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Eigensolutions of multiply connected membranes using the method of fundamental solutions

J.T. Chen[a,*], I.L. Chen[b], Y.T. Lee[a]

[a]Department of Harbor and River Engineering, National Taiwan Ocean University, P.O. Box 7-59, Keelung 20224, Taiwan, ROC
[b]Department of Naval Architecture, National Kaohsiung Marine University, Kaohsiung 81443, Taiwan, ROC

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Abstract

In this paper, the method of fundamental solutions (MFS) of single and double-layer potential approaches for solving the eigenfrequencies of multiply connected membranes is proposed. By employing the fundamental solution, the coefficients of influence matrices are easily determined. The spurious eigensolution accompanied by the true eigensolution appears. It is found that the spurious eigensolution using the MFS depends on the location of the inner boundary where the sources are distributed. To verify this finding, the true and spurious eigenvalues in an annular domain are analytically studied using the degenerate kernels and circulants for an annular membrane. In order to obtain the true eigensolution, the singular value decomposition (SVD) updating techniques and the Burton and Miller method are utilized to filter out the spurious eigensolutions. Two examples are demonstrated analytically and numerically to see the validity of the present method.

Keywords: Multiply connected membrane; Method of fundamental solutions; Helmholtz equation; Circulants; Degenerate kernel; SVD updating techniques; Burton and Miller method

1. Introduction

The method of fundamental solutions (MFS) is a numerical technique as well as finite difference method (FDM), finite element method (FEM) and boundary element method (BEM). It is well known that the MFS can deal with engineering problems when a fundamental solution is known. This method was attributed to Kupradze in 1964 [1]. The MFS has been applied to potential [2], Helmholtz [3], diffusion [4], biharmonic [5] and elasticity problems [1]. The MFS can be seen as one kind of meshless method. The basic idea is to approximate the solution by a linear superposition of fundamental solution with sources located outside the domain of the problem. Moreover, it has some advantages over boundary element method, e.g. no singularity, no boundary integrals and mesh-free model.

In the last decades, Tai and Shaw [6] first employed the complex-valued BEM to solve membrane vibration problem. De Mey [7], Hutchinson and Wong [8] employed only the real-part kernel to solve the membrane and plate vibrations, respectively. Although the complex-valued computation is avoided, they faced the occurrence of spurious eigenequations. One has to investigate the mode shapes in order to identify and reject the spurious ones. If we need to look for the eigenmode as well as eigenvalue, the sorting for the spurious eigenvalues pay a small overhead by identifying the mode shapes. Chen et al. [9] commented that the detection of spurious modes may mislead the judgment of the true and spurious ones, since the spurious mode may have the same nodal line of the true one by observation. This is the reason why Chen and his co-workers have developed many systematic techniques, e.g. dual formulation [9], domain partition [10], SVD updating technique [11], CHEEF method [12], for sorting out the true and the spurious eigenvalues. However, it is true only for the case of problem with a simply connected domain. For multiply connected problems, spurious eigenvalues still occur even

* Corresponding author. Tel.: +886 2 2462 2192x6177; fax: +886 2 2463 2375.
E-mail address: jtchen@mail.ntou.edu.tw (J.T. Chen).
URL: http://ind.ntou.edu.tw/~msvlab.

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though the complex-valued BEM is utilized. This occurrence of spurious eigenvalues and their treatments in BEM have been studied in the membrane and acoustic problems [13,14].

In meshless method, the multiply connected problem has been discussed in [15]. Although the MFS has been applied to solve many engineering problems, its validity for solving the eigensolutions of multiply connected problems was not addressed in the literature to the authors' knowledge. We may wonder whether the spurious solution occurs as BEM does. For the purpose of analytical derivation, some mathematical techniques are utilized, e.g. degenerate kernel and circulants [16]. Recently, circulant was utilized to deal with some problems in MFS [17–19]. Here, an annular case is considered to examine the appearance of true and spurious eigensolutions.

In this paper, the MFS for solving the eigenfrequencies of multiply connected membrane is proposed. The occurring mechanism of the spurious eigensolution of an annular membrane and its treatment in MFS are studied analytically and numerically instead of that of BEM in the published papers [11,12,14]. The degenerate kernels and circulants are employed to derive the spurious eigensolution. In order to filter out the spurious eigenvalues, singular value decomposition updating techniques and Burton and Miller method are utilized. To demonstrate the validity of our proposed methods, two numerical examples are presented.

2. Formulation of multiply connected eigenproblems using the method of fundamental solutions

The governing equation for membrane vibration in Fig. 1 is the Helmholtz equation as follows

$$(\nabla^2 + k^2)u(x) = 0, \quad x \in D, \tag{1}$$

where ∇^2 is the Laplacian operator, D is the domain of interest and k is the wave number.

