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# Recent development of dual BEM in acoustic problems ☆

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**Abstract**

In this paper, recent development of the dual BEM in acoustic problem is presented. The role of hypersingular integral equation in the dual BEM for the problems with a degenerate boundary is examined. First, the dual integral formulation is proposed. Based on the formulation, we develop four methods – the complex-valued dual BEM, the real-part dual BEM, the real-valued dual MRM, and the complete complex-valued MRM. For the real-part dual BEM and the real-valued dual MRM, the spurious eigenvalues occur and can be filtered out by using residual technique or singular value decomposition method. It is also found that the complete MRM with infinite terms of series is equivalent to the complex-valued dual BEM if the constant potential in the zeroth-order fundamental solution is chosen to be an appropriate complex value. The dual formulation can be applied to solve acoustic problems with degenerate boundaries. An illustrative example for acoustic eigenfrequencies of a cavity with an incomplete partition is shown to verify the validity of the four methods. The results are compared with those of FEM and experiment. Good agreement is made. 

*Keywords:* Dual integral formulation; Dual boundary element method; Acoustic problem; Degenerate boundary; Multiple reciprocity method

## 1. Introduction

The boundary element method (BEM), sometimes referred to as the boundary integral equation method, is now establishing a position as a natural alternative to the FEM in many fields of engineering. The dual boundary element method, or so-called the dual boundary integral equation method developed by Chen and Hong [1], is particularly suited for problems with a degenerate boundary. Mathematically speaking, the hypersingular integral equation was first formulated by Hadamard [2] to treat the cylindrical wave equation by spherical means of descent. In the meantime, Mangler derived the same mathematical form in solving the thin airfoil problem [1]. In fact, the original idea came from the applications of the continuous and discontinuous properties of the single and double layer potentials and their derivations when the field point approaching the boundary in the dual integral equations. These properties are classical results and the so-called 'dual integral equations' appeared in many literatures (they are known as Calderon projection), although they may have interpretations by modern mathematical tools. The improper integral was then defined by Tuck [4] as the 'Hadamard principal value'. In aerodynamics, it was termed the 'Mangler's principal value' [3,5]. Such a divergent integral naturally arises in the dual formulation especially for problems with a degenerate boundary, e.g., crack problems in elasticity [6–11], heat flow through a baffle [12], Darcy flow around a cutoff wall [13,14] the aerodynamic problem of a thin airfoil [5] and acoustic

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waves impinging on a screen [15,16,27,32,33]. The dual formulation also plays an important role in some other problems, e.g., the corner problem [17], adaptive BEM [18], and the exterior problem [19]. A general application of the hypersingular integral equation in mechanics was discussed in [20], and a review paper on recent development of dual BEM was presented in [21]. Combining the conventional integral equation, e.g., the Green's Identity or Somigliana Identity, with the hypersingular integral equation, we call the two equations 'dual integral equations' due to the symmetry and transpose symmetry properties of the kernels [1,22,23]. From the above point of view, the definition of the dual integral equations is quite different from the conventional one used in crack elastodynamics by Buecker [24]. The dual equations in the present paper are independent with respect to each other for the undetermined coefficients of the complementary solution. However, the dual integral equations defined by Buecker resulted from the same equation but by collocating different points. The present formulation totally has four kernel functions, which make possible a unified theory encompassing different schemes, various derivations and interpretations. For elasticity, a detailed derivation can be found in [6]. The singularity order of hypersingularity for the kernel in the normal derivative of the double layer potential is stronger than that of the Cauchy type kernel by one. The paradox of the divergent (nonintegrable) kernel is introduced due to the illegal change of the integral and trace operators from the point of view of the dual integral formulation [1]. In order to ensure a finite value, the Leibnitz's rule should be considered as the derivative of CPV so that the boundary term $2/\epsilon$ can be included to compensate for the minus infinity. Many researchers have paid attention to regularization techniques [25] for hypersingularity and nearly hypersingular integrals. Therefore, the value for the finite part can be determined by means of regularization techniques. Based on the theory of dual integral equations, the dual boundary element method can be implemented [10,11]. The dual integral representation for the Laplace equation was proposed in [22] and a general program, BEPO2D, was developed [1]. In the same way, the acoustic problem with a degenerate boundary also requires the dual integral formulation. A large number of papers have focused on the nonphysical solution for the exterior problem of the Helmholtz equation by using BEM. Burton and Miller [28] first combined the dual integral equations to deal with fictitious eigenvalues. Furthermore, the conventional multiple reciprocity method (MRM) also encounters spurious eigenvalues for the interior problem of the Helmholtz equation [29]. Based on the dual MRM, the spurious eigenvalues have been successfully filtered out in one-dimensional [26] and two-dimensional problems [27]. Both cases, the exterior problem by BEM and the interior problem by MRM, have the problems of nonuniqueness for solution. However, for the interior problem with a degenerate boundary, the conventional BEM also results in a singular system, and the problem of nonuniqueness also occurs. Terai [15] and Wu et al. [16] solved the three-dimensional acoustic problem with a screen by using the dual integral formulation. Based on the complex-valued dual formulation, a general program, DUALHAK, was developed to solve the acoustic frequencies and modes for a cavity with an incomplete partition in [32,33].

In this paper, the dual integral equations are constructed for acoustic problems with degenerate boundaries. Based on the dual formulations, four methods, the complex-valued dual BEM, the real-valued dual MRM, the real-part dual BEM and the complete complex-valued MRM, are proposed. An illustrative example for the acoustic eigenfrequencies of a cavity with an incomplete partition will be demonstrated to show the validity of the four methods. Results are compared with those of FEM by Petyt [34,35] and ABAQUS. Also, the experimental data by Petyt [34,35] are available.

