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Engineering Analysis with Boundary Elements 23 (1999) 339–360

ENGINEERING ANALYSIS with BOUNDARY ELEMENTS

Applications of dual MRM for determining the natural frequencies and natural modes of an Euler–Bernoulli beam using the singular value decomposition method

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Received 25 February 1998; accepted 26 September 1998

Abstract

In this paper, a dual multiple reciprocity method (MRM) is employed to solve the natural frequencies and natural modes for an Euler–Bernoulli beam. It is found that the conventional MRM using an essential integral equation results in spurious eigenvalues and modes. By using the natural integral equation of dual MRM, the spurious eigendata can be filtered out. Four numerical examples are given to verify the validity of the present formulation. In one of these four examples, fixed–fixed supported beam, it is found that the boundary eigenvector cannot be determined by either the essential or natural integral equation alone since the rank of the corresponding leading coefficient matrix is insufficient. The singular value decomposition method is then used to solve the eigenproblem after combining the essential and natural integral equations. This method can avoid the spurious eigenvalue problem and possible indeterminancy of boundary eigenvectors at the same time. © 1999 Elsevier Science Ltd. All rights reserved.

Keywords: Dual multiple reciprocity method; Beam; Natural modes; Singular value decomposition

1. Introduction

Dual BEM has been applied in boundary value problems with a degenerate boundary [1–5], corner problem [6], exterior problem [7] and error estimation for adaptive mesh generation. By combining the singular integral equation in conventional BEM and the hypersingular integral equation, many problems can be solved more directly and efficiently. The roles of the hypersingular integral equation in BEM were reviewed in Ref. [8,18].

For a Helmholtz equation, the complex fundamental solution has been employed to solve eigenproblems [11]. To avoid computation in the domain of a complex number, the multiple reciprocity method (MRM) has been employed to solve the Helmholtz problem in the real domain [12–14]. In this algorithm, the Helmholtz equation is treated as a Poisson equation with an external source; therefore, the fundamental solution of the Laplace equation is considered. However, the domain integral is present due to the integration of the external source. MRM can transform this domain integral into boundary integrals iteratively such that the domain cell is not necessary when the remainder term of the domain integral can be neglected. In the literature, the conventional singular integral equation (essential integral equation) was used only in MRM [12]. In Ref. [8], the role of the hypersingular integral equation (natural integral equation) in MRM was discussed and applied to deal with spurious eigenvalues and modes for a rod. The terms of essential and natural integral equations are named according to the nature of the field quntities concerned, where the equation with the primary field is named as the essential integral equation and the equation with the secondary field is named as the natural integral equation. This will be explained in detail later. Kamiya et al. [11] found, using a two dimensional case, that the kernels in MRM were no more than real parts of the kernels in the complex-valued formulation. Yeih et al. [15] proved that MRM can be constructed such that it is fully equivalent to complex-valued formulation by adding a complex constant into the zeroth fundamental solution for the Laplace operator when the radiation condition is satisfied. Further, they clearly explained why the spurious eigenvalue problem is encountered in the conventional dual MRM.

In this paper, the role of the hypersingular integral equation for the natural frequencies and modes of an Euler–Bernoulli

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PII: S0955-7997(98)00084-8

beam using dual MRM is examined. The spurious eigenvalue problem is also encountered in this probelm, and combined use of equations derived from dual MRM can help us to filter out the spurious eigenvalues in a way similar to that in the work of Chen et al. [8]. Further, there exist more equations than unknowns when one determines the boundary eigenvectors under the framework of the dual MRM. Unfortunately, either the essential integral equations or the natural integral equations may fail in some special cases since the rank of the leading coefficient matrix is insufficient. To solve the eigenproblem more efficiently, the singular value decomposition method (SVD) is adopted. The SVD method can avoid the spurious eigenvalue problem and find the boundary eigenvector more efficiently in the sense of the least square error. Four examples will be solved using the dual MRM to demonstrate the validity of the current research. The reason why we select a one-dimensional structure as the beginning point of our study is that the analytical solution of 1D structures can be easily obtained, thus one can easily check the numerical solutions with analytical ones. Phenomena such as spurious eigenvalues can be easily checked out and explained in the one-dimensional structure, which may not be so easy to notice when one begins with 2D or 3D complicated strctures. A way to filter out the spurious eigenvalues proposed in this paper is based on the theory of linear algbra; therefore, this method can be easily applied to 2D or 3D structures without any difficulty. Although only a one-dimensional beam problem is used to show the validity of the proposed method, the extension from 1D cases to 2D or 3D cases has no difficulty theoretically. Some extension work of the proposed method to solve the natural eigenfrequencies and judge their multiplicities for the membrane was carried out [9,10,19] at the same time. This study can be viewed as the prelude of future extended work to higher dimensional structures.

