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Generality and Special Cases of Dual Integral Equations of Elasticity

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ABSTRACT

In this paper the generality of the theory of dual integral equations derived earlier is explored. It is found that many integral equations and potential methods can be deduced from the dual equations and regarded as special cases of the theory. Some useful new formulations are also invented therefrom. These two equations are independent and totally have four kernel functions, which make it possible a unified theory encompassing different schemes and various derivations and interpretations.

彈性破裂力學對偶積分式的廣義性與特例

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摘要

本文導出彈性體的對偶邊界積分方程式，指出這一對式子可視為彈性力學邊界值問題的通用列式，足以解析各種二維與三維的彈性力學與彈性破裂力學問題。積分方程式組可以傳統勢位理論來解釋，含有單層勢位、單層勢位導數、雙層勢位、與雙層勢位導數；對應四個核函數。文中重點在於對此積分方程式組的廣義性加以闡述，發現許多現有方法均為其特例，並且推展出數種新的方法。

INTRODUCTION

In References [3, 8] Hong and Chen derived a theory of dual integral equations of elasticity. The derivation was rigorous in that it was completed from six routes and the results of the six were proved the same. The six routes consisted of three approaches: Betti's law, the weighted residual method, and the theory of potential, and of two operations, traction and trace, in different orders of application. It was pointed out that a single integral equation of elasticity, even with two kernels, such as the Somigliana identity is too slim to solve general elastic problems, especially those with degenerate geometry which encloses no area or volume. The observation led the authors to seek an additional integral equation, ending up with the so called dual integral equations.

In this paper the theory itself is examined more closely and its generality is demonstrated through eight, existing or newly established, special formulations derived from the theory. The eight methods are (1) single layer potential method – indirect boundary element method of the first kind, (2) double layer potential method – indirect boundary element method of the second kind, (3) mixed layer potential method – indirect boundary element method of the third kind, (4) direct boundary element method of the first kind, (5) direct boundary element method of the second kind, (6) displacement discontinuity method of the constant element type, (7) displacement discontinuity method of the linear element type, and (8) dislocation model method. Among all the methods, the direct and indirect methods are popularly referred. The displacement discontinuity method is based on a special Green's function which may be determined by the Neuber-Papkovitch potential. In a similar way the dislocation model relies itself on a complex stress function and also has a special Green's function which may also be determined from the Neuber-Papkovitch potential. In the present study it is found that both Green's functions of the last two methods can be obtained from the dual integral equations by integrating kernel functions over certain intervals.

DERIVATION OF DUAL DOMAIN INTEGRAL EQUATIONS

For completeness and later convenience in reference a brief derivation of dual integral equations developed in [8] is given below. For more details the readers are referred to the work cited.

1. By Betti's law

Let (b_i, t_i, u_i) and (b_i^*, t_i^*, u_i^*) be two equilibrium states in a linearly elastic body where b_i, b_i^* are the body forces, t_i, t_i^* are the boundary tractions, and u_i, u_i^* are the displacements. Betti's law of reciprocality gives

$$\int_D (u_i b_i^* - u_i^* b_i) dV = -\int_B (u_i t_i^* - u_i^* t_i)\, dB \tag{1}$$

where D is a domain with boundary B. It can be recast into the theory of self-adjoint operator $\mathcal{L}$ simply as

$$(\mathcal{L}u/v) = (u/\mathcal{L}v) \tag{2}$$

where

$$\mathcal{L} = \begin{bmatrix} D & 0 \\ 0 & -B \end{bmatrix} \tag{3}$$

If the linearly elastic material is isotropic, the operator D is expressed explicitly as

$$D_{ij} = (\lambda + G)\, \partial_i \partial_j + G\, \delta_{ij}\, \partial_k \partial_k \tag{4}$$

while B the traction operator

$$B_{ij} = \lambda n_i \partial_j + G(n_j \partial_i + \delta_{ij} n_k \partial_k) \tag{5}$$

where λ and G are Lame's constants, n_j are direction cosines of the unit outward normal to the boundary, δ_{ij} is Kronecker symbol and ∂_i is the partial differential operator. For an anisotropic material the expressions for D_{ij} and B_{ij} can be found readily, too. Note that the equations of equilibrium

$$D_{ij} u_j + b_i = 0 \tag{4a}$$

$$D_{ij} u_j^* + b_i^* = 0 \tag{4b}$$

in D and the Cauchy formulae

$$B_{ij}\, u_j = t_i \tag{5a}$$

$$B_{ij} u_j^* = t_i^* \tag{5b}$$

on B and $u_j^* \equiv v_j$ have been used in arriving at Eq. (2)