## 2. Dual integral formulation for an acoustic problem with a degenerate boundary

Consider an acoustic problem which has the following governing equation:

$$\nabla^2\phi(\mathbf{x}) + k^2\phi(\mathbf{x}) = 0, \quad \mathbf{x} \in D, \tag{1}$$

where $D$ is the domain of interest, $\mathbf{x}$ is the domain point, $\phi$ is the acoustic pressure and $k$ is the wave number defined by the angular frequency divided by the sound speed. The homogeneous boundary conditions are shown as follows:

$$\phi(\mathbf{x}) = 0, \quad \mathbf{x} \in B_1, \tag{2}$$

$$\frac{\partial \phi(\mathbf{x})}{\partial n_{\mathbf{x}}} = 0, \quad \mathbf{x} \in B_2, \tag{3}$$

where $B_1$ is the essential boundary in which the acoustic pressure is prescribed, $B_2$ is the natural boundary where the normal derivative of the acoustic pressure in the $n_x$ direction is specified, and $B_1$ and $B_2$ construct the whole boundary of the domain $D$.

The first equation of the dual boundary integral equations for the domain point can be derived from Green's third identity:

$$2\pi\phi(\mathbf{x}) = \int_B T(\mathbf{s},\mathbf{x})\phi(\mathbf{s})\,\mathrm{d}B(\mathbf{s}) - \int_B U(\mathbf{s},\mathbf{x})\frac{\partial \phi(\mathbf{s})}{\partial n_{\mathbf{s}}}\,\mathrm{d}B(\mathbf{s}), \quad \mathbf{x} \in D, \tag{4}$$

where $T(\mathbf{s},\mathbf{x})$ is defined by

$$T(\mathbf{s},\mathbf{x}) \equiv \frac{\partial U(\mathbf{s},\mathbf{x})}{\partial n_{\mathbf{s}}}, \tag{5}$$

in which $n_{\mathbf{s}}$ represents the outnormal direction at point $\mathbf{s}$ on the boundary and $U(\mathbf{s},\mathbf{x})$ is the fundamental solution which satisfies

$$\nabla^2 U(\mathbf{x},\mathbf{s}) + k^2 U(\mathbf{x},\mathbf{s}) = 2\pi\delta(\mathbf{x}-\mathbf{s}), \quad \mathbf{x} \in D, \tag{6}$$

where $\delta(\mathbf{x}-\mathbf{s})$ is the Dirac delta function. After taking the normal derivative with respect to Eq. (4), the second equation of the dual boundary integral equations for the domain point can be derived:

$$2\pi\frac{\partial \phi(\mathbf{x})}{\partial n_{\mathbf{x}}} = \int_B M(\mathbf{s},\mathbf{x})\phi(\mathbf{s})\,\mathrm{d}B(\mathbf{s}) - \int_B L(\mathbf{s},\mathbf{x})\frac{\partial \phi(\mathbf{s})}{\partial n_{\mathbf{s}}}\,\mathrm{d}B(\mathbf{s}), \quad \mathbf{x} \in D, \tag{7}$$

where

$$L(\mathbf{s},\mathbf{x}) \equiv \frac{\partial U(\mathbf{s},\mathbf{x})}{\partial n_{\mathbf{x}}}, \tag{8}$$

$$M(\mathbf{s},\mathbf{x}) \equiv \frac{\partial^2 U(\mathbf{s},\mathbf{x})}{\partial n_{\mathbf{x}}\partial n_{\mathbf{s}}}, \tag{9}$$

in which $n_{\mathbf{x}}$ represents the outnormal direction at point $\mathbf{x}$. The explicit forms of the four kernel functions will be elaborated later on. By moving the field point $x$ in Eqs. (4) and (7) to the smooth boundary, the dual boundary integral equations for the boundary point can be obtained as follows:

$$\pi\phi(\mathbf{x}) = \mathrm{CPV}\int_B T(\mathbf{s},\mathbf{x})\phi(\mathbf{s})\mathrm{d}B(\mathbf{s}) - \mathrm{RPV}\int_B U(\mathbf{s},\mathbf{x})\frac{\partial \phi(\mathbf{s})}{\partial n_{\mathbf{s}}}\mathrm{d}B(\mathbf{s}), \quad \mathbf{x} \in B, \tag{10}$$

$$\pi\frac{\partial \phi(\mathbf{x})}{\partial n_{\mathbf{x}}} = \mathrm{HPV}\int_B M(\mathbf{s},\mathbf{x})\phi(s)\mathrm{d}B(\mathbf{s}) - \mathrm{CPV}\int_B L(\mathbf{s},\mathbf{x})\frac{\partial \phi(\mathbf{s})}{\partial n_{\mathbf{s}}}\mathrm{d}B(\mathbf{s}), \quad \mathbf{x} \in B, \tag{11}$$

where RPV is the Riemann principal value, CPV is the Cauchy principal value and HPV is the Hadamard (Mangler) principal value.

It must be noted that Eq. (11) can be derived simply by applying the normal derivative operator to Eq. (10). Differentiation of the Cauchy principal value should be carried out carefully using Leibnitz's rule. The commutative property provides us with two alternatives for calculating the Hadamard principal value in the same way used for crack problems [6]. For the problem including a normal boundary $S$ and degenerate boundary $C^+ + C^-$, i.e., $B = S + C^+ + C^-$, Eqs. (10) and (11) can be reformulated as follows.

