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Abstract

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Desingularized meshless method for solving Laplace equation with over-specified boundary conditions using regularization techniques

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Abstract The desingularized meshless method (DMM) has been successfully used to solve boundary-value problems with specified boundary conditions (a direct problem) numerically. In this paper, the DMM is applied to deal with the problems with over-specified boundary conditions. The accompanied ill-posed problem in the inverse problem is remedied by using the Tikhonov regularization method and the truncated singular value decomposition method. The numerical evidences are given to verify the accuracy of the solutions after comparing with the results of analytical solutions through several numerical examples. The comparisons of results using Tikhonov method and truncated singular value decomposition method are also discussed in the examples.

Keywords Desingularized meshless method · Tikhonov method · Truncated singular value decomposition method · Inverse problem · Subtracting and adding-back technique

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1 Introduction

The boundary-value problems subjected to the over-specified boundary conditions (B.C.s) can be viewed as one of the inverse problems. The unreasonable results of traditional numerical methods often occur in the inverse problems undergoing the measured and contaminated errors on the over-specified B.C.s because of the ill-posed behavior in the linear algebraic system [4,17]. Mathematically speaking, the influence matrix in the inverse problem is ill-posed since the solution is very sensitive to the given data. Such a divergent problem could be avoided by using regularization methods [1,2,4,7,15,18,19,21–24]. For examples, the truncated singular value decomposition technique (TSVD) [10,11,17, 19], the zeroth order and first order techniques of Tikhonov regularization technique [1,2,9,12,13,19,20] have been applied to deal with divergent problems. The three techniques can obtain a convergence solution more precisely and reasonably. The numerical methods combined with the regularization techniques of the TSVD method and Tikhonov method, respectively [1,2,10,17,25], had been successfully applied to overcome the ill-posed problem of the Laplace equation. In this paper, the desingularized meshless method (DMM) in conjunction with the two regularization techniques is employed to solve the inverse problem. To obtain a better regularization method, the comparison of two regularization techniques is made through several numerical examples.

For the inverse problem, the influence matrix is often ill-posed such that the regularization techniques which regularize the influence matrix are necessary. The TSVD can alleviate the ill-posed behavior of the solution prone to divergence by the input data errors by choosing an appropriate truncated number, i. Similarly, the Tikhonov regularization technique transforms into a well-posed one by choosing an appropriate parameter for λ [22]. An appropriate truncated

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number (or parameter) can be determined according to a compromise point between regularization errors (due to data smoothing) and perturbation errors (due to noise disturbance) by implementing the L2 norm [4,11]. The L2 norm determines the optimal value of λ (or i) which will be employed to provide the compromise point and will be elaborated on later. But we are well aware that many real problems usually have no analytical solution. To find out the optimal solution reasonably, it is needed to employ the error criterion technique in the case of no exact solution. An alternative technique, called L-curve technique [1,14] is introduced. It is implemented in case 3.

During the last decade, scientific researchers have paid attention to the method of fundamental solutions (MFS) for solving engineering problems [3,6,8,16,25], in which the mesh or element is free. The DMM is one kind of modified MFS and has been applied to solve some potential problems of elliptic operators [3,5–7,14,16,25,26]. By employing the desingularization technique of subtracting and adding-back technique to regularize the singularity and hypersingularity of the kernel functions [26], the proposed method can distribute the observation and source points on the coincident locations of the real boundary and still maintain the spirit of the MFS. Therefore, the DMM provides a significant and promising alternative to dominant numerical methods such as the FEM and BEM. Since neither domain nor surface meshing is required for the meshless methods, they could be more attractive for engineers to use.

In this paper, we will employ the DMM in conjunction with the TSVD method and the zeroth order and first order techniques of Tikhonov regularization method to circumvent the ill-posed problems. The results of the examples contaminated with artificial noises on the over-specified B. C. are given to illustrate the validity of the proposed technique.

2 Formulation

2.1 Governing equation subject to over-specified B.C.s

The inverse problem for the Laplace equation subject to over-specified B.C.s as shown in Fig. 1 can be modeled by:

$$\nabla^2 \phi(x) = 0, \quad x \in D, \tag{1}$$

subjected to the B. C. on B_1 as

$$\phi(x) = \overline{\phi}, \quad x \in B_1, \tag{2}$$

$$\psi(x) = \overline{\psi}, \quad x \in B_1, \tag{3}$$

where ∇^2 is the Laplacian operator, D is the domain of interest, $\psi(x) = \partial\phi(x)/\partial n_x$ in which n_x is the normal vector at x, B_1 is the known boundary (B_1) of B in which B is the whole boundary which consists of boundary (B_1) with specified BCs, and the boundary (B_2) with unknown BCs.

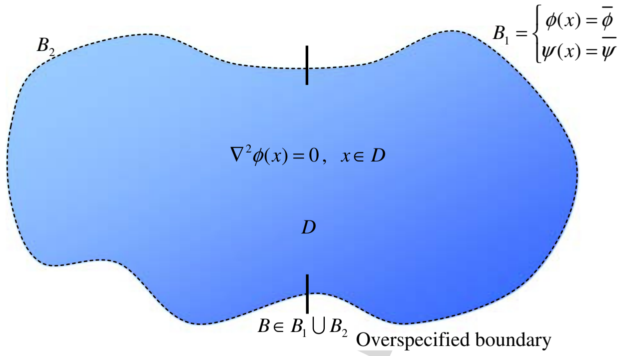

Fig. 1 Problem sketch for the inverse Laplace problem

2.2 Methods of the solution

2.2.1 Review of conventional method of fundamental solutions

By employing the radial basis function (RBF) concept [6–8,16], the representation of the solution for interior problem can be approximated in terms of the strengths α_j of the singularities s_j as

$$\phi(x_i) = \sum_{j=1}^{N+M} A(s_j, x_i)\alpha_j, \tag{4}$$

$$\psi(x_i) = \sum_{j=1}^{N+M} B(s_j, x_i)\alpha_j, \tag{5}$$

where $A(s_j, x_i)$ is RBF, $B(s_j, x_i) = \partial A(s_j, x_i)/\partial n_{x_i}$, α_j is the jth unknown coefficient (strength of the singularity), s_j is the jth source point (singularity), x_i is the ith observation point. The indexes, N and M, are numbers of the boundary points on B_1 and B_2, respectively. The chosen RBFs of Eqs. 4 and 5 in this paper are the double-layer potentials in the potential theory as

$$A(s_j, x_i) = \frac{-\langle (x_i - s_j), n_j \rangle}{r_{ij}^2}, \tag{6}$$

$$B(s_j, x_i) = \frac{2\langle (x_i - s_j), n_j \rangle \langle (x_i - s_j), \overline{n_i} \rangle}{r_{ij}^4} - \frac{\langle n_j, \overline{n_i} \rangle}{r_{ij}^2}, \tag{7}$$

where $\langle\, , \rangle$ is the inner product of two vectors, r_{ij} is $|s_j - x_i|$, n_j is the normal vector at s_j, and $\overline{n_i}$ is the normal vector at x_i.