To elaborate Eq. (2), we state explicitly:

If $D \subset R^n$ $(n=1, 2, 3)$ is a regular or other appropriately conditioned domain with adequately conditioned boundary B, and if the functions $u_i(x)$ and $v_i(x)$ are elements of $C^1(\overline{D}) \cap C^2(D)$ or appropriate Sobolev space and have bounded support,

$$\int_D (v_i D_{ij} u_j - u_i D_{ij} v_j) dV = \int_B (v_i B_{ij} u_j - u_i B_{ij} v_j) dB \tag{6}$$

If B is unbounded, the condition of bounded support can be replaced by a radiation condition.

Now choose specifically:

$$t_i^*(x) = B_{ik}v_k(x) = B_{ik}\, U_{kj}(x, s)\, e_j^*(s) = T_{ij}(x, s)\, e_j^*(s) \quad (7)$$

$$b_i^*(x) = -D_{ik}v_k(x) = -D_{ik}\, U_{kj}(x, s)\, e_j^*(s) = \delta_{ij}(x, s)\, e_j^*(s) \quad (8)$$

$$u_i^*(x) = v_i(x) = U_{ij}(x, s)\, e_j^*(s) \quad (9)$$

where $U_{ij}(x, s)$ and $T_{ij}(x, s)$ are the free space Green's functions (or fundamental solutions) of displacement and traction, respectively, due to a concentrated load in the j direction at the point s, and $e_j^*(s)$ represents an arbitrary concentrated unit load at the point s. Then we have Somigliana's identity [1] if $b_i=0$:

$$\int_B (U_{ij}(x, s)\, t_i(x) - T_{ij}(x, s)\, u_i(x))\, dB(x) = \begin{cases} u_j(s), & s \in D \\ 0 & s \notin D \end{cases} \quad (10)$$

which, for later comparison purposes, is changed to

$$\int_B (U_{ki}(s, x)\, t_k(s) - T_{ki}(s, x) u_k(s))\, dB(s) = \begin{cases} u_i(x) & x \in D \\ 0 & x \notin D \end{cases} \quad (11)$$

In deriving Eq. (10) we have omitted the unit vector e_j^* from both sides of the equations because of its arbitrariness. Now in order to have an additional, independent equation, we apply the traction operator B_{pi} to Eq. (11) and define

$$B_{pi}(\partial, n_x)\ \{U_{ki}(s, x)\} = L_{kp}(s, x) \quad (12)$$

$$B_{pi}(\partial, n_x)\ \{T_{ki}(s, x)\} = M_{kp}(s, x) \quad (13)$$

then it follows that

$$\int_B (L_{kp}(s, x) t_k(s) - M_{kp}(s, x) u_k(s))\, dB(s) = \begin{cases} t_p(x) & x \in D \\ 0 & x \notin D \end{cases} \quad (14)$$

Eqs. $(11)_1$ and $(14)_1$ are the dual equations [2] for any point x in the domain.

2. By physical meaning

The method to be presented is based on the superposition principle due to the linear operator theory and can be understood easily in physical sense. It may be cataloged under the theory of potential. In BIEM terminology it is frequently called the indirect method. First we determine four free space Green's functions or fundamental solutions as shown in Table 1.

Table 1. Meanings of Fundamental Solutions; (j) means in the j Direction.