For $\mathbf{x} \in S$, Eqs. (10) and (11) become

$$\pi\phi(\mathbf{x}) = \mathrm{CPV}\int_S T(\mathbf{s},\mathbf{x})\phi(\mathbf{s})\,\mathrm{d}B(\mathbf{s}) - \mathrm{RPV}\int_S U(\mathbf{s},\mathbf{x})\frac{\partial\phi(\mathbf{s})}{\partial n_{\mathbf{s}}}\,\mathrm{d}B(\mathbf{s}) + \int_{C^+} T(\mathbf{s},\mathbf{x})\Delta\phi(\mathbf{s})\,\mathrm{d}B(\mathbf{s}) - \int_{C^+} U(\mathbf{s},\mathbf{x})\Sigma\frac{\partial\phi(\mathbf{s})}{\partial n_{\mathbf{s}}}\,\mathrm{d}B(\mathbf{s}), \tag{12}$$

$$\pi\frac{\partial\phi(\mathbf{x})}{\partial n_{\mathbf{x}}} = \mathrm{HPV}\int_S M(\mathbf{s},\mathbf{x})\phi(\mathbf{s})\,\mathrm{d}B(\mathbf{s}) - \mathrm{CPV}\int_S L(\mathbf{s},\mathbf{x})\frac{\partial\phi(\mathbf{s})}{\partial n_{\mathbf{s}}}\,\mathrm{d}B(\mathbf{s}) + \int_{C^+} M(\mathbf{s},\mathbf{x})\Delta\phi(\mathbf{s})\mathrm{d}B(\mathbf{s}) - \int_{C^+} L(\mathbf{s},\mathbf{x})\Sigma\frac{\partial\phi(\mathbf{s})}{\partial n_{\mathbf{s}}}\,\mathrm{d}B(\mathbf{s}), \tag{13}$$

where

$$\Delta\phi(\mathbf{s}) = \phi(\mathbf{s}^+) - \phi(\mathbf{s}^-), \tag{14}$$
$$\Sigma\frac{\partial\phi}{\partial n}(\mathbf{s}) = \frac{\partial\phi}{\partial n}(\mathbf{s}^+) + \frac{\partial\phi}{\partial n}(\mathbf{s}^-). \tag{15}$$

For $\mathbf{x} \in C^+$, Eqs. (10) and (11) reduce to

$$\pi\Sigma\phi(\mathbf{x}) = \mathrm{CPV}\int_{C^+} T(\mathbf{s},\mathbf{x})\Delta\phi(\mathbf{s})\,\mathrm{d}B(\mathbf{s}) - \mathrm{RPV}\int_{C^+} U(\mathbf{s},\mathbf{x})\Sigma\frac{\partial\phi(\mathbf{s})}{\partial n_{\mathbf{s}}}\,\mathrm{d}B(\mathbf{s}) + \int_S T(\mathbf{s},\mathbf{x})\phi(\mathbf{s})\,\mathrm{d}B(\mathbf{s}) - \int_S U(\mathbf{s},\mathbf{x})\frac{\partial\phi(\mathbf{s})}{\partial n_{\mathbf{s}}}\,\mathrm{d}B(\mathbf{s}), \tag{16}$$

$$\pi\Delta\frac{\partial\phi(\mathbf{x})}{\partial n_{\mathbf{x}}} = \mathrm{HPV}\int_{C^+} M(\mathbf{s},\mathbf{x})\Delta\phi(\mathbf{s})\,\mathrm{d}B(\mathbf{s}) - \mathrm{CPV}\int_{C^+} L(\mathbf{s},\mathbf{x})\Sigma\frac{\partial\phi(\mathbf{s})}{\partial n_{\mathbf{s}}}\,\mathrm{d}B(\mathbf{s}) + \int_S M(\mathbf{s},\mathbf{x})\phi(\mathbf{s})\,\mathrm{d}B(\mathbf{s}) - \int_S L(\mathbf{s},\mathbf{x})\frac{\partial\phi(\mathbf{s})}{\partial n_{\mathbf{s}}}\mathrm{d}B(\mathbf{s}), \tag{17}$$

where

$$\Sigma\phi(\mathbf{x}) = \phi(\mathbf{x}^+) + \phi(\mathbf{x}^-), \tag{18}$$
$$\Delta\frac{\partial\phi}{\partial n}(\mathbf{x}) = \frac{\partial\phi}{\partial n}(\mathbf{x}^+) - \frac{\partial\phi}{\partial n}(\mathbf{x}^-). \tag{19}$$

Eqs. (14), (15), (18) and (19) indicate that the unknowns on the degenerate boundary double, and that the additional hypersingular integral equation (17) is correspondingly necessary; i.e., the dual boundary integral equations can provide us with sufficient constraint relations for the doubled boundary unknowns on the degenerate boundary.

Based on the dual integral formulation, the following four methods can be treated as special cases and are discussed in the following subsections.

