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Study of free-surface seepage problems using hypersingular equations

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SUMMARY

Free-surface boundary is not only unknown in *a priori* but also the boundary conditions are overspecified. In this paper, the Laplace problem with overspecified boundary conditions on the free surface is solved by using the hypersingular equation instead of singular equation used conventionally in boundary element method. The free surface can be determined using an iterative procedure starting from an initial guess. By introducing the hypersingular equation, the convergence rate of free surface can be accelerated. Finally, numerical examples including rectangular dams and canals were demonstrated and were compared with others to show the validity of the present method. Copyright © 2006 John Wiley & Sons, Ltd.

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KEY WORDS: hypersingular equation; dual BEM; free surface; canal

1. INTRODUCTION

The analysis of seepage problems is strongly influenced by porous media, hydraulic gradient, and pore pressure. In order to study these problems, accurately defining the position of free surface is very important and necessary. In this decade, many researchers utilized boundary element methods (BEM) to solve free-surface seepage problems but only singular equation was used.

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Finite element method [1] needs many iterations after the initial guess. In addition, the solution of problems involving a free surface is generally obtained through an iterative scheme. Since the dual boundary integral element method has been proposed for the Laplace equation problem by Chen and Hong [2, 3], hypersingular equation can be considered as an alternative to solve the seepage problems with free surface. Dual BEM was applied to deal with electrostatic problems by Chyuan *et al*. [4]. Also, the extension to fracture mechanics was done by Hong and Chen [5]. A detailed review article can be found in [6]. We can wonder whether the hypersingular equation may speed up the convergence rate for free-surface problems.

In the past, many different methods were used to determine the location of free surface. For example, Aitchison used the finite difference method (FDM) to determine the position of free surface [7], and Caffrey and Bruch [8] also used FDM and FEM to analyse it. Then, the BEM was used to study the seepage flow through the porous media by Liggett and Liu [9]. After that, the finite element method (FEM) was proposed to solve the problems by Westbrook [1] and by Gioda and Gentile [10], respectively. Cheng *et al*. [11] also determined the free surface for sluice and spillway flows. Demetracopoulos and Hadjitheodorou [12] solved the free-surface problems of canals. Regarding to previous methods, free surface can be determined by using all of these methods but in a different rate of convergence. Hypersingular equation will be proposed to examine its superiority over the available methods.

Comparing with previous methods, domain-based approach spends much time on mesh generation and also need more computer memory storage. In the FEM, the domain was divided into many elements. Mesh generation is a time-consuming job. Especially, it needs to remesh in the domain for each iteration after updating the free surface. Therefore, BEM was proposed to analyse the problem, because it only discretizes the boundary of domain. Mesh generation becomes much easier by this method and it can also save much computer memory storage at the same time. In this study, constant elements are considered for simplicity.

In this paper, free surface is regarded as a moving boundary with the overspecified boundary conditions. The main purpose of this paper is to employ the hypersingular equation for determining the location of free surface. The rate of convergence by using the hypersingular equation is also examined after comparing with that of FEM and traditional BEM, i.e. singular equation. In the following section, singular equation and hypersingular equation will be reviewed and several examples will be tested to demonstrate its validity.

2. PROBLEM STATEMENT

In this paper, the steady-state flow through the homogeneous dam is considered. The problem is to find the potential ϕ which satisfies the Laplace equation

$$\nabla^2\phi = 0 \tag{1}$$

Referring to Figure 1, the boundary conditions can be expressed as

$$\phi = h_1 \quad \text{on } \Gamma_2 \tag{2}$$

$$\phi = h_2 \quad \text{on } \Gamma_5 \tag{3}$$

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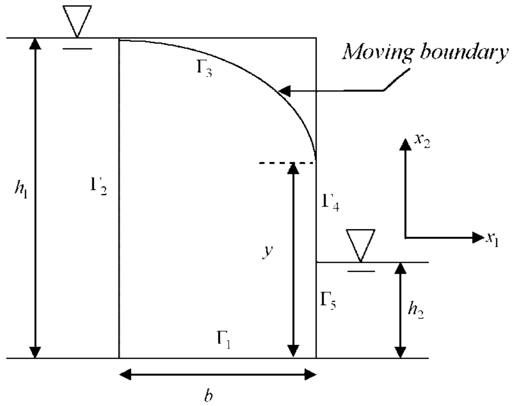

Figure 1. Flow through a rectangular dam.

In the Γ_2 and Γ_5, the potential ϕ is constant and it can be written as

$$\phi = y + \frac{p}{\gamma} \tag{4}$$

where y is the position head, p is the pressure head and γ is the unit weight of fluid. Since the bottom of dam is impermeable, the boundary condition of the earth dam is zero

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{on } \Gamma_1 \tag{5}$$

The boundary Γ_4 is the seepage surface and the pressure head is zero, thus it can be expressed as

$$\phi = y(\underset{\sim}{x}) \quad \text{on } \Gamma_4 \tag{6}$$

The boundary Γ_3 is the free surface which is the interface between saturated region and dry region. The free surface is regarded as the overspecified boundary conditions of

$$\frac{\partial \phi}{\partial n} = 0, \quad \phi = y(\underset{\sim}{x}) \quad \text{on } \Gamma_3 \tag{7}$$

where the $\phi = y(\underset{\sim}{x})$ is unknown in *a priori* and needs to be determined iteratively after the initial guess of free surface. In the following section, the free surface will be solved by using the dual boundary integral equations in conjunction with the initial guess of free surface.