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When the collocation point x_i approaches to the source point s_j, Eqs. 4 and 5 become singular. Equations 4 and 5 for the interior problems need to be regularized by using the subtracting and adding-back technique [5,26] as follows:

$$\phi(x_i) = \sum_{j=1}^{N+M} A^{(I)}(s_j, x_i)\alpha_j - \sum_{j=1}^{N+M} A^{(E)}(s_j, x_i)\alpha_i, \quad x_i \in B, \tag{8}$$

in which

$$\sum_{j=1}^{N+M} A^{(E)}(s_j, x_i)\alpha_i = 0, \quad x_i \in B, \tag{9}$$

where the superscript (I) and (E) denotes the inward and outward normal vectors, respectively. The detailed derivation of Eq. 9 had been given in the Ref. [26]. Therefore, we can obtain

$$\phi(x_i) = \sum_{j=1}^{i-1} A^{(I)}(s_j, x_i)\alpha_j + \sum_{j=i+1}^{N+M} A^{(I)}(s_j, x_i)\alpha_j + \left[\sum_{m=1}^{N+M} A^{(I)}(s_m, x_i) - A^{(I)}(s_i, x_i)\right]\alpha_i, \quad x_i \in B. \tag{10}$$

Similarly, the boundary flux is obtained as

$$\psi(x_i) = \sum_{j=1}^{N+M} B^{(I)}(s_j, x_i)\alpha_j - \sum_{j=1}^{N+M} B^{(E)}(s_j, x_i)\alpha_i, \quad x_i \in B, \tag{11}$$

in which

$$\sum_{j=1}^{N+M} B^{(E)}(s_j, x_i)\alpha_i = 0, \quad x_i \in B. \tag{12}$$

The detailed derivation of Eq. 12 had been demonstrated in the Ref. [26]. Therefore, we can obtain

$$\psi(x_i) = \sum_{j=1}^{i-1} B^{(I)}(s_j, x_i)\alpha_j + \sum_{j=i+1}^{N+M} B^{(I)}(s_j, x_i)\alpha_j - \left[\sum_{m=1}^{N+M} B^{(I)}(s_m, x_i) - B^{(I)}(s_i, x_i)\right]\alpha_i, \quad x_i \in B. \tag{13}$$

According to the dependence of the normal vectors for inner and outer boundaries [26], their relationships are

$$\begin{cases} A^{(I)}(s_j, x_i) = -A^{(E)}(s_j, x_i), & i \neq j \\ A^{(I)}(s_j, x_i) = A^{(E)}(s_j, x_i), & i = j \end{cases} \tag{14}$$

$$\begin{cases} B^{(I)}(s_j, x_i) = B^{(E)}(s_j, x_i), & i \neq j \\ B^{(I)}(s_j, x_i) = B^{(E)}(s_j, x_i), & i = j \end{cases} \tag{15}$$

where the left-hand and right-hand sides of the equal sign in Eqs. 14 and 15 denote the kernels for observation and source point with the inward and outward normal vectors, respectively.

By using the proposed technique, the singular terms in Eqs. 4 and 5 have been transformed into regular terms $\left[\sum_{m=1}^{N+M} A^{(I)}(s_m, x_i) - A^{(I)}(s_i, x_i)\right]$ and $-\left[\sum_{m=1}^{N+M} B^{(I)}(s_m, x_i) - B^{(I)}(s_i, x_i)\right]$ in Eqs. 10 and 13, respectively. The terms of $\sum_{m=1}^{N+M} A^{(I)}(s_m, x_i)$ and $\sum_{m=1}^{N+M} B^{(I)}(s_m, x_i)$ are the adding-back terms and the terms of $A^{(I)}(s_i, x_i)$ and $B^{(I)}(s_i, x_i)$ are the subtracting terms in two brackets for the special treatment technique. After using the abovementioned method of regularization of subtracting and adding-back technique [5,26], we have removed the singularity and hypersingularity of the kernel functions.

2.2.2 *Derivation of diagonal coefficients of influence matrices*

The following linear algebraic system can be derived after collocating N observation points on B_1 and M observation points on B_2, $\{x_i\}_{i=1}^{N+M}$, in Eq. 10 as

$$\left\{ \begin{array}{c} \left\{ \begin{array}{c} \phi_1 \\ \vdots \\ \phi_N \end{array} \right\} \\ \left\{ \begin{array}{c} \phi_{N+1} \\ \vdots \\ \phi_{N+M} \end{array} \right\} \end{array} \right\}_{(N+M)\times 1} = \begin{bmatrix} [A_1]_{N\times(N+M)} \\ [A_2]_{M\times(N+M)} \end{bmatrix} \left\{ \begin{array}{c} \left\{ \begin{array}{c} \alpha_1 \\ \vdots \\ \alpha_N \end{array} \right\} \\ \left\{ \begin{array}{c} \alpha_{N+1} \\ \vdots \\ \alpha_{N+M} \end{array} \right\} \end{array} \right\}_{(N+M)\times 1}, \tag{16}$$