The fundamental solution $U(s,x)$ is considered as

$$U(s,x) = i\mathrm{H}_o^{(1)}(kr), \tag{2}$$

where $H_o^{(1)}$ is the zeroth order Hankel function of the first kind. According to the dual formulation [20], we have the four kernels

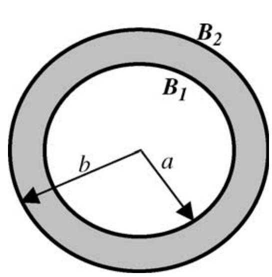

Fig. 1. Figure sketch for an annular problem.

$$U(s,x) = \mathrm{i}J_0(kr) - Y_0(kr), \tag{3}$$

$$T(s,x) = \frac{\partial U(s,x)}{\partial n_s} = -k\frac{\mathrm{i}J_1(kr) - Y_1(kr)}{r} y_i n_i, \tag{4}$$

$$L(s,x) = \frac{\partial U(s,x)}{\partial n_x} = k\frac{\mathrm{i}J_1(kr) - Y_1(kr)}{r} y_i \bar{n}_i,$$

$$M(s,x) = \frac{\partial^2 U(s,x)}{\partial n_x \partial n_s} = k\left(\frac{k(-\mathrm{i}J_2(kr) + Y_2(kr))}{r^2} y_i y_j n_i \bar{n}_j + \frac{\mathrm{i}J_1(kr) - Y_1(kr)}{r} n_i \bar{n}_i\right), \tag{6}$$

where $r \equiv |s-x|$ is the distance between the source and collocation points; n_i is the ith component of the outnormal vector at s; $\bar{n}_i$ is the ith component of the outnormal vector at x, J_m and Y_m denote the first kind and second kind of the mth order Bessel function, respectively, and $y_i \equiv s_i - x_i$, $i = 1, 2$, are the differences of the ith components of s and x, respectively. Based on the indirect method using the dual formulation, we can represent the field solution by

Single-layer potential approach

$$u(x_i) = \sum_j U(s_j, x_i)\phi_j, \tag{7}$$

$$t(x_i) = \sum_j L(s_j, x_i)\phi_j. \tag{8}$$

Double-layer potential approach

$$u(x_i) = \sum_j T(s_j, x_i)\psi_j, \tag{9}$$

$$t(x_i) = \sum_j M(s_j, x_i)\psi_j. \tag{10}$$

The matrix forms of Eqs. (7)–(10) are

Single-layer potential approach

$$\{u_i\} = [U_{ij}]\{\phi_j\}, \tag{11}$$

$$\{t_i\} = [L_{ij}]\{\phi_j\}. \tag{12}$$

Double-layer potential approach

$$\{u_i\} = [T_{ij}]\{\psi_j\}, \tag{13}$$

$$\{t_i\} = [M_{ij}]\{\psi_j\}, \tag{14}$$

where $\{\phi_j\}$ and $\{\psi_j\}$ are the generalized unknowns by using the single and double-layer potential approaches, respectively. For the purpose of deriving the exact eigensolution, we consider the problem with an annular domain. The radii of inner and outer circles are a and b for the real boundary, respectively. The source strengths are distributed on the inner and outer fictitious circular radii a' and b' in Fig. 2,

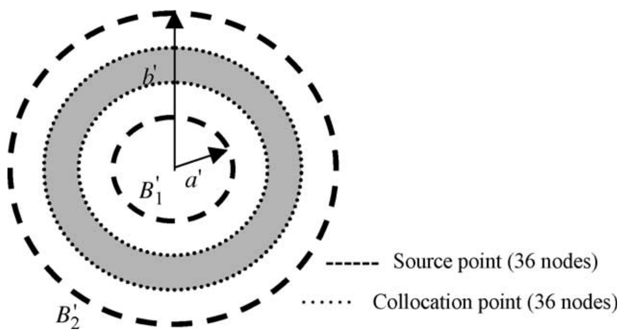

Fig. 2. Figure sketch for node distribution.

respectively. For simplicity, the boundary condition is the Dirichlet–Dirichlet (clamped–clamped) type, $\bar{u}=0$ on all the boundaries. We distributed $2N$ collocation points at each real boundary and $2N$ source points at each fictitious boundary. By matching the boundary condition, the equations can be obtained using the single-layer potential approach of Eq. (7) as shown below

$$\{0\} = [U_{ij}^{11}]\{\phi_j^1\} + [U_{ij}^{12}]\{\phi_j^2\}, \tag{15}$$

$$\{0\} = [U_{ij}^{21}]\{\phi_j^1\} + [U_{ij}^{22}]\{\phi_j^2\}, \tag{16}$$

where the first superscript and second superscripts in U_{ij} denotes the collocating boundary and source boundary (first superscript: 1 for B_1 and 2 for B_2; second superscript: 1 for B_1' and 2 for B_2'), $\{\phi_j^1\}$ and $\{\phi_j^2\}$ are the unknown coefficients on the inner and outer boundaries, respectively. By assembling Eqs. (15) and (16) together, we have

$$[\mathrm{SM}_1]\begin{Bmatrix}\phi_j^1\\ \phi_j^2\end{Bmatrix} = \begin{bmatrix}U_{ij}^{11} & U_{ij}^{12}\\ U_{ij}^{21} & U_{ij}^{22}\end{bmatrix}\begin{Bmatrix}\phi_j^1\\ \phi_j^2\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix}. \tag{17}$$

The determinant of the matrix must be zero to obtain the nontrivial eigensolution, i.e.

$$\det[\mathrm{SM}_1] = 0. \tag{18}$$

By plotting the determinant versus the wave number, the curve drops at the positions of eigenvalues.