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3. DUAL BOUNDARY ELEMENT METHOD FOR FREE-SURFACE SEEPAGE PROBLEMS

In this section, the hypersingular formulation for free-surface seepage problems is briefly described. First, the Green's identity is reviewed, which can be expressed as

$$\int_D (\phi\nabla^2\psi - \psi\nabla^2\phi)\,\mathrm{d}D = \int_B \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)\mathrm{d}B \tag{8}$$

where D is the domain, B is the boundary, ϕ and ψ are the two scalar potentials of seepage flow, and ∇^2 denotes the Laplacian operator. By utilizing the Green's identity, one of the dual boundary integral equations for the domain point x is

$$2\pi\phi(x) = \int_B T(s,x)\phi(s)\,\mathrm{d}B(s) - \int_B U(s,x)\frac{\partial\phi(s)}{\partial n_s}\,\mathrm{d}B(s), \quad x\in D \tag{9}$$

Equation (9) is termed the singular formula. Then taking the normal derivative of Equation (9) for the domain point x, the other equation of the dual boundary integral equations, i.e. hypersingular formula, can be derived

$$2\pi\frac{\partial\phi(x)}{\partial n_x} = \int_B M(s,x)\phi(s)\,\mathrm{d}B(s) - \int_B L(s,x)\frac{\partial\phi(s)}{\partial n_s}\,\mathrm{d}B(s), \quad x\in D \tag{10}$$

where $U(s,x)=\ln r$, $T(s,x)=\partial U(s,x)/\partial n_s$, $L(s,x)=\partial U(s,x)/\partial n_x$, $M(s,x)=\partial^2 U(s,x)/\partial n_s\partial n_x$, r denotes the distance between source point s and collocation point x, n_s is the unit outer normal at point s on the boundary, and n_x is the unit outer normal on the boundary. By tracing the domain point x to the boundary, Equations (9) and (10) reduce to

$$\alpha\phi(x) = \mathrm{CPV}\int_B T(s,x)\phi(s)\,\mathrm{d}B(s) - \mathrm{RPV}\int_B U(s,x)\frac{\partial\phi(s)}{\partial n_s}\,\mathrm{d}B(s), \quad x\in B \tag{11}$$

$$\alpha\frac{\partial\phi(x)}{\partial n_x} = \mathrm{HPV}\int_B M(s,x)\phi(s)\,\mathrm{d}B(s) - \mathrm{CPV}\int_B L(s,x)\frac{\partial\phi(s)}{\partial n_s}\,\mathrm{d}B(s), \quad x\in B \tag{12}$$

where CPV, RPV, HPV are the Cauchy principal value, Riemann principal value, and Hadamard principal value, respectively, and $\alpha=\pi$ in the case of a smooth boundary [13]. Equations (11) and (12) are named the dual boundary integral equations for the boundary point. Based on the theory of dual boundary integral equations, a program BEPO2D was developed by Chen and Hong [2] to solve the Laplace equation. In numerical implementation, Equation (12) is discretized to yield a linear algebraic system.

$$U\underset{\sim}{t} = T\underset{\sim}{u} \tag{13}$$

Due to the overspecified boundary condition of Equation (7) for free-surface problem, we need an initial guess for the free surface and then iterates. By solving the hypersingular equation of (10)

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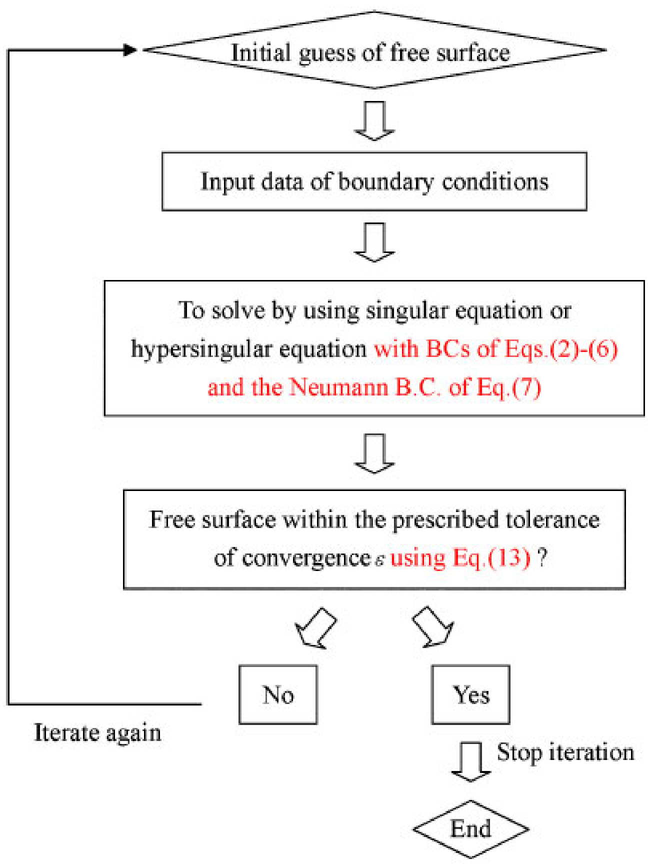