Fundamental solution	Source Point	Type	Field point	Type
U_{ij} (x, s)	s	load (j)	x	displacement (i)
U^*_{ij} (x, s)	s	dislocation (j)	x	displacement (i)
T_{ij} (x, s)	s	load (j)	x	traction (i)
T^*_{ij} (x, s)	s	dislocation (j)	x	traction (i)

Then invoking the superposition principle, we have

$$u_i(x) = \int_B U_{ik}(x, s)\, \phi_k(s)\, dB(s) + \int_B U^*_{ik}(x, s)\, \psi_k(s)\, dB(s) \quad (15)$$

$$t_i(x) = \int_B T_{ik}(x, s)\, \phi_k(s)\, dB(s) + \int_B T^*_{ik}(x, s)\, \psi_k(s)\, dB(s) \quad (16)$$

Here the load $\phi_k(s)$ on the boundary B must be understood to be the relative traction; that is, the difference between the traction applied on the boundary to the domain D_i under consideration and that applied to the exterior D_e which is complement to the considered domain. If the traction applied to the exterior is assumed to vanish as shown later in the right-hand side of Eq. $(17)_2$, the load $\phi_k(s)$ of Eq. (15) turns out to be the traction $t_k(s)$ applied to the domain considered. Similarly, in Eqs. (15) and (16) the dislocation $\psi_k(s)$ on the boundary is synonymous to the relative displacement and can be interpreted to be the displacement $u_k(s)$ of the points of the bounding boundary of the domain which is being considered if the displacement of the boundary bounding the exterior is taken to be zero as to be done in the right-hand side of Eq. $(18)_2$. Hence

$$\int_B U_{ik}(x, s)\, t_k(s)\, dB(s) + \int_B U^*_{ik}(x, s)\, u_k(s)\, dB(s) = \begin{cases} u_i(x) & x \in D_i \\ 0 & x \in D_e \end{cases} \quad (17)$$

$$\int_B T_{ik}(x,s)\,t_k(s)\,dB(s) + \int_B T^*_{ik}(x,s)\,u_k(s)\,dB(s) = \begin{cases} t_i(x) & x \in D_i \\ 0 & x \in D_e \end{cases} \tag{18}$$

We note that the kernel functions of Eqs. (11), (14), (15) and (16) appear to be different in the orders of subscripts and arguments; however, we shall soon see in the following section where four important theorems are proved, that they are equivalent in essence.

FOUR LEMMAS AND CONSISTENCY OF THE DERIVATIONS

To show the consistencies of Eqs. (11) and (15) and of (14) and (16), which are derived from different approaches, we have to prove the following four lemmas.

Lemma (a) $U_{ik}(x, s) = U_{ki}(s, x)$

Lemma (b) $U^*_{ik}(x, s) = -T_{ki}(s, x)$

Lemma (c) $T_{ik}(x, s) = L_{ki}(s, x)$

Lemma (d) $T^*_{ik}(x, s) = -M_{ki}(s, x)$

Lemmas (a) and (b) can be proved using Betti's law. Lemma (a) is well known.

To prove Lemma (b), we refer to Table 2 and recall Betti's law, having

$$1 {*} u_i(s) + \int_S (T_{ji}(x,s)\,u_j^-(x) - T_{ji}(x,s)\,u_j^+(x), \; dS(x) = \int_S t(x)\,u(x)\,dS(x)$$

Set

$$g_j(x) = u_j^-(x) - u_j^+(x)$$

then

$$u_i(s) = -\int_S T_{ji}(x,s)\,g_j(x)\,dS(x)$$

When

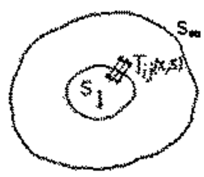
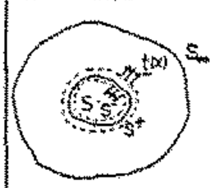
$$g_j(x) = \delta_{ij}(x - x')$$

we have

$$u_i(s) = -T_{ji}(x', s) = U^*_{ij}(s, x')$$

Hence Lemma (b) is proved. After the proof of Lemmas (a) and (b), we are in a position to prove Lemmas (c) and (d). By definition of Eq. (12) and using Lemma (a), we have

Table 2. Proof of Lemma (b).

	System 1	System 2
Domain	infinite	infinite
Figure		
Boundary	S_∞, S^+, S^-	S_∞, S^+, S^-
Force	unit force at s $T_{ji}(x, s)$	$t(x)$ on S^+, S^-
Disp.	$u(x)$, x on S^+, S^-	$u_i(s), u_j^+(x), u_j^-(x)$

$$L_{kp}(s,x) = B_{pi}(\partial, n_x)\,\{U_{ki}(s,x)\} = B_{pi}(\partial, n_x)\,\{U_{ik}(x,s)\} = T_{pk}(x,s)$$

Similarly, by definition of Eq. (13) and Lemma (b),

$$M_{kp}(s,x) = B_{pi}(\partial, n_x)\,\{T_{ki}(s,x)\} = B_{pi}(\partial, n_x)\,\{-U^*_{ik}(x,s)\} = -T^*_{pk}(x,s)$$

This completes the proof of the four Lemmas.