3. Mathematical analysis of the true and spurious eigenvalues for an annular membrane in MFS

For an annular membrane, we can express $x=(\rho,\phi)$ and $s=(R,\theta)$ in terms of polar coordinate. The U kernel can be expressed in terms of degenerate kernels as shown below

$$U(s,x) = \begin{cases} U^{\mathrm{I}}(\theta,\phi) = \sum_{m=-\infty}^{\infty} J_m(k\rho)[\mathrm{i}J_m(kR) - Y_m(kR)]\cos(m(\theta-\phi)), & R>\rho,\\ U^{\mathrm{E}}(\theta,\phi) = \sum_{m=-\infty}^{\infty} J_m(kR)[\mathrm{i}J_m(k\rho) - Y_m(k\rho)]\cos(m(\theta-\phi)), & R<\rho, \end{cases} \tag{19}$$

where the superscripts 'I' and 'E' denote the interior ($R>\rho$) and exterior domains ($R<\rho$), respectively. Since, the rotation symmetry is preserved for a circular boundary with uniform nodes, the four influence matrices, $[U^{11}]$, $[U^{12}]$, $[U^{21}]$ and $[U^{22}]$ are all symmetric circulants. By superimposing $2N$ lumped strength along each boundary, we have the influence matrix

$$[U^{11}] = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{2N-2} & a_{2N-1}\\ a_{2N-1} & a_0 & a_1 & \cdots & a_{2N-3} & a_{2N-2}\\ a_{2N-2} & a_{2N-1} & a_0 & \cdots & a_{2N-4} & a_{2N-3}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ a_1 & a_2 & a_3 & \cdots & a_{2N-1} & a_0 \end{bmatrix}, \tag{20}$$

where the elements of the first row are obtained by

$$a_{j-i} = U^{11}(s_j, x_i). \tag{21}$$

The matrix $[U^{11}]$ in Eq. (20) is found to be a circulants [16] since the rotational symmetry for the influence coefficients is considered. By using the degenerate kernel and the orthogonal property, the eigenvalue of the matrix $[U^{11}]$ can be obtained as follows [21]

$$\lambda_m^{[U^{11}]} = 2NJ_m(ka')[\mathrm{i}J_m(ka) - Y_m(ka)], \tag{22}$$

where $m=0,\pm1,\pm2,\ldots,\pm(N-1),N$. Similarly, the eigenvalue of matrices, $[U^{12}]$, $[U^{21}]$ and $[U^{22}]$ are shown below:

$$\lambda_m^{[U^{12}]} = 2NJ_m(ka)[\mathrm{i}J_m(kb') - Y_m(kb')], \tag{23}$$

$$\lambda_m^{[U^{21}]} = 2NJ_m(ka')[\mathrm{i}J_m(kb) - Y_m(kb)], \tag{24}$$

$$\lambda_m^{[U^{22}]} = 2NJ_m(kb)[\mathrm{i}J_m(kb') - Y_m(kb')]. \tag{25}$$

By using the similar transformation, we can decompose the $[U^{11}]$ matrix into

$$[U^{11}] = \Phi\Sigma_{[U^{11}]}\Phi^H, \tag{26}$$

where $\Sigma_{[U]} = \mathrm{diag}(\lambda_0^{[U^{11}]}, \lambda_1^{[U^{11}]}, \lambda_{-1}^{[U^{11}]}, \ldots, \lambda_{(N-1)}^{[U^{11}]}, \lambda_{-(N-1)}^{[U^{11}]}, \lambda_N^{[U^{11}]})$ and 'H' is the Hermitian conjugate, and

$$\Phi = \frac{1}{\sqrt{2N}} \begin{bmatrix} 1 & (e^{2\pi i/2N})^0 & (e^{-2\pi i/2N})^0 & \cdots & (e^{-2(N-1)\pi i/2N})^0 & (e^{2N\pi i/2N})^0 \\ 1 & (e^{2\pi i/2N})^1 & (e^{-2\pi i/2N})^1 & \cdots & (e^{-2(N-1)\pi i/2N})^1 & (e^{2N\pi i/2N})^1 \\ 1 & (e^{2\pi i/2N})^2 & (e^{-2\pi i/2N})^2 & \cdots & (e^{-2(N-1)\pi i/2N})^2 & (e^{2N\pi i/2N})^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & (e^{2\pi i/2N})^{2N-2} & (e^{-2\pi i/2N})^{2N-2} & \cdots & (e^{-2(N-1)\pi i/2N})^{2N-2} & (e^{2N\pi i/2N})^{2N-2} \\ 1 & (e^{2\pi i/2N})^{2N-1} & (e^{-2\pi i/2N})^{2N-1} & \cdots & (e^{-2(N-1)\pi i/2N})^{2N-1} & (e^{2N\pi i/2N})^{2N-1} \end{bmatrix}. \quad (27)$$