Figure 2. Flowchart of iteration procedure.

subject to BCs of Equations (2)–(6) and the Neumann BCs of Equation (7), for the initial guess of free surface, we can solve ϕ. However, the solved ϕ is not the same with the initial guess of the free surface. Then, by setting the ϕ value for the second guess of free surface, we can solve the hypersingular equation again. For the stop condition of convergence, we use

$$\varepsilon = \frac{\sqrt{\sum_{i=1}^{m} (\phi_i^{(n+1)} - \phi_i^{(n)})^2}}{\sqrt{\sum_{i=1}^{m} (\phi_i^{(n)})^2}} < 10^{-4} \tag{14}$$

where the symbol m is the number of elements on the free surface, $\phi_i^{(n+1)}$ is the location of free surface for the $(n+1)$th number of iteration, and the allowable tolerance used in this paper is 10^{-4}. Stop condition of Equation (14) determines the number of iteration.

In this paper, the program BEPO2D was used to iterate the location of free surface, and the flowchart is shown in Figure 2.

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4. NUMERICAL EXAMPLES

4.1. Case 1. Rectangular dam

The free surface of the homogeneous rectangular dam is considered in Figure 1, where the upper hydraulic head $h_1 = 24$, the lower hydraulic head $h_2 = 4$, and the width of the earth dam $b = 16$. In this case, the boundary of domain is divided into 39 elements. In order to compare with the available results [7], the initial guess of free surface is divided into eight elements; all the elements are the same in length, which are shown in Figure 3. By utilizing the singular and hypersingular formulations to solve the location of free surface under the stop condition of Equation (14), respectively, results are compared with those of Aitchison and Westbrook as shown in Table I, Figures 4 and 5. It can be seen in Table I that the data (12.68) is better to match the numerical solution of 12.79 by Aitchison than that (12.61) of using the singular formulation.

For the free-surface problem, it is one kind of inverse problem since boundary is not known in *a priori*. It is realizable that the analytical solution of this problem is not easy to find. Based on the semi-analytical nature of Aitchison's solution by using the complex variable, his data is more believable than other numerical results. The separation point at $b = 16$ is interesting and important since the location is a singular point due to the intersection of the fixed and free boundaries. Besides, the separation point plays an important role in dam stability and is our main concern for civil engineers. Therefore, the data of the point ($b = 16$) by Aitchison is chosen to compare with

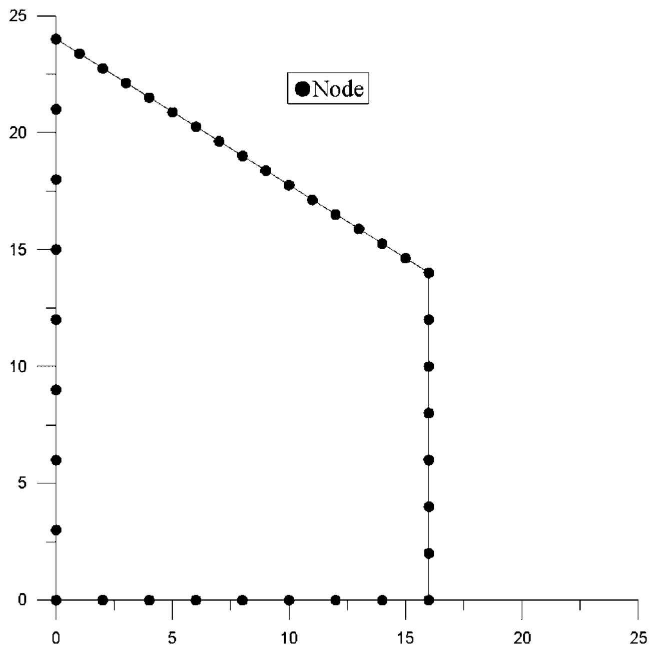

Figure 3. Boundary element mesh of case 1.