It is worthy noting that, in real calculations, only three kernels need to be determined because of Lemma (c). We also note that from above it can be shown that

$$B_{pi}(\partial, n_x)\,\{U_{ik}(x,s)\} = T_{pk}(x,s)$$

$$B_{pk}(\partial, n_s)\,\{U_{ik}(x,s)\} = -U^*_{ip}(x,s)$$

This subtle result cautions us that the traction derivative of an influence function with respect to the coordinate of the field point x, indeed, represents another influence function, which describes the state of a different response due to the same sigularity source. On the contrary, the traction derivative of an influence function with respect to a source point coordinate is another influence function of the same response as the original influence function but due to a different sigularity source [7].

SPECIAL CASES

In this section, we shall show that most of the currently available methods can be deduced from exploiting the four kernel functions of the dual equations.

1. Single layer potential method - Indirect boundary element method of the first kind (IBEM-1)

Conventional indirect boundary element procedures unanimously employ only one type of potential, either the single layer potential and its derivative, or the double layer potential and its derivative. Their basic idea consists of superimposing singular solutions of fundamental equations of the theory of elasticity for the infinite medium or the half space in a suitable fashion in order to generate an elastic state of deformation coinciding with that in a real, elastic body. For explanatory purposes, we record the formulation here,

$$u_i(x) = \int_B U_{ik}(x,s)\,\phi_k(s)\,dB \tag{19}$$

$$t_i(x) = \int_B T_{ik}(x,s)\,\phi_k(s)\,dB \tag{20}$$

where ϕ_k is the unknown single layer density function. As the point x approaches and finally locates on the boundary,

$$u_i(x) = \int_B U_{ik}(x,s)\,\phi_k(s)\,dB$$

$$t_i(x) = \beta_{ij}\,\phi_j(x) + \text{C.P.V.} \int_B T_{ik}(x,s)\,\phi_k(s)\,dB$$

Although it is not always impossible to attach some physical interpretation to ϕ_k, the layer source ϕ_k is sometimes called the "fictitious source" in literature [e.g., 4], the method hence called the fictitious stress or traction method.

We observe that Eqs. (19) - (20) can be obtained from Eqs. (15) – (16) by setting the double layer function ψ_k to be zero.

2. Double layer potential method - Indirect boundary element method of the second kind (IBEM-2)

The formulation of the double layer potential method is

$$u_i(x) = \int_B U^*_{ik}(x,s)\,\psi_k(s)\,dB(s)$$

$$t_i(x) = \int_B T^*_{ik}(x,s)\,\psi_k(s)\,dB(s)$$

where ψ_k is the unknown double layer potential function. Taking the trace process, we have

$$u_i(x) = \beta_{ij}\,\psi_j + \text{C.P.V.} \int_B U^*_{ik}(x,s)\,\psi_k(s)\,dB(s)$$

$$t_i(x) = \text{H.P.V.} \int_B T^*_{ik}(x,s)\,\psi_k(s)\,dB(s)$$

It is easy to see that the method is merely a special case of the theory of the dual integral equations, Eq. (15) and (16), by setting ϕ_k to be zero.

When applying this method to solve crack problems in the infinite domain, the fictitious source ψ_k has the physical meaning of displacement discontinuity or relative displacement of the upper and lower crack surfaces ($\psi_k = \Delta u_k = u_k^+ - u_k^-$). In references of mathematical physics [5, 9, 10], the single layer potential method is employed to solve Neumman (or traction) problems. On the other hand, Dirichlet (or displacement) problems are often solved by the double layer potential method. Nevertheless, we would like to add that either the IBEM-1 or the IBEM-2 can be utilized by itself to solve any uncracked elasticity problems, no matter how boundary conditions are prescribed. However, the IBEM-2 can solve crack problems, but the IBEM-1 can not.