Similarly, $[U^{12}]$, $[U^{21}]$ and $[U^{22}]$ can be decomposed. Eq. (17) can be decomposed into

$$[\mathrm{SM}_1] = \begin{bmatrix} \Phi \sum_{[U^{11}]} \Phi^H & \Phi \sum_{[U^{12}]} \Phi^H \\ \Phi \sum_{[U^{21}]} \Phi^H & \Phi \sum_{[U^{22}]} \Phi^H \end{bmatrix} = \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix} \begin{bmatrix} \Sigma_{[U]} & \Sigma_{[U]} \\ \Sigma_{[U]} & \Sigma_{[U]} \end{bmatrix} \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix}^H. \quad (28)$$

Since, Φ is unitary, the determinant of $[\mathrm{SM}_1]$ is

$$\det[\mathrm{SM}_1] = \sigma_0(\sigma_1\sigma_2, \cdots, \sigma_{N-1})^2 \sigma_N = 0, \quad (29)$$

where

$$\sigma_m = \lambda_m^{[U^{11}]}\lambda_m^{[U^{22}]} - \lambda_m^{[U^{12}]}\lambda_m^{[U^{21}]} = 4N^2 J_m(ka')[-iJ_m(kb') + Y_m(kb')] \times \{J_m(kb)Y_m(ka) - J_m(ka)Y_m(kb)\}, \quad (30)$$

for the annular membrane with the Dirichlet–Dirichlet boundary conditions by using the single-layer potential approach. After comparing with the analytical solution [14], we can obtain the true and spurious eigenequations in Eq. (29). Since, the middle bracket $[-iJ_m(kb') + Y_m(kb')]$ of Eq. (30) is never zero for any k, the spurious eigenequation of $J_m(ka') = 0$ and the true eigenequation $J_m(kb)Y_m(ka) - J_m(ka)Y_m(kb) = 0$ are also obtained. Similarly, we can obtain the true and spurious eigenequations for different boundary conditions and using different formulations. All the results are derived analytically as shown in Table 1. It is found that the occurrence of spurious eigenvalues depends on the formulation and the location of inner source point instead of the specified boundary condition, while the true eigenequation is independent of the formulation and is relevant to the specified boundary condition. For the multiply connected membrane, the single-layer potential approach produces spurious eigenvalues which are associated with the interior eigenvalue with the essential homogeneous boundary conditions, while the double-layer potential approach produces spurious eigenvalues which are associated with the interior eigenvalue with the natural homogeneous boundary conditions.

4. Treatments of spurious eigenvalues

4.1. SVD updating techniques

4.1.1. SVD updating document

In order to extract out the true eigenvalues, the SVD updating document is utilized. Other than the single-layer potential approach to obtain Eq. (17), we can also select the double-layer potential approach and obtain

$$[\mathrm{SM}_2]\begin{Bmatrix} \psi_j^1 \\ \psi_j^2 \end{Bmatrix} = \begin{bmatrix} T^{11} & T^{12} \\ T^{21} & T^{22} \end{bmatrix} \begin{Bmatrix} \psi_j^1 \\ \psi_j^2 \end{Bmatrix} = \{0\}. \quad (31)$$

By employing the relation in the degenerate kernels between the direct and indirect methods [22], the SVD updating document (Indirect method) to extract out the true eigenequation is equivalent to the SVD updating term (Direct method). We have

$$[C] = \begin{bmatrix} (\mathrm{SM}_1)^H \\ (\mathrm{SM}_2)^H \end{bmatrix}. \quad (32)$$

Where, the rank of the matrix $[C]$ must be smaller than $4N$ for true eigenvalues. By using the property of Eq. (26),

Table 1
The true and spurious eigenequations for different boundary conditions by using the single- and double-layer potential approaches

Inner–outer boundary		Single-layer potential approach	Double-layer potential approach
Dirichlet–Dirichlet	True	$J_m(kb)Y_m(ka) - J_m(ka)Y_m(kb) = 0$	$J_m(kb)Y_m(ka) - J_m(ka)Y_m(kb) = 0$
	Spurious	$J_m(ka') = 0$	$J'_m(ka') = 0$
Dirichlet–Neumann	True	$J'_m(kb)Y_m(ka) - J_m(ka)Y'_m(kb) = 0$	$J'_m(kb)Y_m(ka) - J_m(ka)Y'_m(kb) = 0$
	Spurious	$J_m(ka') = 0$	$J'_m(ka') = 0$
Neumann–Dirichlet	True	$J_m(kb)Y'_m(ka) - J'_m(ka)Y_m(kb) = 0$	$J_m(kb)Y'_m(ka) - J'_m(ka)Y_m(kb) = 0$
	Spurious	$J_m(ka') = 0$	$J'_m(ka') = 0$
Neumann–Neumann	True	$J'_m(kb)Y'_m(ka) - J'_m(ka)Y'_m(kb) = 0$	$J'_m(kb)Y'_m(ka) - J'_m(ka)Y'_m(kb) = 0$
	Spurious	$J_m(ka') = 0$	$J'_m(ka') = 0$