### 2.1. Complex-valued dual BEM

For simplicity, a two-dimensional case is considered here. The closed forms of the four kernels in the dual complex-valued BEM are shown below:

$$U(\mathbf{s},\mathbf{x}) = \frac{-\mathrm{i}\pi H_0^{(1)}(kr)}{2}, \tag{20}$$

$$T(\mathbf{s},\mathbf{x}) = \frac{-\mathrm{i}k\pi}{2} H_1^{(1)}(kr)\frac{y_i n_i}{r}, \tag{21}$$

$$L(\mathbf{s},\mathbf{x}) = \frac{\mathrm{i}k\pi}{2} H_1^{(1)}(kr)\frac{y_i \bar{n}_i}{r}, \tag{22}$$

$$M(\mathbf{s},\mathbf{x}) = \frac{-\mathrm{i}k\pi}{2}\left\{ -k\frac{H_2^{(1)}(kr)}{r^2} y_i y_j n_i \bar{n}_j + \frac{H_1^{(1)}(kr)}{r} n_i \bar{n}_i \right\}, \tag{23}$$

where $\mathrm{i}^2 = -1,\ r = |\mathbf{x}-\mathbf{s}|, H_n^{(1)}(kr)$ denotes the first kind Hankel function with order $n$, and $n_i$ and $\bar{n}_i$ denote the $i$th components of the normal vectors at $\mathbf{s}$ and $\mathbf{x}$, respectively.

### 2.2. Real-valued dual MRM

By employing the conventional MRM [26,30,37], we have the two kernels

$$U(\mathbf{s},\mathbf{x}) = U^0(\mathbf{s},\mathbf{x}) - k^2 U^1(\mathbf{s},\mathbf{x}) + k^4 U^2(\mathbf{s},\mathbf{x}) + \cdots, \tag{24}$$

$$T(\mathbf{s},\mathbf{x}) = T^0(\mathbf{s},\mathbf{x}) - k^2 T^1(\mathbf{s},\mathbf{x}) + k^4 T^2(\mathbf{s},\mathbf{x}) + \cdots, \tag{25}$$

where the explicit forms of $U^j(\mathbf{s},\mathbf{x})$ and $T^j(\mathbf{s},\mathbf{x})$ will be introduced later. In order to filter out the spurious solutions, the dual MRM proposed the hypersingular integral equation with the following two kernels:

$$L(\mathbf{s},\mathbf{x}) = L^0(\mathbf{s},\mathbf{x}) - k^2 L^1(\mathbf{s},\mathbf{x}) + k^4 L^2(\mathbf{s},\mathbf{x}) + \cdots, \tag{26}$$

$$M(\mathbf{s},\mathbf{x}) = M^0(\mathbf{s},\mathbf{x}) - k^2 M^1(\mathbf{s},\mathbf{x}) + k^4 M^2(\mathbf{s},\mathbf{x}) + \cdots, \tag{27}$$

in which

$$L^j(\mathbf{s},\mathbf{x}) = \frac{\partial U^j(\mathbf{s},\mathbf{x})}{\partial n_{\mathbf{x}}}, \quad j = 0,1,2,\ldots, \tag{28}$$

$$M^j(\mathbf{s},\mathbf{x}) = \frac{\partial^2 U^j(\mathbf{s},\mathbf{x})}{\partial n_{\mathbf{x}} \partial n_{\mathbf{s}}}, \quad j = 0,1,2,\ldots. \tag{29}$$

The explicit forms of the $j$th terms in the four kernels by using the real-valued dual MRM are

$$U^j(\mathbf{s},\mathbf{x}) = r^{2j}\ln(r)A(j) - r^{2j}B(j), \tag{30}$$

$$T^j(\mathbf{s},\mathbf{x}) = -[(2j\ln(r)+1)r^{2j-2}y_i n_i]A(j) + [2jr^{2j-2}y_i n_i]B(j), \tag{31}$$

$$L^j(\mathbf{s},\mathbf{x}) = +[(2j\ln(r)+1)r^{2j-2}y_i \bar{n}_i]A(j) - [2jr^{2j-2}y_i \bar{n}_i]B(j), \tag{32}$$

$$\begin{aligned} M^j(\mathbf{s},\mathbf{x}) = & -[(4j(j-1)\ln(r)+4j-2)r^{2j-4}y_i n_i y_k \bar{n}_k]A(j) - [(2j\ln(r)+1)r^{2j-2}n_i \bar{n}_i]A(j) \\ & + [4j(j-1)r^{2j-4}y_i n_i y_k \bar{n}_k]B(j) + [2jr^{2j-2}n_i \bar{n}_i]B(j), \end{aligned} \tag{33}$$

where $A(j)$ and $B(j)$ in Eq. (30) can be found in [30,37]. After constructing the hypersingular integral equation, the dual MRM can filter out the spurious eigenvalues and eigenmodes. Nevertheless, the dual MRM cannot solve the problems with impedance boundary conditions since the information on the imaginary part is lost. Also, this is the reason why the conventional MRM cannot solve for the exterior problems since the method cannot satisfy the radiation condition automatically. The applications of dual MRM to the vibration problems of a rod and a beam can be found in [26,38].

### 2.3. Real-part dual BEM

According to the findings by Yeih et al. [29] and Kamiya et al. [36], the series forms of the kernels in the real-valued dual MRM are no more than the real-parts of the closed-form kernels in the complex-valued dual BEM. The real-part for the kernels in the complex-valued dual BEM are shown below:

$$U(\mathbf{s},\mathbf{x}) = \mathrm{Re}\left\{\frac{-\mathrm{i}\pi H_0^{(1)}(kr)}{2}\right\}, \tag{34}$$

$$T(\mathbf{s},\mathbf{x}) = \mathrm{Re}\left\{\frac{-\mathrm{i}k\pi}{2} H_1^{(1)}(kr)\frac{y_i n_i}{r}\right\}, \tag{35}$$