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where

$$[A_1]=\begin{bmatrix} \sum_{m=1}^{N+M} a_{1,m}-a_{1,1} & a_{1,2} & \cdots & a_{1,N} & \cdots & a_{1,N+M} \\ a_{2,1} & \sum_{m=1}^{N+M} a_{2,m}-a_{2,2} & \cdots & a_{2,N} & \cdots & a_{2,N+M} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{N,1} & a_{N,2} & \cdots & \sum_{m=1}^{N+M} a_{N,m}-a_{N,N} & \cdots & a_{N,N+M} \end{bmatrix}_{N\times(N+M)}, \tag{17}$$

$$[A_2]=\begin{bmatrix} a_{N+1,1} & \cdots & \sum_{m=1}^{N+M} a_{N+1,m}-a_{N+1,N+1} & \cdots & a_{N+1,N+M} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{N+M,1} & \cdots & a_{N+M,N+1} & \cdots & \sum_{m=1}^{N+M} a_{N+M,m}-a_{N+M,N+M} \end{bmatrix}_{M\times(N+M)}, \tag{18}$$

in which

$$a_{ij}=A^{(I)}(s_j,x_i), \quad i,j=1,2,\ldots,N+M. \tag{19}$$

In a similar way, Eq. 13 yields

$$\left\{\begin{array}{c} \left\{\begin{array}{c} \overline{\psi_1} \\ \vdots \\ \overline{\psi_N} \end{array}\right\} \\ \left\{\begin{array}{c} \psi_{N+1} \\ \vdots \\ \psi_{N+M} \end{array}\right\} \end{array}\right\}_{(N+M)\times 1} = \begin{bmatrix} [B_1]_{N\times(N+M)} \\ [B_2]_{M\times(N+M)} \end{bmatrix} \left\{\begin{array}{c} \left\{\begin{array}{c} \alpha_1 \\ \vdots \\ \alpha_N \end{array}\right\} \\ \left\{\begin{array}{c} \alpha_{N+1} \\ \vdots \\ \alpha_{N+M} \end{array}\right\} \end{array}\right\}_{(N+M)\times 1}, \tag{20}$$

where

$$[B_1]=\begin{bmatrix} -\left[\sum_{m=1}^{N+M} b_{1,m}-b_{1,1}\right] & b_{1,2} & \cdots & b_{1,N} & \cdots & b_{1,N+M} \\ b_{2,1} & -\left[\sum_{m=1}^{N+M} b_{2,m}-b_{2,2}\right] & \cdots & b_{2,N} & \cdots & b_{2,N+M} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{N,1} & b_{N,2} & \cdots & -\left[\sum_{m=1}^{N+M} b_{N,m}-b_{N,N}\right] & \cdots & b_{N,N+M} \end{bmatrix}_{N\times(N+M)}, \tag{21}$$

$$[B_2]=\begin{bmatrix} b_{N+1,1} & \cdots & -\left[\sum_{m=1}^{N+M} b_{N+1,m}-b_{N+1,N+1}\right] & \cdots & b_{N+1,N+M} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{N+M,1} & \cdots & b_{N+M,N+1} & \cdots & -\left[\sum_{m=1}^{N+M} b_{N+M,m}-b_{N+M,N+M}\right] \end{bmatrix}_{M\times(N+M)}, \tag{22}$$

in which

$$b_{ij} = B^{(I)}\left(s_j, x_i\right), \quad i, j = 1, 2, \ldots, N + M. \tag{23}$$

2.2.3 Derivation of influence matrix

We can rearrange the influence matrices of Eqs. 16 and 20 into the linear algebraic system as

$$\left\{ \begin{matrix} \{\phi_1\}_{N\times 1} \\ \{\psi_1\}_{N\times 1} \end{matrix} \right\} = \begin{bmatrix} [A_1]_{N\times(N+M)} \\ [B_1]_{N\times(N+M)} \end{bmatrix} \{\alpha\}_{(N+M)\times 1}. \tag{24}$$

The linear algebraic system in Eq. 24 can be generally written as

$$D = CX. \tag{25}$$

For the inverse problem, the influence matrix C is often ill-posed such that the regularization techniques are necessary to regularize the ill-posed matrix.

2.3 Regularization techniques for the inverse problem

2.3.1 Truncated singular value decomposition method (TSVD)

In the singular value decomposition (SVD), the matrix C can be decomposed into

$$C = [U][\Sigma][V]^T, \tag{26}$$

where $[U] = \left[u_1, u_2, \ldots, u_{(N+M)}\right]$ and $[V] = \left[v_1, v_2, \ldots, v_{(N+M)}\right]$ are column orthonormal matrices, with column vectors called left and right singular vectors, respectively, T denotes the matrix transposition, and $[\Sigma] = diag(\sigma_1, \sigma_2, \ldots, \sigma_{(N+M)})$ is a diagonal matrix with nonnegative diagonal elements in non-increasing order, which are the singular values of C.

A convenient measure of the conditioning of the matrix C is the condition number defined as

$$Cond = \frac{\sigma_1}{\sigma_{(N+M)}}, \tag{27}$$

where σ_1 is the maximum singular value and $\sigma_{(N+M)}$ is the minimum singular value, i.e., the ratio between the largest singular value and the smallest singular value. By means of the SVD, the solution a^0 can be written as

$$a^0 = \sum_{i=1}^{k} \frac{u_i^T d}{\sigma_i} v_i, \tag{28}$$

where k is the rank of C, u_i is the element of the left singular vector and v_i is the element of the right singular vector. For an ill-conditioned matrix, there are small singular values, therefore the solution is dominated by contributions from small singular values when the noise is present in the input data. One simple remedy to treat the difficulty is to leave out contributions from small singular values, i.e., taking a^p as an approximate solution, where a^p is defined as

$$a^p = \sum_{i=1}^{p} \frac{u_i^T d}{\sigma_i} v_i, \tag{29}$$

where $p \le k$ is the regularization parameter, which determines when one starts to leave out small singular values. Note that if $p = k$, the approximate solution is exactly the least squares solution. This method is known as TSVD in the inverse problem community [10,17,19].