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Table I. Free surface obtained by using different methods.

x	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Aitchison	23.74	23.41	23.02	22.59	22.12	21.60	21.04	20.43	19.78	19.08	18.31	17.48	16.57	15.54	14.39	12.79
Westbrook	23.64	23.32	23.06	22.52	22.12	21.55	21.07	20.36	19.81	19.07	18.26	17.45	16.54	15.51	14.33	—
Present (singular equation)	23.76	23.42	23.03	22.59	22.12	21.60	21.04	20.43	19.78	19.07	18.30	17.47	16.56	15.50	14.15	12.61
Present (hyper-singular equation)	23.74	23.40	23.01	22.52	22.09	21.57	21.00	20.39	19.73	19.02	18.24	17.39	16.45	15.39	14.09	12.68

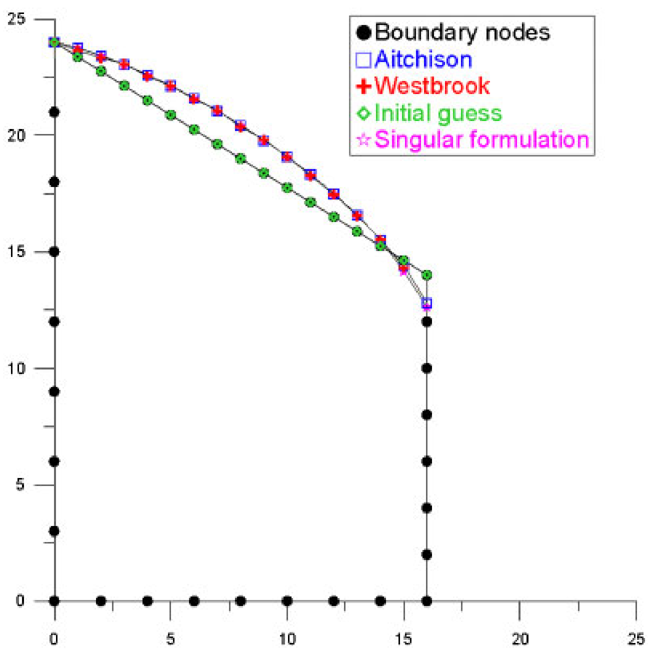

Figure 4. Free surface (singular equation).

our results. It is also found that the hypersingular-approach solution cannot obtain better solution through all the range. At least, we provide an alternative approach to solve the free-surface seepage problems and predict the separation point properly. After comparing with the references, the number of iterations of present method is fewer than that of FEM and singular BEM as shown in Table II.

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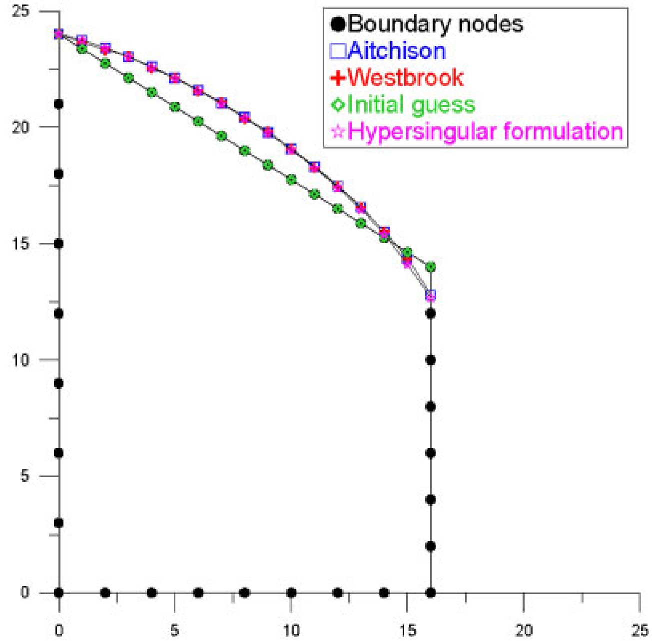

Figure 5. Free surface (hypersingular equation).

Table II. Number of iterations by using different methods (Case 1).

Method	Mesh	Number of iterations
FEM [1]	17×25	49
Singular equation	39	14
Present (hypersingular equation)	39	13

4.2. Case 2. Rectangular dam

Referring to the Figure 1 again, another case with different water depth is considered. The homogeneous rectangular dam with base $b=1$, the upper hydraulic head $h_1=1$, and the lower hydraulic head $h_2=0$. The boundary mesh is shown in Figure 6, and the free surface is determined by using the singular and hypersingular equations, respectively, as shown in Figures 7 and 8. Table III shows the profile of free surface by using different methods. The number of iterations of singular and hypersingular equations is also shown in Table IV. Table V shows the final position of the separation point above the dam base obtained by using different methods. Although Polubarinova-Kochina developed a free surface for the rectangular dam [14], it was adjusted by Cryer later [15]. Ozis used Cryer's formulation and improved the evaluations of integrals in 1981 [16] and the location of 12.7070 is seen as the best solution of separation point. Then, Bruch used linear boundary elements and an iterative technique to determine the separation point [17]. Singular