the matrix can be written as

$$[C]=\begin{bmatrix}\Phi&0&0&0\\0&\Phi&0&0\\0&0&\Phi&0\\0&0&0&\Phi\end{bmatrix}\begin{bmatrix}\Sigma_{U^{11}}&\Sigma_{U^{21}}\\\Sigma_{U^{12}}&\Sigma_{U^{22}}\\\Sigma_{T^{11}}&\Sigma_{T^{21}}\\\Sigma_{T^{12}}&\Sigma_{T^{22}}\end{bmatrix}\begin{bmatrix}\Phi^{-\mathrm{H}}&0\\0&\Phi^{-\mathrm{H}}\end{bmatrix}. \tag{33}$$

Based on the equivalence between the SVD technique and the least-squares method [22], we can obtain the true eigenequation $(J_m(kb)Y_m(ka)-J_m(ka)Y_m(kb)=0)$. This indicates that only the true eigenvalues for the annular membrane are imbedded in the SVD updating matrix.

4.1.2. SVD updating term

In order to sort out the spurious eigenvalues, the SVD updating term is utilized. For the Neumann problem using

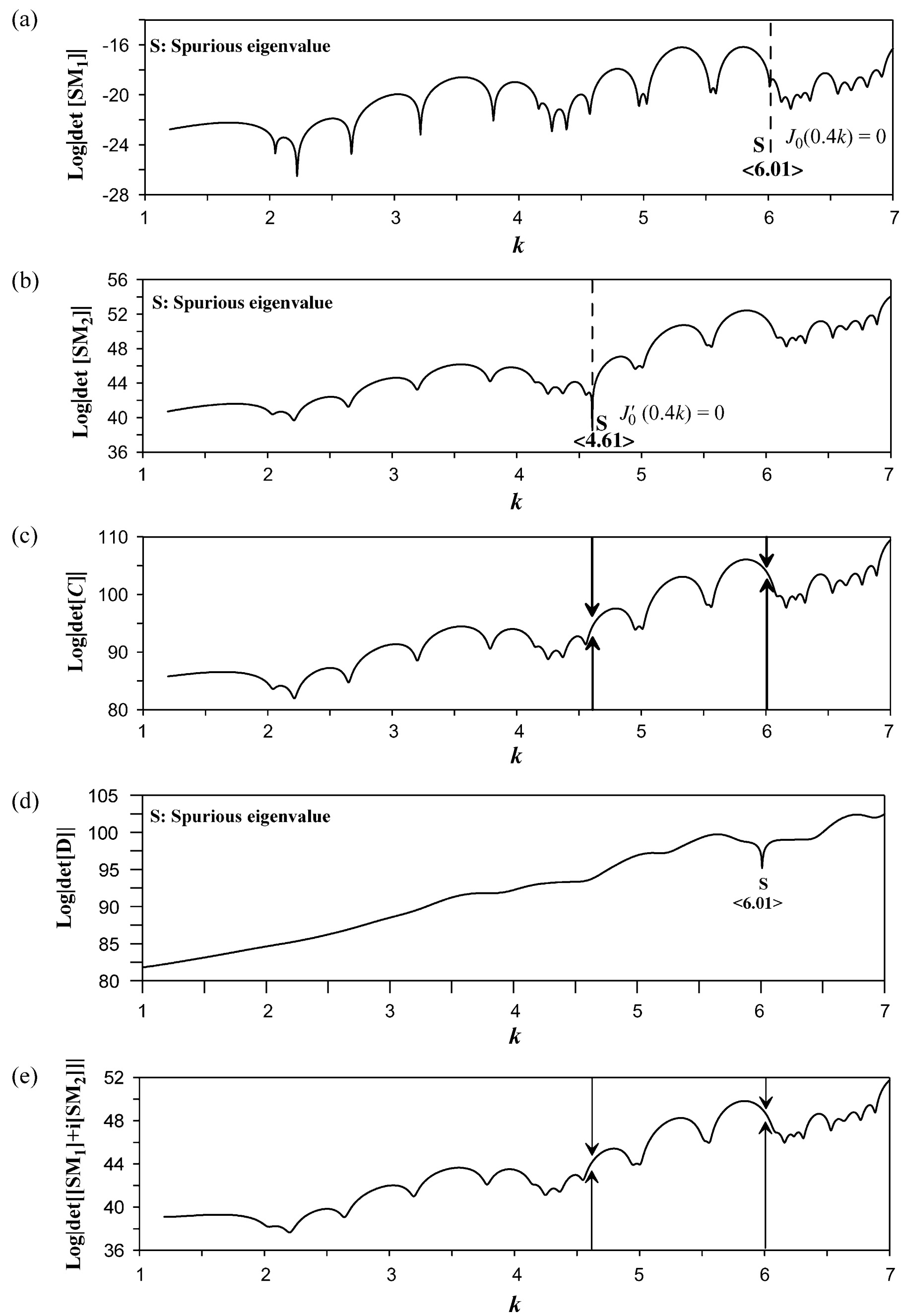

Fig. 3. (a) The determinant versus the wave number by using the single-layer potential approach. (b) The determinant versus the wave number by the using double-layer potential approach. (c) The determinant versus the wave number by using the SVD updating document. (d) The determinant versus the wave number by using the SVD updating term. (e) The determinant versus the wave number by using the Burton and Miller method.