2.3.2 Tikhonov regularization technique

Tikhonov proposed a method [1,2,9,12,13,19,20] to transform an ill-posed problem into a well-posed one. Instead of solving Eq. 25 directly, the solution of Tikhonov technique regularized as follows:

$$f_\lambda(X_\lambda) = \min_{X\in R^M} f_\lambda(X), \tag{30}$$

where the λ is the regularization parameter and f_λ is the k-th order Tikhonov function as given

$$f_\lambda(X) = \|CX - D\|^2 + \lambda^2 \left\| R^{(k)} X \right\|^2,$$
$$R^{(k)} \in R_{(M-k)\times M}, \quad k = 0, 1, 2 \ldots. \tag{31}$$

Solving $\nabla f_\lambda(X) = 0$, we can obtain the Tikhonov regularized solution X_λ of the Eq. 30 which is given as the solution of the regularized equation

$$\left(C^T C + \lambda^2 {R^{(k)}}^T R^{(k)}\right) X = C^T D, \quad k = 0, 1, 2, 3 \ldots, \tag{32}$$

where T denotes matrix transposition. The matrix, $R^{(k)}$,in Eq. 31 is a matrix that defines a (semi) norm of solution vector in which the superscript, k, represents k-th derivative operator on R. $R^{(k)}$is an identity matrix when $k = 0$ and the influence matrix, $(C^T C + \lambda^2 {R^{(k)}}^T R^{(k)})$, in Eq. 32 can only be regularized in a diagonal term by λ. $R^{(k)}$ is a banded matrix when $k = 1$ and the influence matrix in Eq. 32 can be regularized in a diagonally banded term. In this paper, the zeroth order and first order techniques of Tikhonov regularization method are considered, respectively. The matrices of $R^{(0)}$ and $R^{(1)}$ of zeroth order and first order techniques of Tikhonov regularization method are given by

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$$R^{(0)} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{M \times M}, \tag{33}$$

$$R^{(1)} = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 1 \end{bmatrix}_{(M-1) \times M}. \tag{34}$$

An ill-posed matrix will be transformed into a well one by employing the proposed regularization techniques. If too much regularization, i.e., $\lambda$is large, the solution will be too smoothing. If too little regularization, i.e., λ is small, the solution will be unreasonable by the contributions from the input data with perturbation error in measurements. The choice of the regularization parameter in Eq. 32 is vital for obtaining a reasonable and convergent solution and this is obtained on the next section.

2.4 Determining the optimal parameter

(1) L2 norm technique
To aid us in selecting the optimal parameter λ (or I, truncated number), the value of L2 norm is implemented as the y-axis and parameter λ (or I, truncated number) as the x-axis. The L2 norm is defined as $\|\phi - \phi_e\| = \int |\phi - \phi_e|^2 \, dB$, where ϕ is the numerical result and ϕ_e is the analytical result. When the L2 norm $\|\phi - \phi_e\|$ tends to be very small versus the regularization parameter, it is the optimal parameter. The L-curve shape can be similarly observed in the figure. The corner point of the L-curve shape is a local minimum norm and is the appropriate choice for the optimal parameter (or the optimal truncated number of TSVD).

(2) L-curve technique
The L-curve technique is a log-log plot of the norm of regularized solution versus the norm of corresponding residual norm [1,11]. The norm of regularized solution is defined as

$$Log \, \|CX - D\|^2, \tag{30}$$

and the norm of corresponding residual norm as follows

$$Log \, \|X\|^2, \tag{31}$$

The x-axis is the solution norm, and y-axis is the residual norm. The former is the index of how smooth the solution is treated, and the latter is the distance index between the predicted output and real output. The corner point of L-curve technique is a compromise between the regularization errors due to data smoothing and perturbation errors in measurements or other noise, even though an analytical solution is not available. The L-curve technique belongs to a error criterion technique and does not need to compare the results with analytical solution.

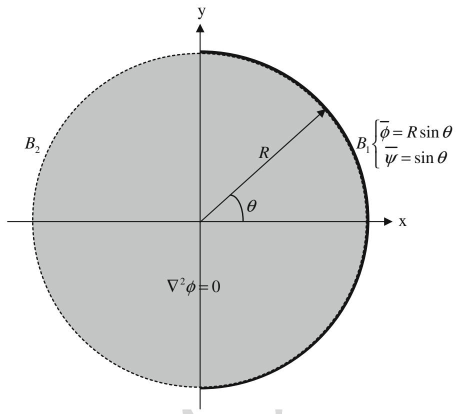

Fig. 2 Problem sketch for the case 1

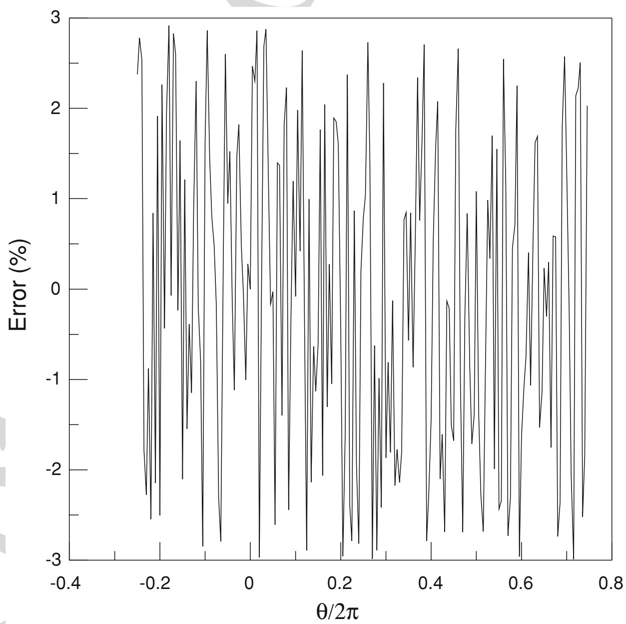

Fig. 3 Relative noise error for the case 1

3 Numerical examples

To show the accuracy and validity of the proposed method and obtain a better regularization method, three cases with circular, square and infinite strip domains subjected to the over-specified B.C.s are considered.