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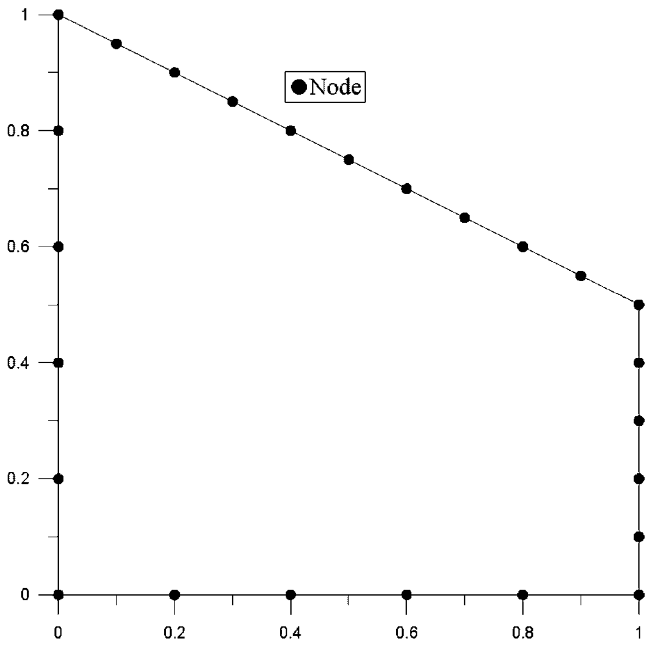

Figure 6. Boundary element mesh of case 2.

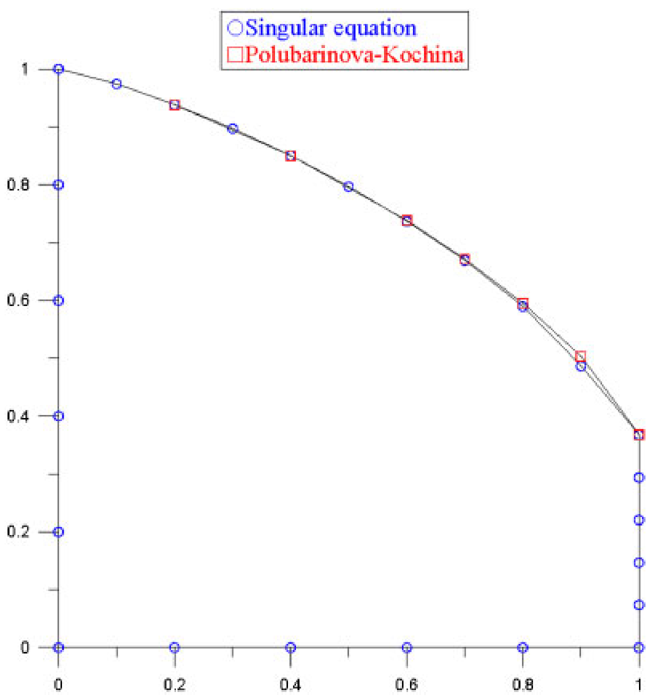

Figure 7. Free surface (singular equation).

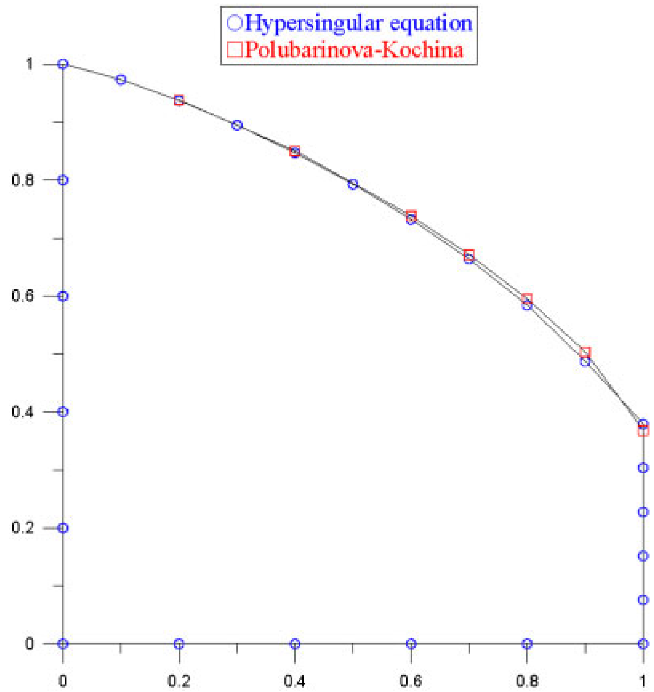

Figure 8. Free surface (hypersingular equation).

Table III. Free surface obtained by using different methods.

x	0.2	0.4	0.6	0.8	1.0
Polubarinova-Kochina [14]	0.938	0.850	0.738	0.595	0.368
Singular equation	0.939	0.850	0.737	0.590	0.368
Hypersingular equation	0.937	0.847	0.732	0.584	0.379

Table IV. Iteration number by using the singular equation and hypersingular equation (Case 2).

Method	Mesh	Number of iteration
Singular equation	25	12
Present (hypersingular equation)	25	9

and hypersingular equations using constant elements are both employed to solve the problem as shown in Table V. It is found that the solution (12.68) using the hypersingular formulation agrees better than that (12.61) by using the singular formulation after comparing with the Ozis' solution (12.70). Instead of using higher-order elements, hypersingular formulation yields a better solution.