$$L(\mathbf{s},\mathbf{x}) = \mathrm{Re}\left\{\frac{\mathrm{i}k\pi}{2} H_1^{(1)}(kr)\frac{y_i \bar{n}_i}{r}\right\}, \tag{36}$$

$$M(\mathbf{s},\mathbf{x}) = \mathrm{Re}\left\{\frac{-\mathrm{i}k\pi}{2}\left\{ -k\frac{H_2^{(1)}(kr)}{r^2} y_i y_j n_i \bar{n}_j + \frac{H_1^{(1)}(kr)}{r} n_i \bar{n}_i\right\}\right\}, \tag{37}$$

where Re denotes the real-part. In the same way, this method has the problem of spurious modes as the real-valued MRM does if the singular integral equation ($UT$ equation) is used only. Also, this method as well as the real-valued dual MRM, cannot treat the exterior problems and interior problems with impedance boundary conditions. The main advantage of this method is that it can solve problems in the real domain without the lengthy derivation of the series kernels in the real-valued dual MRM.

### 2.4. Complete complex-valued MRM

Recently, Yeih et al. [29] proposed a complete MRM which can recover the information of the imaginary part. The main difference between the complete MRM and the complex-valued dual BEM is the kernel representation. The kernels in the complete MRM are the same as those of the complex-valued dual BEM after series expansion. The series forms can be represented by

$$U(\mathbf{s},\mathbf{x}) = \sum_{j=0}^{\infty}(-k^2)^j U_j(\mathbf{s},\mathbf{x}), \tag{38}$$

$$T(\mathbf{s},\mathbf{x}) = \frac{\partial U(\mathbf{s},\mathbf{x})}{\partial n_{\mathbf{s}}}, \tag{39}$$

$$L(\mathbf{s},\mathbf{x}) = \frac{\partial U(\mathbf{s},\mathbf{x})}{\partial n_{\mathbf{x}}}, \tag{40}$$

$$M(\mathbf{s},\mathbf{x}) = \frac{\partial^2 U(\mathbf{s},\mathbf{x})}{\partial n_{\mathbf{x}}\partial n_{\mathbf{s}}}, \tag{41}$$

where

$$U_0(\mathbf{s},\mathbf{x}) = \ln(r) + \left(\gamma + \ln\left(\frac{k}{2}\right)\right) - \frac{\pi}{2\mathrm{i}}, \quad j = 0, \tag{42}$$

$$U_j(\mathbf{s},\mathbf{x}) = 4F_j\big(\ln(r) - S_j\big) + F_j\left(\gamma + \ln\left(\frac{k}{2}\right)\right) + \frac{\pi\mathrm{i}}{2}F_j, \quad j = 1,2,3,\ldots, \tag{43}$$

in which

$$\gamma = \lim_{j\to\infty}\left(\sum_{l=1}^{j}\frac{1}{l} - \ln(j)\right), \tag{44}$$

$$F_j = \frac{r^{2j}}{(j!)^2 4^j}, \tag{45}$$

$$S_j = \sum_{l=1}^{j}\frac{1}{l}. \tag{46}$$

It is interesting to find that the difference between $U^0(\mathbf{s},\mathbf{x})$ in Eq. (30) for $j=0$ and $U_0(\mathbf{x},\mathbf{s})$ in Eq. (42) is only a complex constant which can make the kernel in the complete MRM satisfy the radiation condition. Also, the kernel functions of the complete MRM in Eq. (38) with infinite terms can be proved to be equal to that of the complex-valued dual BEM in Eq. (20) after series expansion.

## 3. Detection of the spurious roots for the real-valued MRM and the real-part BEM

### 3.1. Residual method

According to Eqs. (10) and (11), we can obtain the eigenvalues independently for the problem without degenerate boundaries. However, spurious roots are imbedded. As mentioned by Kamiya et al. [36], the equation derived using MRM is no more than a real-part in the complex-valued formulation. The loss of the imaginary part in MRM results in the spurious roots. Chen and Wong [26] and Yeih et al. [29] extended the general proof for any dimensional problems and demonstrated it using a one-dimensional case. The imaginary part in the complex-valued formulation is not present in MRM, and the number of constraints for the eigenequation is insufficient. These findings can explain the reason why the spurious roots occur using the MRM when either Eq. (10) or (11) is employed alone, i.e., the mechanism of the spurious roots can be understood in this way.

Since only the real-part is concerned in MRM, one approach to obtaining enough constraints for the eigenequation instead of obtaining the imaginary part of the complex-valued formulation is obtained by differentiation with respect to the conventional MRM. This method results in the hypersingular formulation for MRM. For simplicity, we deal with the Neumann problem. After discretizing the dual integral equations, we have

$$[T(k)]\{u\} = 0, \tag{47}$$

$$[M(k)]\{u\} = 0, \tag{48}$$

where $[T]$ and $[M]$ are the influence matrices for $T$ and $M$ kernels. More detail can be found in [27]. An approach to detecting the spurious roots is the criterion of satisfying both Eqs. (47) and (48). The spurious roots from Eq. (48) will not satisfy Eq. (47). Also, the spurious roots from Eq. (51) will not satisfy Eq. (50) in controversia. Therefore, two residuals can be defined as follows:

$$\epsilon_T = [T(k_M)]\{u_M\}, \tag{49}$$

where $\{u_M\}$ satisfies $[M(k_M)]\{u_M\} = 0$

$$\epsilon_M = [M(k_T)]\{u_T\}, \tag{50}$$

where $\{u_T\}$ satisfies $[T(k_T)]\{u_T\} = 0$, and $\epsilon_T$ and $\epsilon_M$ are the residuals induced by Eqs. (49) and (50), respectively, and $k_M$ and $k_T$ are the eigenvalues obtained by Eqs. (47) and (48), respectively. By setting an appropriate value of the threshold, we can determine whether the root is true or spurious. To double check, the acoustic modes are examined by means of the distribution of nodal lines and orthogonal properties.