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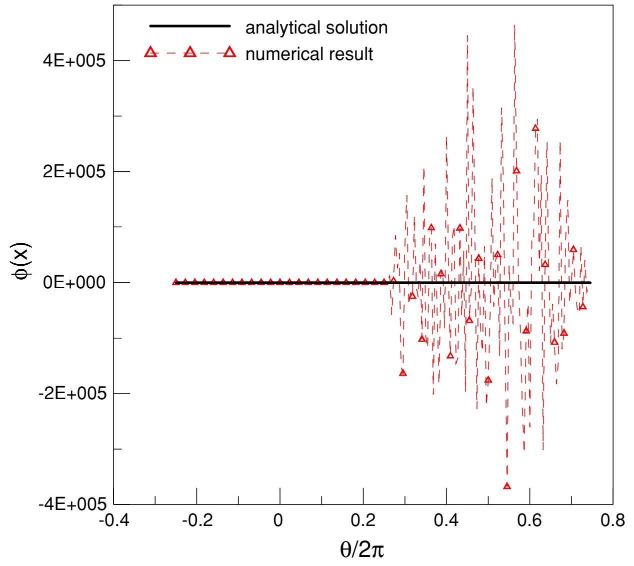

Fig. 4 Numerical result by using the DMM without employing the regularization technique

Case 1: Circular domain case

The problem sketch of the inverse problem with circular domain is drawn in Fig. 2. The unit radius is given and the input data on the over-specified boundary is specified as $\phi_{exact} = \sin\theta$ and $\psi_{exact} = \cos\theta$ in which $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. By using the random data simulation, we can obtain random errors contaminating the input data $\overline{\phi_i} = (\phi_i)_{exact} + (\phi_i)_{exact} rand(i)\varepsilon$ and $\overline{\psi_i} = (\psi_i)_{exact} + (\psi_i)_{exact} rand(i)\varepsilon$ where the random number $rand(i)$ is chosen between $[-1,1]$ (also in Cases 2 and 3) and ε denotes the percentage of the relative noise error, as shown in Fig. 3. If regularization techniques are not employed, the results are unreasonable and divergent as shown in Fig. 4. To see the sensitivity analysis of regularization parameters of the three regularization techniques to obtain a optimal solution, we find out the relationship between the norm error and the value of λ (or i) in which the norm error is defined as $\int_0^{2\pi} |\phi_{exact}(r = 1, \theta) - \phi(r = 1, \theta)|^2 d\theta$. Figure 5a displays the optimal truncated number, 94, for the TSVD technique. Figure 5b, c displays the optimal value of regularization parameters of 0.0001905 and 0.198, respectively, for the zeroth order and first order techniques of Tikhonov regularization method. We obtain three better results with the three optimal parameters by employing the three regularization techniques as shown in Fig. 6 by distributing 200 nodes. The result of the first order technique of Tikhonov method is better than other techniques as shown in Fig. 6. Therefore we adopted the first order technique of Tikhonov method in cases 2 and 3. The result of absolute error with the exact solution of three regularization methods is plotted in Fig. 7. To see the convergent analysis of the DMM in conjunction with the first order Tikhonov regularization method, Fig. 8 is plotted. A convergent result can be obtained after distributing over 100 points.

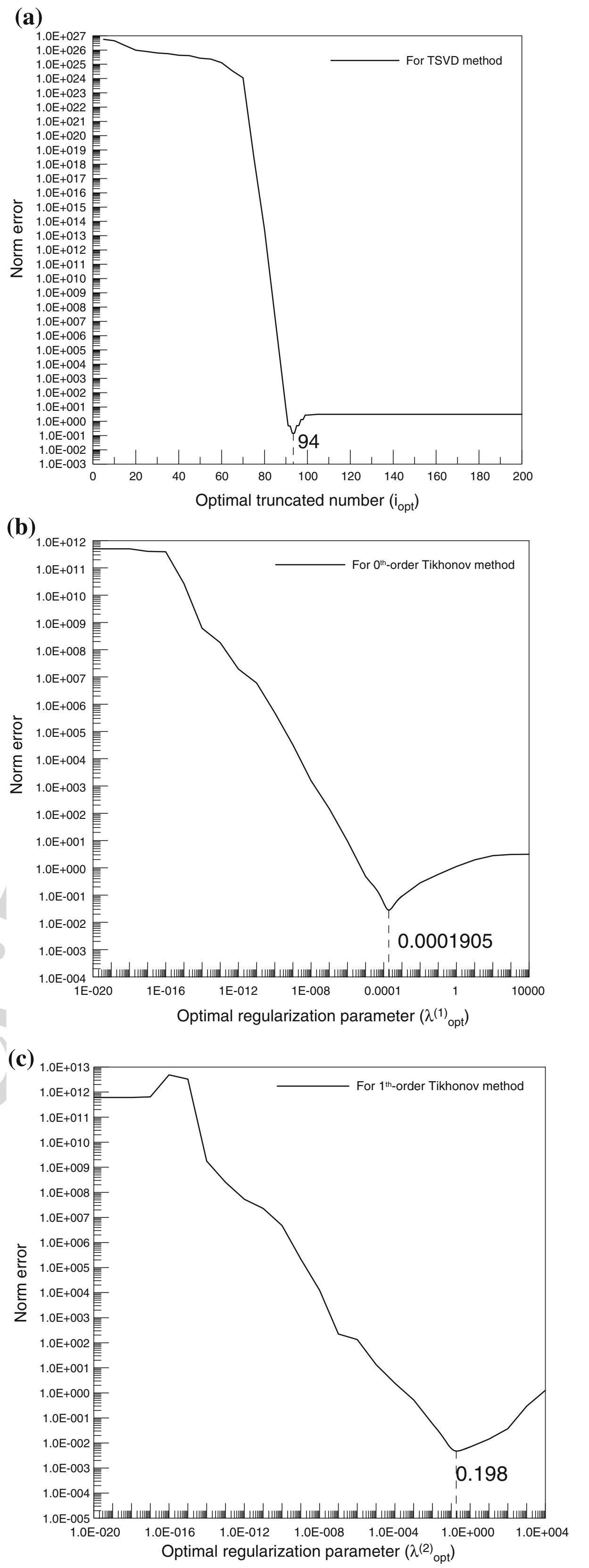

Fig. 5 Optimal truncated number and regularization parameter for **a** TSVD method, **b** zeroth order Tikhonov method, **c** first order Tikhonov method