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Table V. Final position of separation point using different methods.

Reference	Height
Polubarinova-Kochina [14]	12.95
Cryer [15]	12.7132
Ozis [16]	12.7070
Westbrook [1], FEM	NA
Bruch [17], BEM, Linear element	12.98
Cabral and Wrobel [18], BEM, B-spline	12.74
Present, BEM, constant element, singular equation	12.61
Present, BEM, constant element, hypersingular equation	12.68

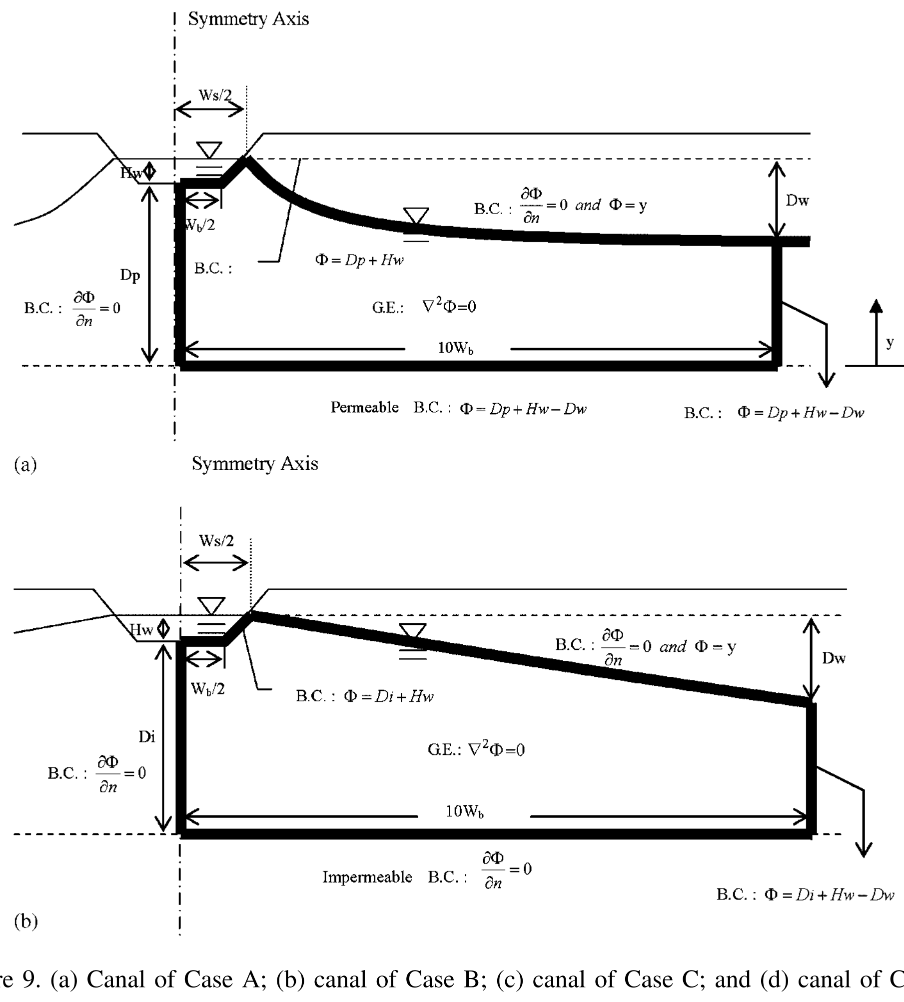

Figure 9. (a) Canal of Case A; (b) canal of Case B; (c) canal of Case C; and (d) canal of Case D.