3. Mixed layer potential method – Indirect boundary element method of the third kind (IBEM-3)

For crack problems as in Fig. 1, it is possible to combine the merits of the IBEM-1 and the IBEM-2 by developing the single layer density ϕ_k on the outer boundary S and the double layer density ψ_k on the crack surface Γ, where S + Γ = B. Hence

$$u_i(x) = \int_S U_{ik}(x,s)\,\phi_k(s)\,dB(s)$$

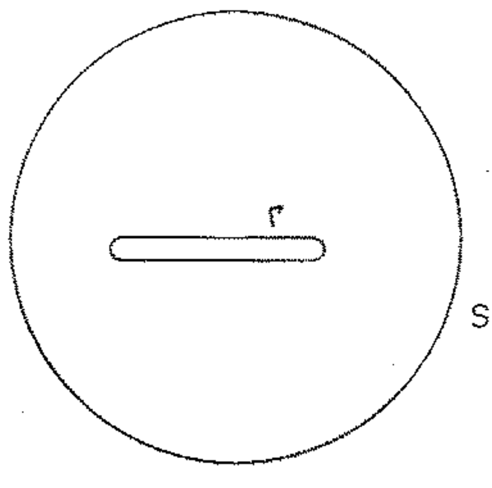


Fig. 1. A crack problem.

$$+\int_{\Gamma} U_{ik}^{*}(x, s)\, \psi_k(s)\, dB(s)$$

$$t_i(x) = \int_S T_{ik}(x, s)\, \phi_k(s)\, dB(s) + \int_{\Gamma} T_{ik}^{*}(x, s)\, \psi_k(s)\, dB(s)$$

where x is in the exterior. If x is pushed to on the outer boundary S,

$$u_i(x) = \int_S U_{ik}(x, s)\, \phi_k(s)\, dB(s) + \int_{\Gamma} U_{ik}^{*}(x, s)\, \psi_k(s)\, dB(s)$$

$$t_i(x) = \beta_{ij}\phi_j(x) + \text{C.P.V.} \int_S T_{ik}(x, s)\, \phi_k(s)\, dB(s) + \int_{\Gamma} T_{ik}^{*}(x, s)\, \psi_k(s)\, dB(s)$$

If x is moved to on the crack boundary Γ,

$$u_i(x) = \int_S U_{ik}(s, x)\, \phi_k(s)\, dB(s) + \beta_{ij}\psi_j(x) + \text{C.P.V.} \int_{\Gamma} U_{ik}^{*}(x, s)\, \psi_k(s)\, dB(s)$$

$$t_i(x) = \int_S T_{ik}(x, s)\, \phi_k(s)\, dB(s) + \text{H.P.V.} \int_{\Gamma} T_{ik}^{*}(x, s)\, \psi_k(s)\, dB(s).$$

The above four equations are sufficient to solve any crack problem and the method using them is called the mixed layer potential method or the indirect boundary element method of the third kind.

If we use both density functions ϕ_k and ψ_k on all the boundaries, then ϕ_k and ψ_k can be identified with the true values of the traction and the displacement [6]. And this general method makes no difference, as had been demonstrated on the occasion of the proof of the four lemmas and the consistency of the derivations, with the general direct boundary element methods, special cases of which are the next topics we are to proceed to.

4. Direct boundary element method of the first kind (DBEM-1)

It is apparent that the first one of the dual integral equations, Eqs. (11) and (14), is the very formulation of the conventional direct boundary element method presented in references, and it is suitable for solving boundary value problems of elasticity without cracks. It can be deemed as a special case of our theory of dual integral equations. For reasons to be clear soon, we call this method the direct boundary element method of the first kind (DBEM-1), which is not suited to solve crack problems.

5. Direct boundary element method of the second kind (DBEM-2)

There is no reason why when we take the first equation to deal with elasticity problems, we can not employ the second (Eq. (14)) to solve the same problem. We call this latter method the direct boundary element method of the second kind (DBEM-2). Indeed, the DBEM-2 gives a better convergence owing to stronger singularities of kernels, since singular integral equations with strongly singular kernels may be desirable because they lead to a diagonally dominant system of linear algebraic equations which can be solved by iteration techniques as opposed to direct elimination schemes. This might provide a computational saving. However, in a domain with degenerate geometry (such as line cracks or crack surfaces), the special methods such as the DBEM-1 or DBEM-2 are not available; the general method consisting of both the dual equations must be used to consider all boundary conditions of the lower and upper surfaces of cracks.