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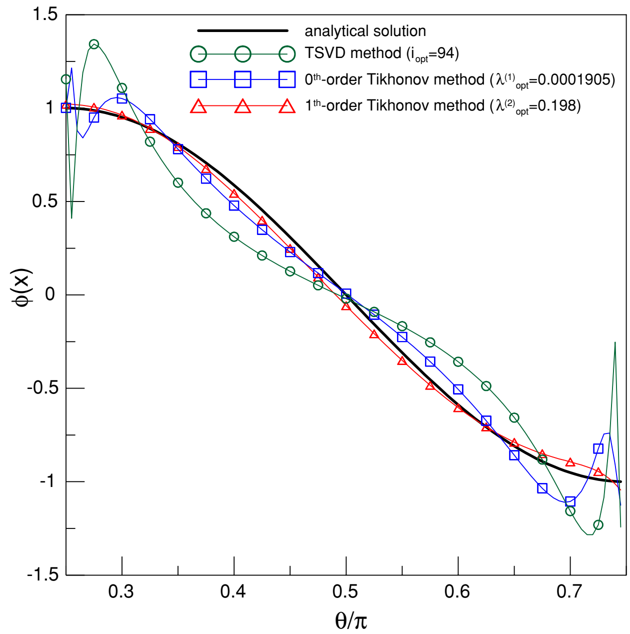

Fig. 6 Numerical results by employing the TSVD method, the zero and first order Tikhonov methods, respectively, and using 200 nodes for the case 1

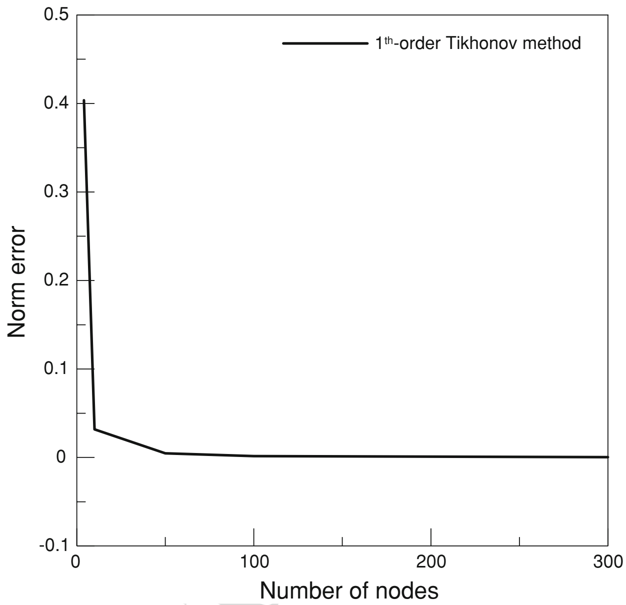

Fig. 8 The norm error along the boundary versus the number of nodes by using the first order Tikhonov method for the case 1

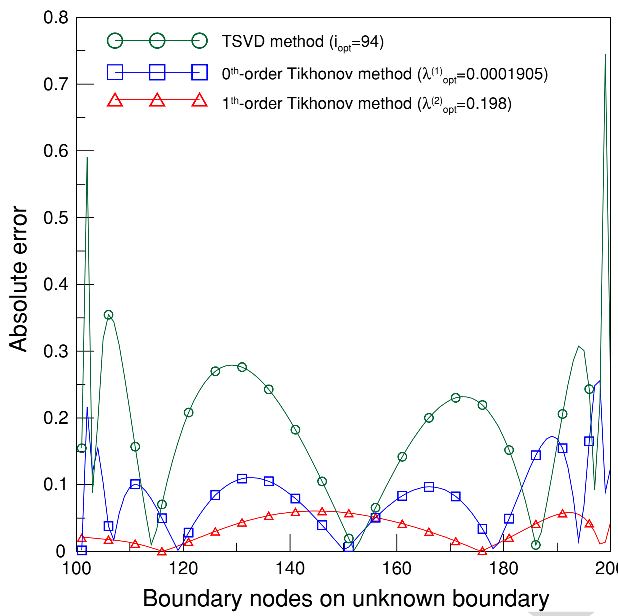

Fig. 7 Absolute error with the exact solution by employing three regularization methods and using 200 nodes for the case 1

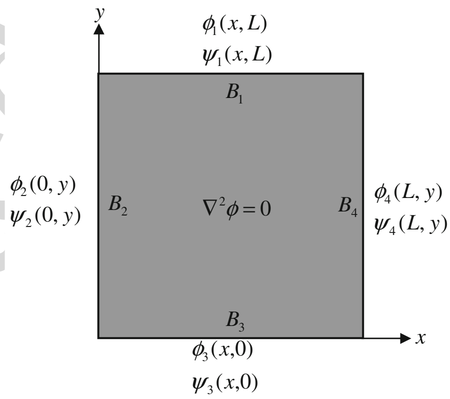

Fig. 9 Problem sketch for the case 2

Case 2: Square domain case

The square domain of the inverse problem and B.C.s are sketched in Fig. 9. The exact solution in the whole domain is $u(x, y) = xy$. The over-specified B.C.s is given on partial boundary. To see the effects on increasing or decreasing the information of known or unknown data and to examine how the diversity of boundary data will affect the solution, three kinds of distributions are given as S1 type: data on boundary 4 is unknown. S2 type: data on boundaries 3 and 4 are unknown. S3 type: data on boundaries 2, 3 and 4 are unknown. The length of square domain is 1.0.

Table 1 Optimal regularization parameters for the S1, S2 and S3 labels of different random errors

Label	Unknown boundary	Optimal regularization parameters		
		0.1% Random error	1% Random error	3% Random error
S1	B_4	7.143	6.953	11.735
S2	B_3, B_4	1.863	2.489	2.699
S3	B_2, B_3, B_4	124.321	132.676	117.817

