate problems. For a circular plate, the true and spurious eigenvalues and eigenequations were derived analytically by using the degenerate kernel, Fourier series and circulants in the continuous and discrete systems. Since either two equations in the plate formulation (4 equations) can be chosen, C_2^4 (6) options can be considered. The occurrence of spurious eigenequation only depends on the formulation instead of the specified boundary condition, while the true eigenequation is independent of the formulation and is relevant to the specified boundary condition. All the results are shown in Table 2. Three cases were demonstrated analytically and numerically to see the validity of the present method. Also, the SVD updating technique is adopted to suppress the ocurrence of the spurious eigenvalues for the clamped plate. Although the circular case lacks generality, it leads significant insight into the occuring mechanism of true and spurious eigenequation. The proof is only limited to circular plates, it is a great help to the readers who may require analytical explanation about the spurious eigenequation. The same algorithm in the discrete system can be applied to solve arbitrary-shaped plate numerically without any difficulty [11]. Nevertheless, mathematical derivation in continuous and discrete systems can not be done anaytically.

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