6. Displacement discontinuity method of the constant element type (DD-1)

Here we use the kernel functions $U_{ik}^{*}(x, s)$ and $T_{ik}^{*}(x, s)$ of the dual equations and take advantage of the concept of constant finite elements, interpolating the double layer density function ψ_k by a multiple-step function. Hence for a constant element with nodal values (D_x, D_y), as in Fig. 2, we have

$$u_x = \int_{-a}^{a} U_{11}^{*}(x, s)\, D_x\, ds + \int_{-a}^{a} U_{12}^{*}(x, s)\, D_y\, ds$$

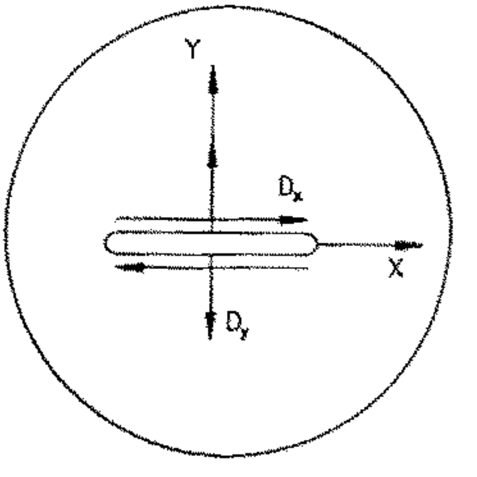


Fig. 2. Constant nodal values of D_x and D_y.

$$u_y = \int_{-a}^{a} U^*_{21}(x,s) D_x ds + \int_{-a}^{a} U^*_{22}(x,s) D_y ds$$

$$t_x = \int_{-a}^{a} T^*_{11}(x,s) D_x ds + \int_{-a}^{a} T^*_{12}(x,s) D_y ds$$

$$t_y = \int_{-a}^{a} T^*_{21}(x,s) D_x ds + \int_{-a}^{a} T^*_{22}(x,s) D_y ds$$

Carrying out the integration over the interval -a to a by MACSYMA, we have

$$u_x = [2(1-\nu) f,y - y\, f,xx]\, D_x + [-(1-2\nu) f,x - y\, f,xy]\, D_y$$

$$u_y = [(1-2\nu) f,x - y\, f,xy]\, D_x + [2(1-\nu) f,y - y\, f,yy]\, D_y$$

in which $f(x,y) = \frac{-1}{4\pi(1-\nu)} \{y(\arctan(y/x-a) - \arctan(y/x+a)) - (x-a)\ln\sqrt{(x-a)^2+y^2} + (x+a)\ln\sqrt{(x+a)^2+y^2}\}$

Taking n = (0, 1) and n = (1, 0), we have

$$\sigma_{xx} = 2G[2 f,xy + y\, f,xyy]\, D_x + 2G[f,yy + y\, f,yyy]\, D_y$$

$$\sigma_{xy} = \sigma_{yx} = 2G[f,yy + y\, f,yyy]\, D_x + 2G[-y\, f,xyy]\, D_y$$

$$\sigma_{yy} = 2G[-y\, f,xyy]\, D_x + 2G[f,yy - y\, f,yyy]\, D_y$$

For definiteness, we give a numerical example of the infinite elastic medium (shear modulus G, Poisson ratio $\nu = 0.1$) with a line crack of length 2b=20 where a uniformly distributed pressure p=0.001*G is exerted to open up the surfaces, as in Fig. 3. The exact solution of the crack profile is

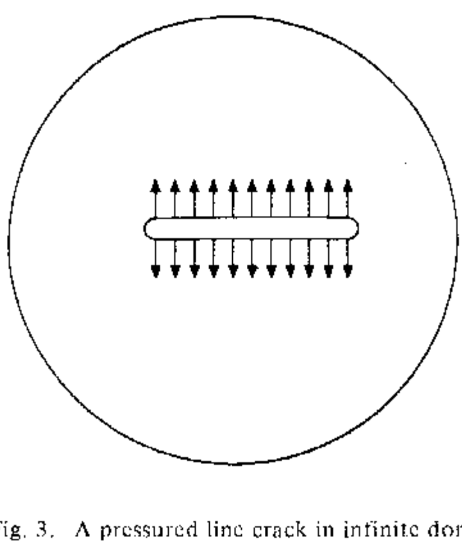


Fig. 3. A pressured line crack in infinite domain.