2. Problem statement and analytical derivations

Consider a one-dimensional Euler–Bernoulli beam vibration problem with the following governing equation:

$$\frac{d^4u(x)}{dx^4} - \lambda u(x) = 0, \ 0 \le x \le 1 \tag{1}$$

where λ and $u(x)$ denote the eigenvalue and eigenmode, respectively. Without loss of generality, it is assumed that the beam has a unit length.

Four examples are considered as follows:

Case 1: A simply supported beam. Boundary conditions are given as $u(0) = 0$, $u''(0) = 0$, $u(1) = 0$ and $u''(1) = 0$, where $'$ denotes differentiation with respect to x.

Case 2: A cantilever beam. Boundary conditions are given as $u(0) = 0$, $u'(0) = 0$, $u''(1) = 0$ and $u'''(1) = 0$.

Case 3: A fixed-roller supported beam. Boundary conditions are given as $u(0) = 0$, $u'(0) = 0$, $u(1) = 0$ and $u''(1) = 0$.

Case 4: A fixed-fixed supported beam. Boundary conditions are given as $u(0) = 0$, $u'(0) = 0$, $u(1) = 0$ and $u'(1) = 0$.

Consider an auxilliary system with a fundamental solution satisfying

$$\frac{d^4U(x,s)}{dx^4} = \delta(x-s), \ -\infty < x < \infty \tag{2}$$

where $U(x,s)$ is a fundamental solution expressed as

$$U(x,s) = \frac{1}{12}|x-s|^3 = U^{(0)}(x,s) \tag{3}$$

In the above expression, $U^{(0)}(x,s)$ is called the zeroth-order fundamental solution.

By integrating by parts, we have

$$\int_0^1 \nabla^4 U^{(0)}(x,s)u(x)dx = \int_0^1 U^{(0)}(x,s)\nabla^4 u(x)dx + \left[u(x)\frac{d^3U^{(0)}(x,s)}{dx^3} - \frac{du(x)}{dx}\frac{d^2U^{(0)}(x,s)}{dx^2} + \frac{d^2u(x)}{dx^2}\frac{dU^{(0)}(x,s)}{dx} - \frac{d^3u(x)}{dx^3}U^{(0)}(x,s)\right]\Big|_{x=0}^{x=1} \tag{4}$$

By transforming the domain integral term on the right hand side of the equal sign in Eq. (4), we have

$$D^{(0)} = \int_0^1 U^{(0)}(x,s)\nabla^4 u(x)\mathrm{d}x = \int_0^1 \nabla^4 U^{(1)}(x,s) b^{(0)}\mathrm{d}x = \int_0^1 U^{(1)}(x,s)\nabla^4 b^{(0)}\mathrm{d}x + \left[b^{(0)} \frac{d^3 U^{(1)}(x,s)}{\mathrm{d}x^3} - \frac{\mathrm{d}b^{(0)}}{\mathrm{d}x}\frac{\mathrm{d}^2 U^{(1)}(x,s)}{\mathrm{d}x^2} + \frac{\mathrm{d}^2 b^{(0)}}{\mathrm{d}x^2}\frac{\mathrm{d}U^{(1)}(x,s)}{\mathrm{d}x} - \frac{\mathrm{d}^3 b^{(0)}}{\mathrm{d}x^3} U^{(1)}(x,s) \right]\Big|_{x=0}^{x=1} \tag{5}$$

where

$$\nabla^4 U^{(1)}(x,s) = U^{(0)}(x,s)$$

$$b^{(0)} = \nabla^4 u(x) = \lambda u(x)$$

By transforming the domain integral term on the right hand side of the equal sign in Eq. (5) again, we have

$$D^{(1)} = \int_0^1 U^{(1)}(x,s)\nabla^4 b^{(0)}\mathrm{d}x = \int_0^1 \nabla^4 U^{(2)}(x,s) b^{(1)}\mathrm{d}x = \int_0^1 U^{(2)}(x,s)\nabla^4 b^{(1)}\mathrm{d}x + \left[b^{(1)} \frac{\mathrm{d}^3 U^{(2)}(x,s)}{\mathrm{d}x^3} - \frac{\mathrm{d}b^{(1)}}{\mathrm{d}x}\frac{\mathrm{d}^2 U^{(2)}(x,s)}{\mathrm{d}x^2} + \frac{\mathrm{d}^2 b^{(1)}}{\mathrm{d}x^2}\frac{\mathrm{d}U^{(2)}(x,s)}{\mathrm{d}x} - \frac{\mathrm{d}^3 b^{(1)}}{\mathrm{d}x^3} U^{(2)}(x,s) \right]\Big|_{x=0}^{x=1} \tag{6}$$