Table 2
The former five true eigenvalues are compared with the different methods with an annular boundary

	k_1	k_2	k_3	k_4	k_5
Analytical solution [14]	2.05	2.23	2.66	3.21	3.80
FEM (ABAQUS) [14]	2.03	2.20	2.62	3.15	3.71
BEM (CHIEF) [14]	2.05	2.23	2.67	3.22	3.81
MFS (single-layer potential approach)	2.05	2.22	2.66	3.21	3.80
MFS (double-layer potential approach)	2.04	2.21	2.65	3.20	3.79
MFS (SVD updating document)	2.05	2.22	2.65	3.20	3.79
MFS (Burton and Miller)	2.04	2.20	2.64	3.20	3.78

the single-layer potential approach, we have

$$[\mathrm{SM}_N]\begin{Bmatrix}\phi_j^1\\ \phi_j^2\end{Bmatrix}=\begin{bmatrix}L^{11} & L^{12}\\ L^{21} & L^{22}\end{bmatrix}\begin{Bmatrix}\phi_j^1\\ \phi_j^2\end{Bmatrix}=\{0\}. \quad (34)$$

In order to obtain an overdetermined system, we can combine $[\mathrm{SM}_1]$ and $[\mathrm{SM}_N]$ matrices by using the SVD updating term. We have

$$[D]=\begin{bmatrix}\mathrm{SM}_1\\ \mathrm{SM}_N\end{bmatrix}. \quad (35)$$

Where the rank of the matrix $[D]$ must be smaller than $4N$ for spurious eigenvalues. By using the property of Eq. (26), the matrix can be written as

$$[D]=\begin{bmatrix}\Phi & 0 & 0 & 0\\ 0 & \Phi & 0 & 0\\ 0 & 0 & \Phi & 0\\ 0 & 0 & 0 & \Phi\end{bmatrix}\begin{bmatrix}\Sigma_{U^{11}} & \Sigma_{U^{12}}\\ \Sigma_{U^{21}} & \Sigma_{U^{22}}\\ \Sigma_{L^{11}} & \Sigma_{L^{12}}\\ \Sigma_{L^{21}} & \Sigma_{L^{22}}\end{bmatrix}\begin{bmatrix}\Phi^{-\mathrm{H}} & 0\\ 0 & \Phi^{-\mathrm{H}}\end{bmatrix}. \quad (36)$$

Based on the equivalence between the SVD technique and the least-squares method [22], we can obtain the spurious eigenequation ($J_m(ka')=0$). This indicates that only the spurious eigenvalues for the annular membrane are imbedded in the SVD updating matrix.

4.2. Burton and Miller method

By employing the Burton and Miller method for dealing with fictitious frequencies, we extend this concept to suppress the appearance of the spurious eigenvalue of the annular membrane in the MFS.

By assembling the Eqs. (17) and (31) with an imaginary number, we have

$$\big[[\mathrm{SM}_1]+\mathrm{i}[\mathrm{SM}_2]\big]\begin{Bmatrix}\varphi_1\\ \varphi_2\end{Bmatrix}=\{0\}, \quad (37)$$

where the φ_1 and φ_2 are the mixed densities. Thus, only the true eigenequation ($J_m(kb)Y_m(ka)-J_m(ka)Y_m(kb)=0$) is obtained by using the Burton and Miller method.

5. Numerical examples

We consider two Dirichlet eigenproblems with the multiply connected domain. The fundamental solution we used is the $U(s,x)=\mathrm{i}J_0(kr)-Y_0(kr)$. The treatments, SVD updating techniques and Burton and Miller method, are also employed to filter out the spurious eigenvalues.