$$u_y(x) = -(1-\nu)\, pb \sqrt{(1-x^2/b^2)}/G$$

Numerical experiments are performed with results shown on Fig. 4. From the above figures and table, we find that the result is in general satisfactory. The convergence can be shown to be in the L_2 sense. Nevertheless, it is questionable why the numerical data are greater than the exact solutions on all collocation points and why convergence at the tip element behaves poorly regardless of the increase of element number. The first reason may be that the subsidiary (auxiliary) condition is not used. The second reason may be that the DD-1 employs discontinuous functions to fit continuous ones so that Gibbs phenomena occur. This may explain that an error of 25% remains at the tip even the element number increases.

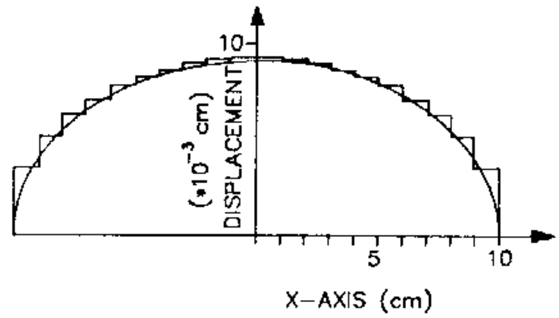


Fig. 4. Crack Profile of Numerical and Analytical Solutions (P/G = 10^{-3})

Crouch [4] obtained exactly an identical result by employing the Neuber-Papkovitch potential function and called his procedure the displacement discontinuity method. One of the most important applications of the method in recent years was to geological materials with cracks, inclusions, joints and faults.

It is obvious that there are no reason we stick to constant elements when we possess so a general theory of dual integral equations. Hence we proceed to the next algorithm.

7. Displacement discontinuity method of the linear element type (DD-2)

As the same procedure of the DD-1 but now using piecewise linear interpolations, we propose the displacement discontinuity method of the linear element type.

Numerical results are shown in Fig. 5, where collocation points are placed at the centers of the elements. Although severe oscillation is present, the values at the collocation points seem to be acceptable

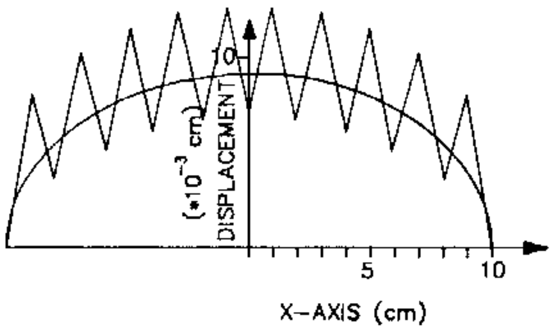


Fig. 5. Crack profile of linear elements colocated to element centers (N = 10).

while the values at other locations, especially at the nodes, are totally intolerable. In view of the nature of the L_2 convergence, we improve the algorithm substantially by collocating to nodal points, instead. Surprisingly excellent results are obtained and shown in Fig. 6.

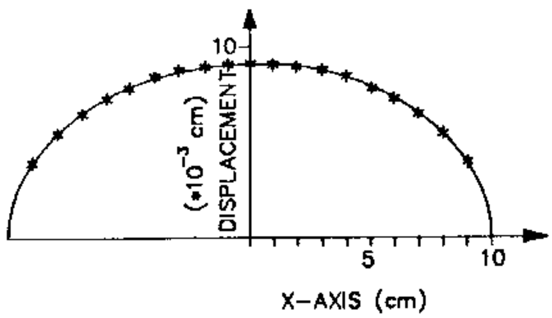


Fig. 6. Crack profile of linear elements colocated to nodes (N = 10).