### 3.2. Singular value decomposition (SVD) technique

It is noted that the residual method needs to find the spurious boundary modes in advance from one equation (either UT or LM equation) in the stage of direct search method, and then substitutes it into another eigenequation from either UT or LM equation to find the residuals. In some cases, e.g., double roots [33] and null matrix for rank deficiency [27], it is not straightforward to determine the boundary modes. Now we will look for a more efficient way to filter out the spurious eigenvalues free from determining the boundary mode in advances.

To distinguish the spurious eigenvalues by the SVD technique, we can merge Eqs. (47) and (48) together to have

$$[C(k)]_{2N\times N}\{u\}_{N\times 1} = \{0\}, \tag{51}$$

where $N$ is the number of unknowns, and $[C(k)]$ matrix is composed from $[T]$ and $[M]$ matrices as shown below:

$$[C(k)]_{2N\times N} = \begin{bmatrix} T(k) \\ M(k) \end{bmatrix}. \tag{52}$$

Even though the $[C]$ matrix has dependent rows resulted from the degenerate boundary, the SVD technique can be employed to find all the true eigenvalues since enough constraints are considered. For the true eigenvalues, the rank for the $[C]$ matrix with dimension $2N \times N$ must at most be $N-1$ to have a nontrivial solution. For the spurious eigenvalues, the rank must be $N$ to have a trivial solution. Based on this criterion, the SVD technique is utilized to detect the true eigenvalues by checking the first minimum singular value to be zero. Since discretization creates error, not exactly zeros will be obtained. In order to avoid the threshold for the zero definition, a more nearer zero will be found in the smaller increment for the critical wave number, $k$. Such a value is confirmed to be a true eigenvalue.

A brief introduction to SVD is given below.

Consider a linear algebra problem with more equations than unknowns:

$$[\mathbf{A}]_{m\times n}\mathbf{x}_{n\times 1} = \mathbf{b}_{m\times 1}, \quad m > n, \tag{53}$$

where $m$ is the number of equations, $n$ is the number of unknowns and $\mathbf{A}$ is the leading matrix, which can be decomposed into

$$[\mathbf{A}]_{m\times n} = \mathbf{U}_{m\times m}\Sigma_{m\times n}\mathbf{V}^*_{n\times n}, \tag{54}$$

where $\mathbf{U}$ is a left unitary matrix constructed by the left singular vectors, $\Sigma$ is a diagonal matrix which has singular values $\sigma_1, \sigma_2, \ldots,$ and $\sigma_n$ allocated in the diagonal line as

$$\Sigma = \begin{bmatrix} \sigma_n & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}, \quad m > n, \tag{55}$$

in which $\sigma_n \geqslant \sigma_{n-1} \cdots \geqslant \sigma_1$ and $\mathbf{V}^*$ is the complex conjugate transpose of a right unitary matrix constructed by the right singular vectors. As we can see in Eq. (53), there exist at most $n$ nonzero singular values. This means that we can find at most $n$ linear independent equations in the system of equations. If we have $s$ zero singular values $(0 \leqslant s \leqslant n)$, this means that the rank of the system of equations is equal to $n - s$. However, the singular value may be very close to zero numerically, resulting in rank deficiency. For a general eigenproblem as shown in this paper, the $[C]$ matrix with dimension $2N \times N$ in Eq. (51) will have the rank to be $N-1$ for the true eigenvalue with multiplicity 1. For the true eigenvalues with multiplicity $M$, the rank will be reduced to $N - M$. In the case of spurious eigenvalues, the rank is $N$. Therefore, only trivial solution for the spurious mode can be obtained.

Determining the eigenvalues of the system of equations has now been transformed into finding the values of $k$ which make the rank of the leading coefficient matrix be smaller than $N$. This means that when $m = 2N$, $n = N$ and $\mathbf{b}_{2N\times 1} = \mathbf{0}$, the eigenvalues will make $s = M$, such that the minimum singular value must be zero or very close to zero. To find the boundary eigenvector associated with the eigevalue of multiplicity 1, we can set one of the elements in the boundary eigenvector to be one and solve it using SVD by the pseudo-inverse matrices. The pseudo-inverse matrix, $\mathbf{A}^+$ of $\mathbf{A}$, is expressed as

$$\mathbf{A}^+_{n\times m} = \mathbf{V}_{n\times n}\Sigma^+_{n\times m}\mathbf{U}^*_{m\times m}, \tag{56}$$

where $\Sigma^+$ is constructed by taking the transpose of $\Sigma$ and then replacing the diagonal singular value terms with its inverse, expressed as

$$\Sigma^{+} = \begin{bmatrix} \frac{1}{\sigma_n} & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{\sigma_1} & \cdots & 0 \end{bmatrix}, \quad m > n. \tag{57}$$

The above-mentioned SVD method has been proved to be equivalent to the least square error solution in determining the unknown vector when the number of equations is larger than the number of unknowns [39]. After introducing the SVD method, we do not need to worry about how to pick a specific group of equations such that the rank of the leading coefficient is sufficient to solve for the boundary eigenvector. On the other hand, we can take all the $2N$ equations in dual model into account, which apparently causes the rank of the leading coefficient matrix to be equal to $N-1$ for spurious eigenvalues. Thus, the boundary eigenvector can be easily found in the sense of the least square error. After finding the first minimum singular value in $[C]$ matrices for different values of $k$, one can easily determine whether the eigenvalue is true or not free from finding the boundary modes in advance as residual method does. For further details concerning the SVD method, please refer to [40]. Another advantage for using SVD is that it can determine the multiplicities for the true eigenvalues by finding the number of near zero in the successive singular values. An example, a rectangular cavity with a zero-thickness partition with the eigenvalues of multiplicity one is shown to demonstrate the SVD technique.