5.1. Case 1: an annular case

The inner and outer radii of an annular membrane are 0.5 m and the outer radius of 2 m, respectively. The fictitious sources are distributed at $a'=0.4$ and $b'=2.2$ m. Thirty-six concentrated singularities locate uniformly at the outer and inner fictitious boundaries as shown in Fig. 2, respectively. Fig. 3(a) and (b) shows the determinant versus the wave number by using the single-layer potential approach and double-layer potential approach, respectively. The drop location indicates the possible eigenvalues. As predicted analytically, the spurious eigenvalue of $k=6.01$ ($J_m(ka')=0$, $m=0$) and $k=4.61$ ($J_m'(ka')=0$, $m=0$) appear for the single and double-layer potential approaches, respectively. It indicates that spurious eigenvalues using the single and double-layer potential approach happen to be the true eigenvalues of the Dirichlet (clamped) and Neumann (free) circular membranes with a radius $a'=0.4$, respectively. Fig. 3(c) shows the determinant versus the wave number by using the SVD updating document where only true eigenvalues are sorted out. Fig. 3(d) shows the determinant versus wave number by using the SVD updating term where the contaminated spurious eigenvalues are extracted out. Fig. 3(e) shows the determinant versus the wave number by using the Burton

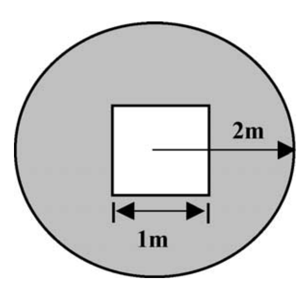

Fig. 4. Sketch of the multiply connected problem.

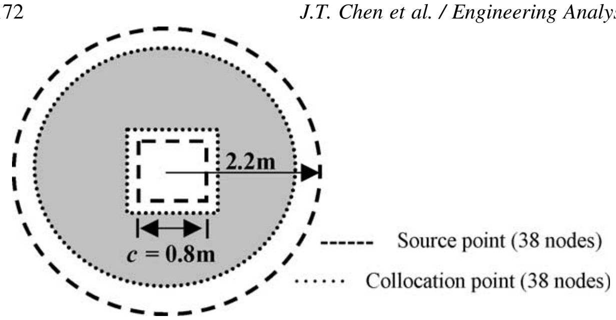

Fig. 5. Figure sketch for node distribution.

and Miller method for the annular membrane where only the true eigenvalues are drawn out. It is found that the spurious eigenvalues are effectively suppressed by using the SVD updating document and the Burton and Miller approaches. Only the true eigenvalues occur in Fig. 3(c) and (e), and only the spurious eigevalues appear in Fig. 3(d). The former five true eigenvalues using the MFS are compared with those using FEM and BEM as shown in