where

$$\nabla^4 U^{(2)}(x,s) = U^{(1)}(x,s)$$

$$b^{(1)} = \nabla^4 b^{(0)} = \lambda(\nabla^4 u(x)) = (\lambda)^2 u(x)$$

Repeating the above process, many boundary terms appear except for one remainder of the domain integral as follows:

$$D^{(0)} = \sum_{j=0}^{N} \left[b^{(j)} \frac{\mathrm{d}^3 U^{(j+1)}(x,s)}{\mathrm{d}x^3} - \frac{\mathrm{d}b^{(j)}}{\mathrm{d}x}\frac{\mathrm{d}^2 U^{(j+1)}(x,s)}{\mathrm{d}x^2} + \frac{\mathrm{d}^2 b^{(j)}}{\mathrm{d}x^2}\frac{d\mathrm{U}^{(j+1)}(x,s)}{\mathrm{d}x} - \frac{\mathrm{d}^3 b^{(j)}}{\mathrm{d}x^3} U^{(j+1)}(x,s) \right]\Big|_{x=0}^{x=1} + R_{N+1} \tag{7}$$

where R_{N+1} is a remainder term, and the body source term and remainder term are found to be

$$b^{(j)}(x) = (\lambda)^{(j+1)} u(x) \tag{8}$$

$$\frac{\mathrm{d}b^{(j)}(x)}{\mathrm{d}x} = (\lambda)^{(j+1)} u'(x) \tag{9}$$

$$R_{N+1} \equiv \int_0^1 U^{(N+1)}(x,s)\nabla^4 b^{(N)}\mathrm{d}x \tag{10}$$

The primary fields, $u(s)$ and $u'(s)$, and secondary fields, $u''(s)$ and $u'''(s)$, can be expressed as

$$u(s) = \Bigg\{ u(x)\frac{\partial^3 U^{(0)}(x,s)}{\partial x^3} - \frac{\mathrm{d}u(x)}{\mathrm{d}x}\frac{\partial^2 U^{(0)}(x,s)}{\partial x^2} + \frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2}\frac{\partial U^{(0)}(x,s)}{\partial x} - \frac{\mathrm{d}^3 u(x)}{\mathrm{d}x^3} U^{(0)}(x,s) + \sum_{j=0}^{N}\left[b^{(j)}\frac{\partial^3 U^{(j+1)}(x,s)}{\partial x^3} - \frac{\mathrm{d}b^{(j)}}{\mathrm{d}x}\frac{\partial^2 U^{(j+1)}(x,s)}{\partial x^2} + \frac{\mathrm{d}^2 b^{(j)}}{\mathrm{d}x^2}\frac{\partial U^{(j+1)}(x,s)}{\partial x} - \frac{\mathrm{d}^3 b^{(j)}}{\mathrm{d}x^3} U^{(j+1)}(x,s) \right]\Bigg\}\Big|_{x=0}^{x=1} + R_{N+1} \tag{11}$$

$$u'(s) = \Bigg\{ u(x)\frac{\partial^4 U^{(0)}(x,s)}{\partial x^3 \partial s} - \frac{\mathrm{d}u(x)}{\mathrm{d}x}\frac{\partial^3 U^{(0)}(x,s)}{\partial x^2 \partial s} + \frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2}\frac{\partial^2 U^{(0)}(x,s)}{\partial x \partial s} - \frac{\mathrm{d}^3 u(x)}{\mathrm{d}x^3}\frac{\partial U^{(0)}(x,s)}{\partial s} + \sum_{j=0}^{N}\left[b^{(j)}\frac{\partial^4 U^{(j+1)}(x,s)}{\partial x^3 \partial s} - \frac{\mathrm{d}b^{(j)}}{\mathrm{d}x}\frac{\partial^3 U^{(j+1)}(x,s)}{\partial x^2 \partial s} + \frac{\mathrm{d}^2 b^{(j)}}{\mathrm{d}x^2}\frac{\partial^2 U^{(j+1)}(x,s)}{\partial x \partial s} - \frac{\mathrm{d}^3 b^{(j)}}{\mathrm{d}x^3}\frac{\partial U^{(j+1)}(x,s)}{\partial s} \right]\Bigg\}\Big|_{x=0}^{x=1} + R'_{N+1} \tag{12}$$