## 4. An illustrative example

To demonstrate the validity of the four methods using dual formulation, an example given by Petyt [34,35] is considered. A two-dimensional cavity enclosed by rigid walls is shown in Fig. 1. The cavity is a rectangle, 236 mm long and 113 mm high, and contains a rigid partition located halfway along the longer side of the cavity. The thickness of the partition is modeled as zero thickness, i.e., the boundary of partition is degenerate. The partition extends from one side of the cavity halfway across to the other wall. The cavity is filled with an acoustic fluid whose density is 1.0 $kg/m^3$ and whose bulk modulus is 0.1183 MPa. The first five acoustic frequencies given in Table 1 were solved using the four methods, and the results were compared with those of ABAQUS program [30,31] and FEM by Petyt [34,35]. In order to filter out the spurious eigenvalues for the real-valued dual MRM and the real-part dual BEM, the residual method and the singular value decomposition technique can be adopted. Fig. 2 shows the residuals by using the real-valued dual MRM. Fig. 3 shows the residuals by using the real-part dual BEM. It is found that the spurious eigenvalues can be easily filtered out by Fig. 3 after choosing an appropriate threshold value and checking the number of nodal lines and the orthogonality of modal shapes. However, Fig. 2 cannot have an appropriate threshold value. The reason may be explained that only ten-terms expansion is considered in the real-valued dual MRM. Figs. 4 and 5 show the first minimum singular value, $\sigma_1$, versus acoustic frequency,

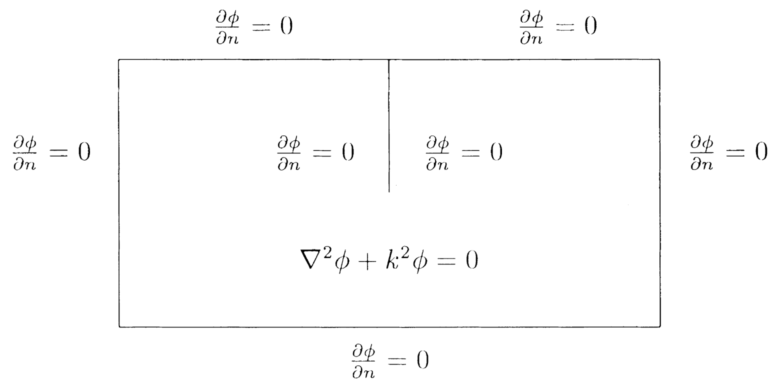


Fig. 1. A cavity with an incomplete partition.

Table 1
The first five acoustic frequencies (Hz) using different methods

| | Mode 1 | Mode 2 | Mode 3 | Mode 4 | Mode 5 |
|---|---|---|---|---|---|
| Complex-valued dual BEM | 584 | 1439 | 1518 | 1537 | 1818 |
| Real-valued dual MRM by residual | 580 | 1452 | 1510 | 1532 | 1762 |
| Real-part dual BEM by residual | 579 | 1445 | 1515 | 1531 | 1818 |
| Real-valued dual MRM by SVD | 577 | 1444 | 1529 | 1534 | 1991 |
| Real-part dual BEM by SVD | 588 | 1444 | 1518 | 1537 | 1827 |
| Complete complex-valued MRM | 576 | 1447 | 1510 | 1521 | 1800 |
| ABAQUS (AC2D4) | 618 | 1421 | 1496 | 1527 | 1780 |
| ABAQUS(AC2D8) | 605 | 1458 | 1536 | 1563 | 1851 |
| FEM by Petyt | 591 | 1478 | 1540 | 1570 | 1861 |
| Measurement by Petyt | 570 | 1470 | 1534 | 1555 | 1840 |

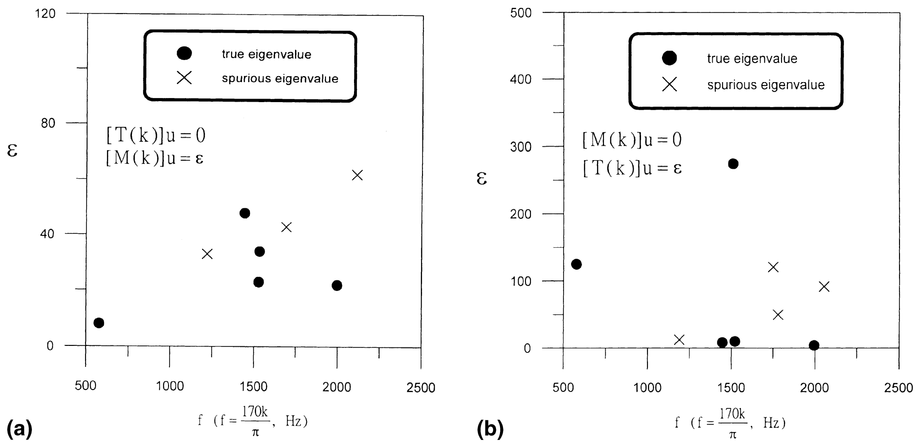


Fig. 2. The residuals for the real-valued dual MRM: (a) $[M(k)]\{u\} = \epsilon$; (b) $[T(k)]\{u\} = \epsilon$.