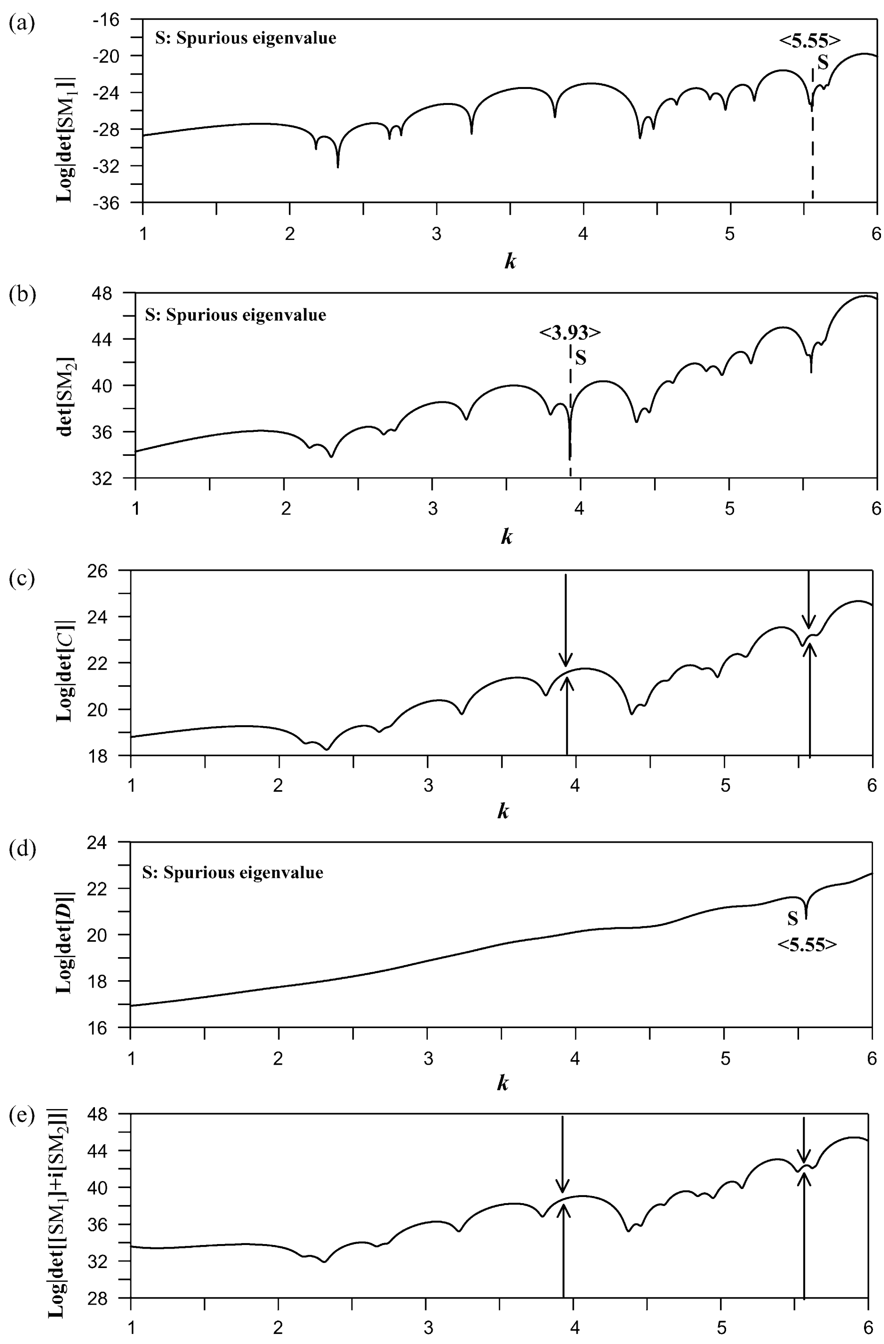

Fig. 6. (a) The determinant versus the wave number by using the single-layer potential approach. (b) The determinant versus the wave number by the using double-layer potential approach. (c) The determinant versus the wave number by using the SVD updating document. (d) The determinant versus the wave number by using the SVD updating term. (e) The determinant versus the wave number by using the Burton and Miller method.

Table 3
The former five true eigenvalues are compared with the different methods with an inner square and outer circle boundary

	k_1	k_2	k_3	k_4	k_5
FEM (ABAQUS) [14]	2.19	2.33	2.67	2.76	3.22
BEM (CHIEF) [14]	2.19	2.33	2.69	2.76	3.24
MFS (single-layer potential approach)	2.18	2.33	2.68	2.76	3.24
MFS (double-layer potential approach)	2.17	2.32	2.67	2.75	3.23
MFS (SVD updating document)	2.18	2.32	2.67	2.76	3.23
MFS (Burton and Miller)	2.17	2.31	2.67	2.76	3.22

Table 2. After comparing the results with the analytical solution, good agreement is made.

5.2. Case 2: inner square and outer circle

A multiply connected domain composed of an inner square and outer circle boundaries is considered in Fig. 4. Thirty-eight singularities are uniformly distributed on the fictitious outer circle boundary and inner square boundary with a length c as shown in Fig. 5. Fig. 6(a) and (b) show the determinant versus wave number by using the single-layer potential approach and double-layer potential approach, respectively. The drop location indicates the possible eigenvalues. The expected spurious eigenvalue of $k=5.55$ ($\lambda_{mn}=\pi\sqrt{m^2+n^2}/c$, $m=1$ and $n=1$) [23] and $k=3.93$ ($\lambda_{mn}=\pi\sqrt{m^2+n^2}/c$, $m=0$ and $n=1$ or $m=0$ and $n=1$) [23] appear in Fig. 6(a) and (b) by using the single and double-layer potential approaches, respectively. Both the figures show that spurious eigenvalues using the single and double-layer potential approaches happen to be the true eigenvalues of the Dirichlet (clamped) and Neumann (free) square membranes with a length of 0.8 m, respectively. Fig. 6(c) and (d) show the determinant versus the wave number by using the SVD updating document and term, respectively. Only the true eigenvalues are presented in Fig. 6(c). Only the spurious eigenvalues appear in Fig. 6(d). Fig. 6(e) shows the determinant versus the wave number by using the Burton and Miller method for the multiply connected membrane where all the spurious eigenvalues are extracted out. It is found that the spurious eigenvalues are effectively suppressed by using the SVD updating document and the Burton and Miller approaches. The former five true eigenvalues using the MFS are compared with those using FEM and BEM as shown in Table 3. Both cases show consistency that single and double-layer potential approaches result in the spurious eigenvalues which are the associated interior eigenvalues of the Dirichlet (clamped) and Neumann (free) membranes bounded by the inner fictitious sources, respectively. Also, the validity of the regularization techniques, SVD updating and Burton and Miller approaches, is demonstrated.

The mathematical study for the spurious eigensolution is suitable for the annular case only. In order to verify the validity of our proposed method, the general case was given and the numerical data can support the existence of spurious eigensolution. The treatments, SVD updating techniques and Burton and Miller method were successfully used to deal with spurious eigenvalues in case 1. The proposed method also works well for the noncircular case as shown in case 2.

For the special case of very small interior region, e.g. $a \ll b$ for annular membrane, the present method fails since no place to distribute singularities can be found. A similar problem has been studied in [24].

6. Conclusions

We have proved that the spurious eigenvalues for annular problems occur by using degenerate kernels and circulants when the MFS is used. The positions of spurious eigenvalues for the annular problem depend on the location of inner fictitious boundary where the sources are distributed. The spurious eigenvalues appearing in the single and double-layer MFS were found to be the interior eigenvalues of subject to the Dirichlet (clamped) and Neumann (free) boundary conditions, respectively. The spurious eigenvalues in the multiply connected problem are found to be the true eigenvalues of the associated simply connected problem enclosing by the inner boundary. Finally, we have employed the SVD updating techniques and Burton and Miller method to filter out the spurious eigenvalues successfully.

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