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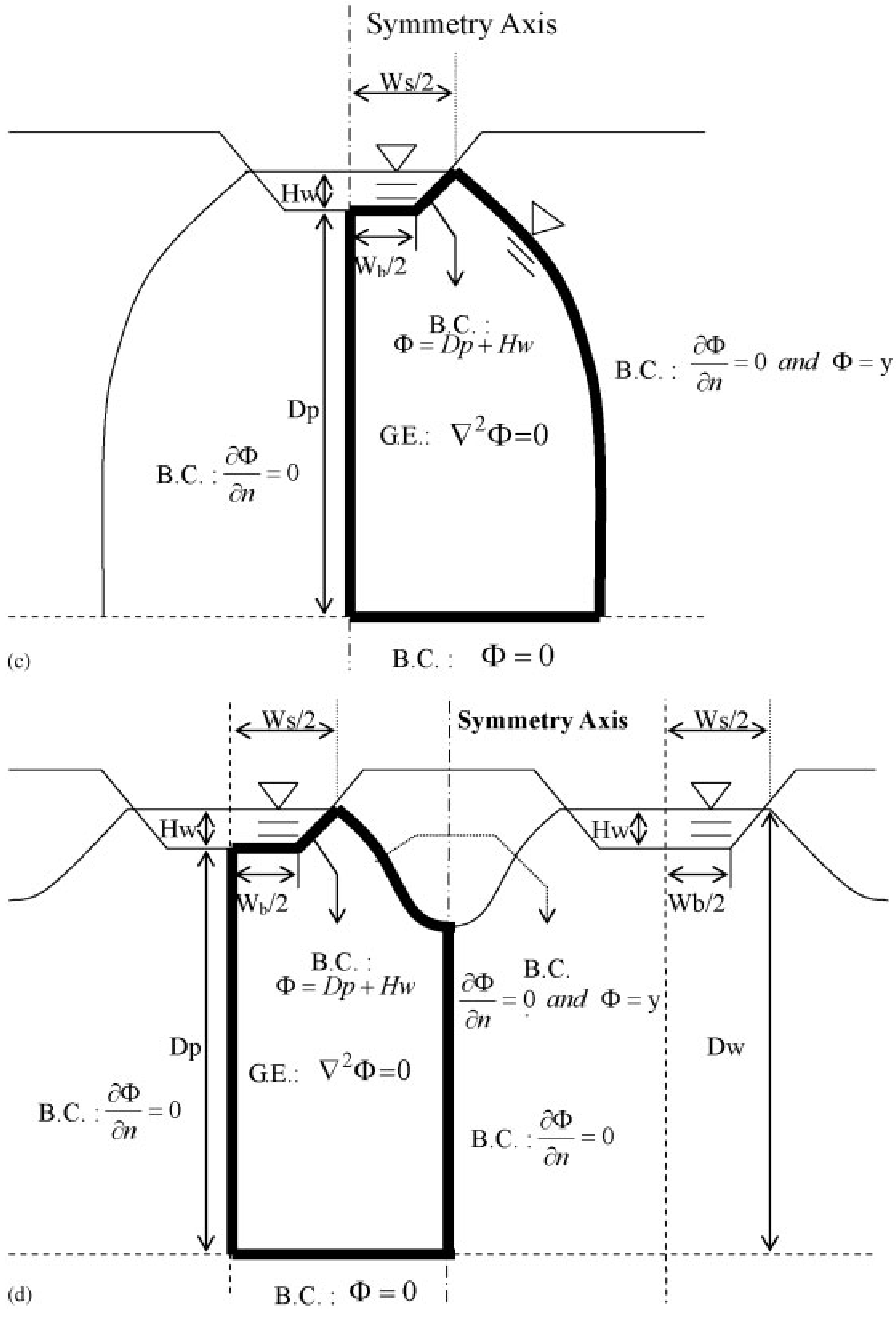

Figure 9. *Continued*.

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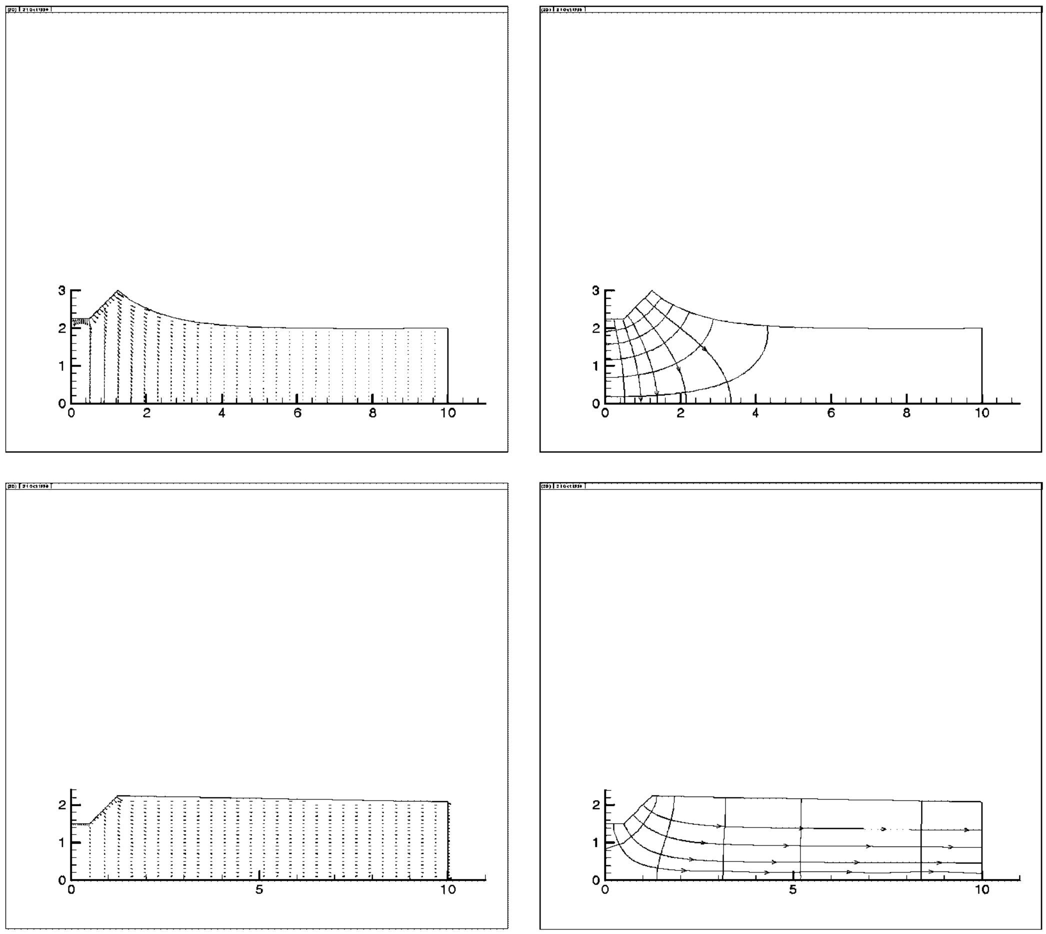