8. Dislocation model method (DMM)

A similar algorithm to the DD-1 and the DD-2, but assuming constant element from 0 to + ∞, instead of -a to a, can be proposed to determine the following special Green's functions,

$$u_x = \frac{G}{4\pi(1-\nu)} \{[2(1-\nu)\arctan(y/x) + xy/(x^2+y^2)] D_x + [\tfrac{1}{2}(1-2\nu) \log(x^2+y^2) + y^2/(x^2+y^2)] D_y\}$$

$$u_y = \frac{G}{4\pi(1-\nu)} \{[-\tfrac{1}{2}(1-2\nu)\log(x^2+y^2) - x^2/(x^2+y^2)] D_x + [-2(1-\nu) \arctan(y/x) - xy/(x^2+y^2)] D_y\}$$

$$u_z = \frac{1}{2\pi} \arctan(y/x) D_z$$

$$\sigma_{xx} = \frac{G}{4\pi(1-\nu)} \{[-2y(3x^2+y^2)/(x^2+y^2)^2] D_x + [2x(x^2-y^2)/(x^2+y^2)] D_y\}$$

$$\sigma_{yy} = \frac{G}{4\pi(1-\nu)} \{[2y(x^2-y^2)/(x^2+y^2)^2) D_x + [2x(x^2+3y^2)/(x^2+y^2)^2] D_y\}$$

$$\sigma_{xy} = \frac{G}{4\pi(1-\nu)} \{[2x(x^2-y^2)/(x^2+y^2)^2] D_x + [2y(x^2-y^2)/(x^2+y^2)^2] D_y\}$$

$$\sigma_{zz} = \frac{-\nu G}{\pi(1-\nu)} y/(x^2+y^2) D_x + \frac{\nu G}{\pi(1-\nu)} x/(x^2+y^2) D_y$$

where D_x, D_y and D_z are Burgur's vectors.

These equations were also presented by Lardner, but his differs from the above by a datum which represents the rigid body motion. He obtained the Green's functions from the complex stress function method.

CONCLUDING REMARKS

In this paper, we was able to not only broaden the application of the BIEM but also present an integrated theory which emcompasses many different thoughts and schemes in the fields of elasticity, linear elastic fracture mechanics, integral equations, and (vector) potential theory.

CONCLUDING REMARKS

It has been shown that the theory developed can encompass many existing schemes of integral equations and potential theories in the fields of elasticity and linear elastic fracture mechanics. Also from the unified view of the theory the relationships of the methods were made clear and, moreover, new methods of IBEM-3, DBEM-2, and DD-1 were thereby invented. Other new methods suitable for particular classes of problems may also be derived and will be presented elsewhere.

NOMENCLATURE

The following symbols are used in this paper.

$A \cap B$	the intersection of A and B
b_i, b_i^*	body forces
B	boundary

B_1, B_2	the boundaries with prescribed displacement and traction, respectively,
$B_{pi}(\partial, n_x)$	traction operator with respect to x
C.P.V.	Cauchy principal value
D	domain
$\overline{D}$	the closure of D
D_x, D_y, D_z	the displacement discontinuity in x, y, z direction
E	Young's modulus
G	shear modulus
g_j	relative displacement in the j direction at x
H.P.V.	Hadamard principal value
$L_{ki}(s, x)$	kernel function of the second of dual integral equations
$M_{ki}(s, x)$	kernel function of the second of dual integral equations
n_i	normal vector of s
$\bar{n}_i$	normal vector of x
p	pressure at crack surface
R^n	the considered n-dimensional domain
S^+, S^-	the boundary near the boundary S
S_∞	the infinite boundary
$s \in D$	s is an element of D
$s \notin D$	s is not an element of D
$T_{ij}(x, s)$	the i component traction at x due to the concentrated j direction load at s
$T^*_{ij}(x, s)$	the i component traction at x due to the j direction dislocation at s
t_i, t_i^*	tractions
$U_{ij}(x, s)$	the i component displacement at x due to the concentrated j direction load at s
$U^*_{ij}(x, s)$	the i component displacement at x due to the j direction dislocation at s
$B_{pi}(\partial, n_x)$	traction operator in classical elasticity
D_{ij}	linear operator in classical elasticity
$\mathcal{L}$	self-adjoint operator in classical elasticity
∂_i	partial differentiation with respect to i component
δ_{ij}	Kronecker delta
β_{ij}	jump terms
$\delta_{ij}(x, s)$	the concentrated source at s in the j direction
λ	Lame's constant
ν	Poisson's ratio
ϕ_k	single layer density
ψ_k	double layer density

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