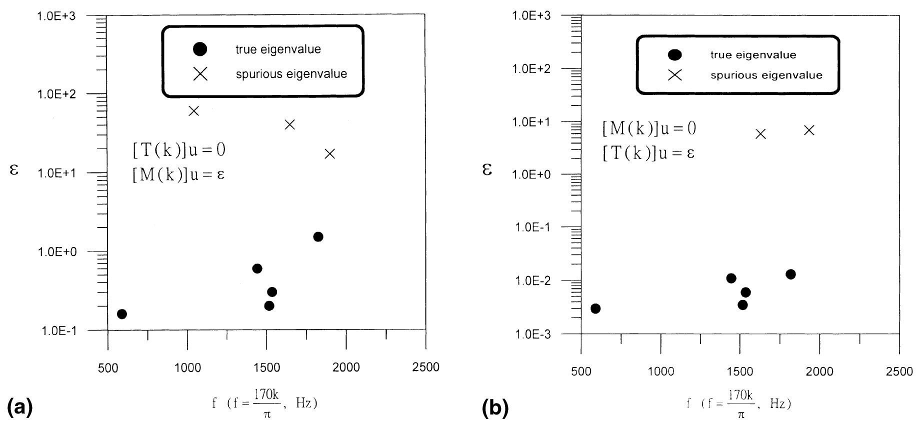


Fig. 3. The residuals for the real-part dual BEM: (a) $[M(k)]\{u\} = \epsilon$; (b) $[T(k)]\{u\} = \epsilon$.

$f$, for the real-valued dual MRM and the real-part dual BEM, respectively. It is found that the true eigenvalues can be easily determined by finding the near-zero singular values. It can be expected that the two figures are the same if infinite-terms expansion for the kernels are considered in the real-valued MRM. Two types of elements in the ABAQUS program, AC2D4 and AC2D8, were considered. Although no mesh

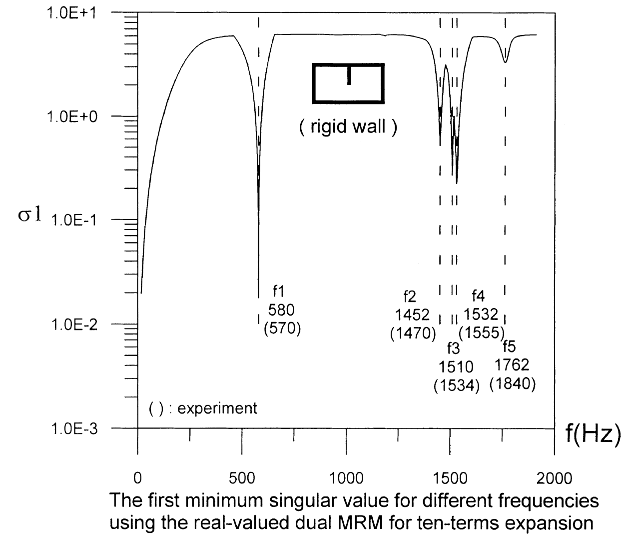


Fig. 4. The first minimum singular value, $\sigma_1$ for different frequencies using the real-valued dual MRM for 10-terms expansion.

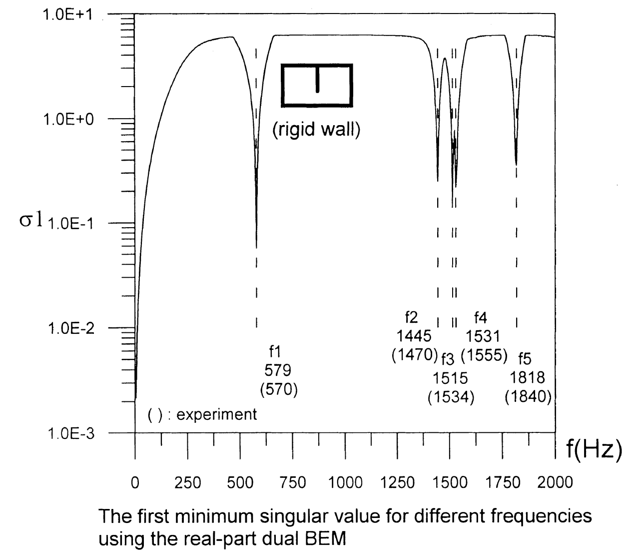


Fig. 5. The first minimum singular value versus, $\sigma_1$ for different frequencies using the real-part dual BEM.

convergence studies have been performed, the close agreement between the acoustic frequencies and the acoustic modes of the present results in coarse mesh and those given by Petyt et al. suggests that the mesh is adequate. For the first mode, the present results are also in better agreement with the experimental data obtained by Petyt than they are with the data obtained using other numerical methods.

## 5. Conclusions

The general formulation of the dual integral equations of the boundary value problem for the two-dimensional Helmholtz equation with a degenerate boundary has been reviewed in this paper. Four methods based on the dual formulation, the complex-valued dual BEM, the real-valued dual MRM, the real-part dual BEM and the complete complex-valued MRM, were proposed. The acoustic frequencies for a rectangular cavity with an incomplete partition has been successfully determined by using the four methods, and the results have been compared well with those obtained using other numerical methods and experiments.

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