(a)

(b)

Figure 10. (a) Velocity field and streamlines of Case A; (b) velocity field and streamlines of Case B; (c) velocity field and streamlines of Case C; and (d) velocity field and streamlines of Case D.

4.3. Case 3. Canals

Four surface canals (Cases A, B, C and D) are also solved. Figure 9 shows the mathematical models for Cases A, B, C and D. The stream lines and velocity fields are plotted and the free surfaces are determined as shown in Figure 10. Fewer number of iteration by using hypersingular formulation is required than that by using the singular one. The results of the four cases are utilized to demonstrate the feasibility of hypersingular formulation.

5. CONCLUSIONS

Free-surface seepage problems were solved by using the hypersingular equation and the results were compared with other solutions. It is found that the iteration number using the present method

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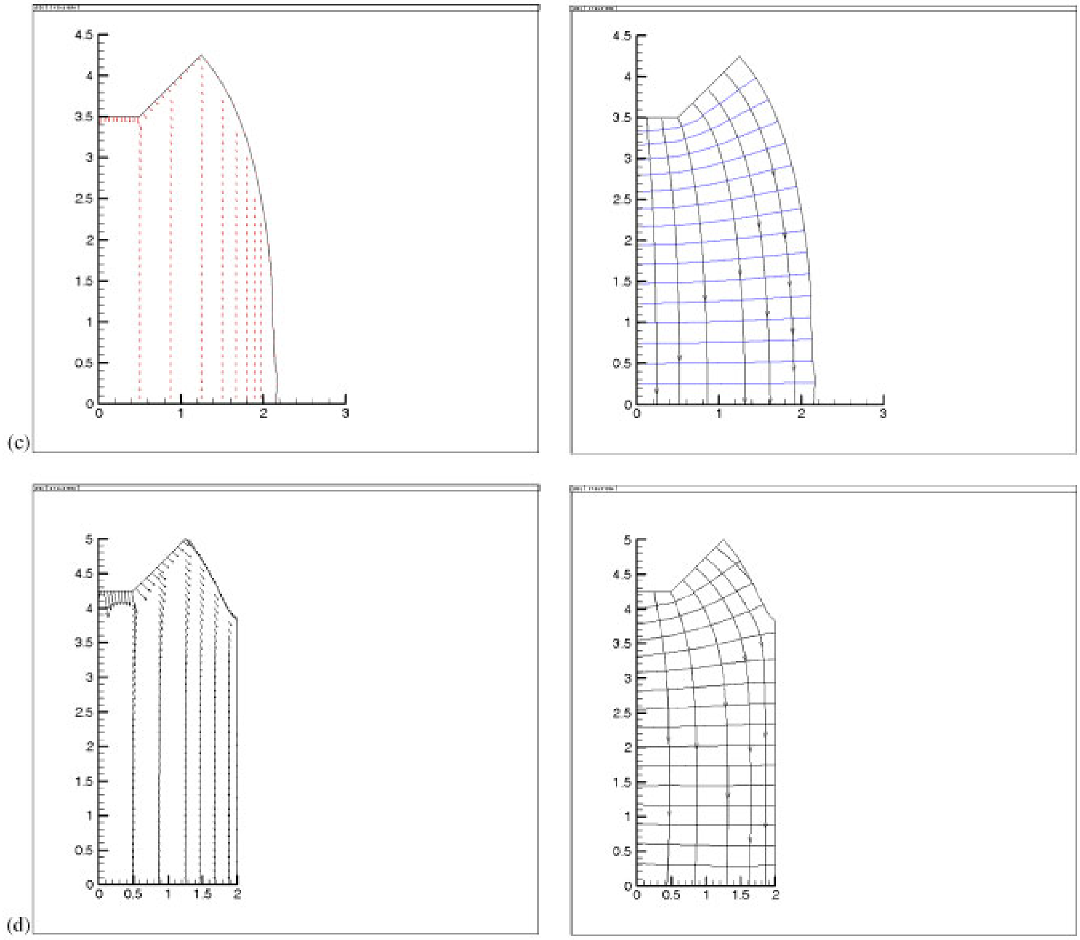

Figure 10. *Continued*.

is fewer than that of the other methods. It is found that increasing the order of kernel singularity yields more accurate results than increasing the order of boundary element. Several examples including rectangular dams and surface canals were demonstrated to check the efficiency of the present method.

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