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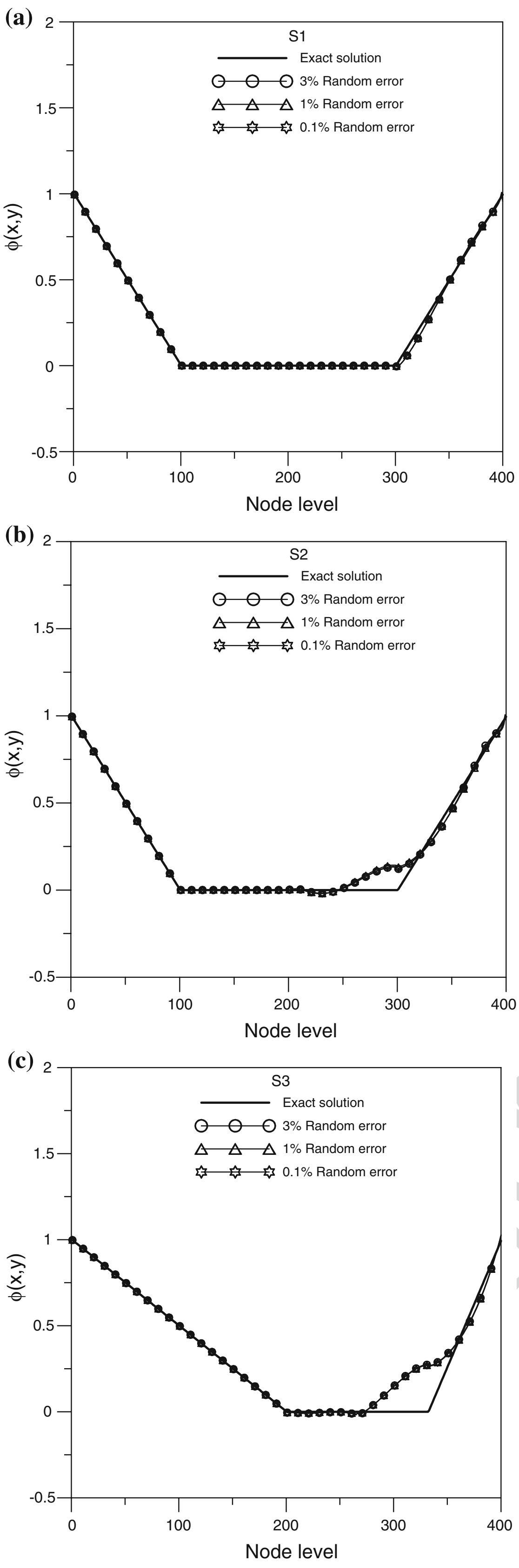

Fig. 10 Numerical results for **a** S1, **b** S2, **c** S3 by using the first order Tikhonov method in different random error

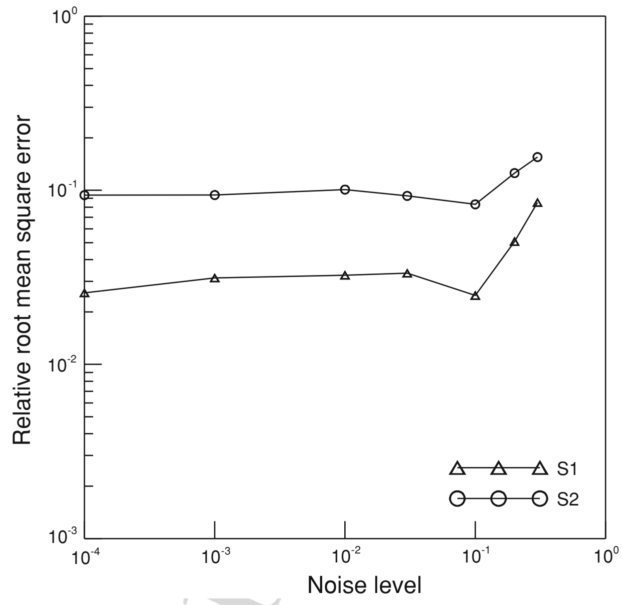

Fig. 11 Relative root mean square errors for the S1 and S2 cases

The optimal regularization parameters for different relative noise levels $\varepsilon = 0.1$–3% are reported in Table 1. The numerical results with different relative noise levels are illustrated in Fig. 10a–c for the three types (S1, S2 and S3 types). The phenomenon is apparent that the more number of the known data are given, the more accurately the results are derived. To compare the result for different relative noise levels with the reference [25], the relative root mean square error [25] respective to the various noise level are graphically shown in Fig. 11. To see the ill-posed sensitivity, the condition number of the influence matrix versus the number of boundary nodes is graphically reported in Fig. 12 for S1 and S2 types. It is shown that the more number of nodes is distributed, the larger the condition number is obtained.

Case 3: Infinite strip region case

The infinite strip region of inverse problem and overspecified boundary conditions, $\phi(x, l) = \overline{\phi}$ and $\phi_y(x, l) = 0$, are given, respectively, as shown in Fig. 13, and the cosine and square waves through the surface of infinite strip region are considered, respectively. We can obtain the optimal regularization parameters, 0.02 and 0.00025, respectively, for the cosine wave and square wave in the surface by using the L-curve technique, which are shown in Figs. 14 and 15. The unknown boundary data, $\phi(x, 0)$, is solved by adopting the optimal parameters. The new specified boundary condition, $\phi(x, 0)$, is given again which is obtained before and the original boundary condition, $\phi_y(x, l) = 0$ is defined as the new boundary condition. The new problem with new specified boundary condition is the well-posed problem. The result of $\phi(x, l)$ is reformulated by using the DMM and compare it

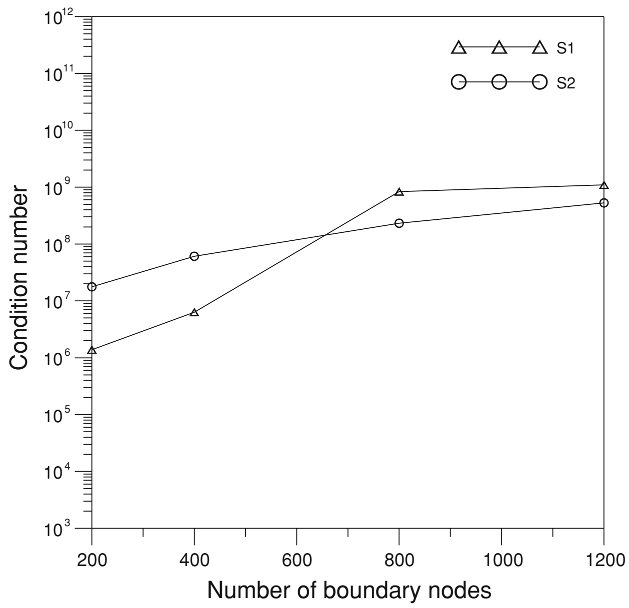

Fig. 12 Condition number for the S1 and S2

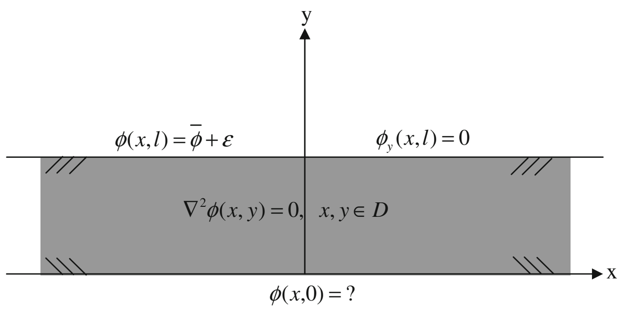

Fig. 13 Problem sketch for the case 3

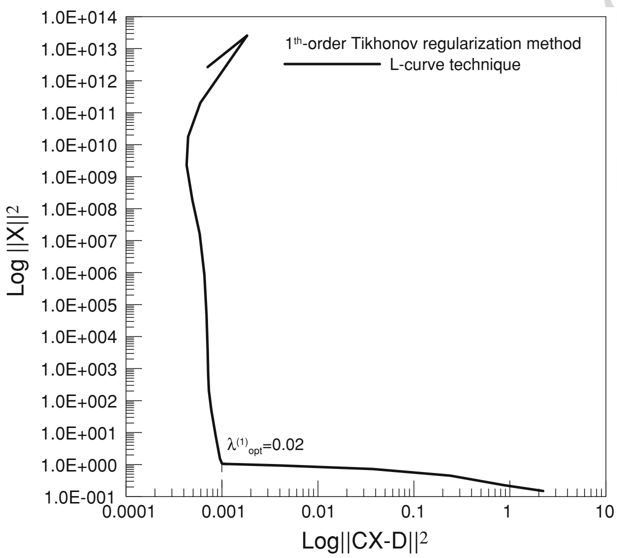

Fig. 14 Optimal regularization parameter of cosine wave by employing the L-curve technique

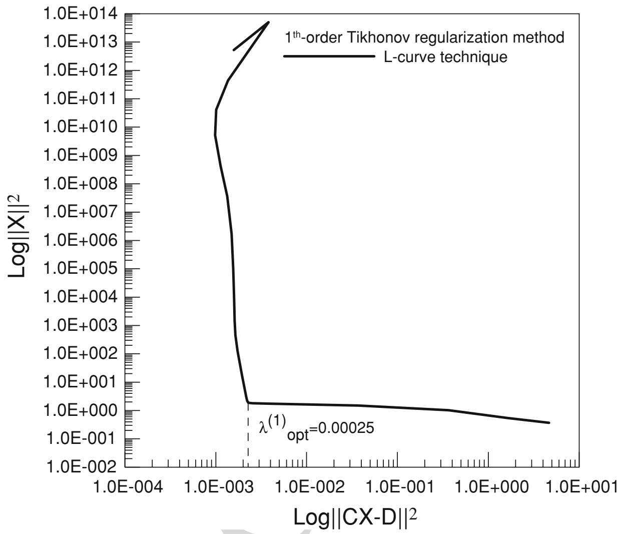

Fig. 15 Optimal regularization parameter of square wave by employing the L-curve technique

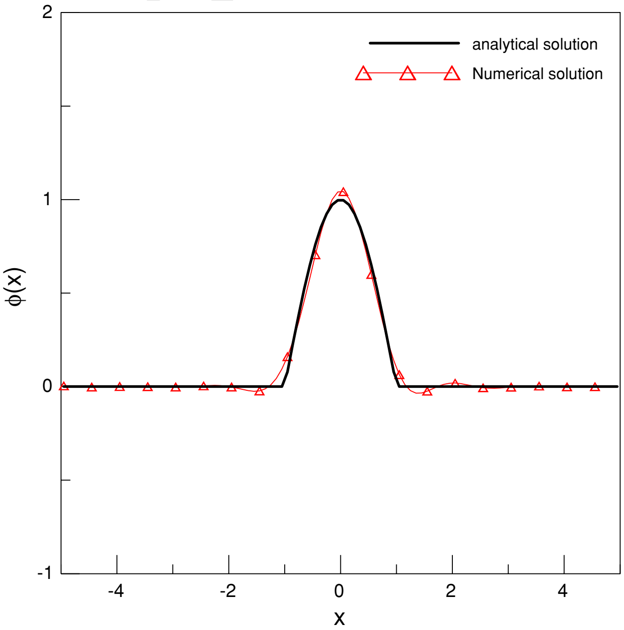

Fig. 16 Numerical result of cosine wave by using the L-curve technique in conjunction with the first order Tikhonov method

with the original boundary condition, $\overline{\phi}$, as plotted in Figs. 16 and 17, respectively.

4 Conclusions

In this paper, we successfully applied the DMM in conjunction with the regularization techniques to solving inverse problems. The source and collocation points can be located on the real boundary at the same time by using the proposed

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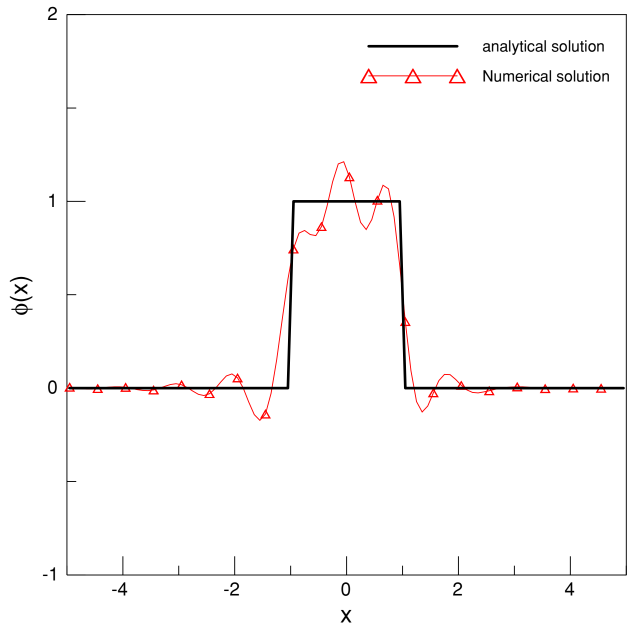

Fig. 17 Numerical result of square wave by employing the L-curve technique in conjunction with the first order Tikhonov method

desingularization technique. The resulting ill-conditioned system of linear algebraic equations has been regularized by using the three regularization techniques. The ill-posed problems can be effectively remedied by using the first order regularization method and the absolute error with the exact solution is smaller than those of other regularization techniques through the given examples.

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