Table 1
Explicit forms for the kernel functions

Kernel	$x > s$	$x < s$	Kernel	$x > s$	$x < s$
$U^{(j+1)}(x,s)$	$\frac{1}{2}\frac{r^{(4j+7)}}{(4j+7)!}(j \geq 0)$	$-\frac{1}{2}\frac{r^{(4j+7)}}{(4j+7)!}(j \geq 0)$	$\frac{\partial^2 U^{(j+1)}(x,s)}{\partial s^2}$	$\frac{1}{2}\frac{r^{(4j+5)}}{(4j+5)!}(j \geq 0)$	$-\frac{1}{2}\frac{r^{(4j+5)}}{(4j+5)!}(j \geq 0)$
$\frac{\partial U^{(j+1)}(x,s)}{\partial x}$	$\frac{1}{2}\frac{r^{(4j+6)}}{(4j+6)!}(j \geq 0)$	$-\frac{1}{2}\frac{r^{(4j+6)}}{(4j+6)!}(j \geq 0)$	$\frac{\partial^3 U^{(j+1)}(x,s)}{\partial x \partial s^2}$	$\frac{1}{2}\frac{r^{(4j+4)}}{(4j+4)!}(j \geq 0)$	$-\frac{1}{2}\frac{r^{(4j+4)}}{(4j+4)!}(j \geq 0)$
$\frac{\partial^2 U^{(j+1)}(x,s)}{\partial x^2}$	$\frac{1}{2}\frac{r^{(4j+5)}}{(4j+5)!}(j \geq 0)$	$-\frac{1}{2}\frac{r^{(4j+5)}}{(4j+5)!}(j \geq 0)$	$\frac{\partial^4 U^{(j+1)}(x,s)}{\partial x^2 \partial s^2}$	$\frac{1}{2}\frac{r^{(4j+3)}}{(4j+3)!}(j \geq 0)$	$-\frac{1}{2}\frac{r^{(4j+3)}}{(4j+3)!}(j \geq 0)$
$\frac{\partial^3 U^{(j+1)}(x,s)}{\partial x^3}$	$\frac{1}{2}\frac{r^{(4j+4)}}{(4j+4)!}(j \geq 0)$	$-\frac{1}{2}\frac{r^{(4j+4)}}{(4j+4)!}(j \geq 0)$	$\frac{\partial^5 U^{(j+1)}(x,s)}{\partial x^3 \partial s^2}$	$\frac{1}{2}\frac{r^{(4j+2)}}{(4j+2)!}(j \geq 0)$	$-\frac{1}{2}\frac{r^{(4j+2)}}{(4j+2)!}(j \geq 0)$
$\frac{\partial U^{(j+1)}(x,s)}{\partial s}$	$-\frac{1}{2}\frac{r^{(4j+6)}}{(4j+6)!}(j \geq 0)$	$\frac{1}{2}\frac{r^{(4j+6)}}{(4j+6)!}(j \geq 0)$	$\frac{\partial^3 U^{(j+1)}(x,s)}{\partial s^3}$	$-\frac{1}{2}\frac{r^{(4j+4)}}{(4j+4)!}(j \geq 0)$	$\frac{1}{2}\frac{r^{(4j+4)}}{(4j+4)!}(j \geq 0)$
$\frac{\partial^2 U^{(j+1)}(x,s)}{\partial x \partial s}$	$-\frac{1}{2}\frac{r^{(4j+5)}}{(4j+5)!}(j \geq 0)$	$\frac{1}{2}\frac{r^{(4j+5)}}{(4j+5)!}(j \geq 0)$	$\frac{\partial^4 U^{(j+1)}(x,s)}{\partial x \partial s^3}$	$-\frac{1}{2}\frac{r^{(4j+3)}}{(4j+3)!}(j \geq 0)$	$\frac{1}{2}\frac{r^{(4j+3)}}{(4j+3)!}(j \geq 0)$
$\frac{\partial^3 U^{(j+1)}(x,s)}{\partial x^2 \partial s}$	$-\frac{1}{2}\frac{r^{(4j+4)}}{(4j+4)!}(j \geq 0)$	$\frac{1}{2}\frac{r^{(4j+4)}}{(4j+4)!}(j \geq 0)$	$\frac{\partial^5 U^{(j+1)}(x,s)}{\partial x^2 \partial s^3}$	$-\frac{1}{2}\frac{r^{(4j+2)}}{(4j+2)!}(j \geq 0)$	$\frac{1}{2}\frac{r^{(4j+2)}}{(4j+2)!}(j \geq 0)$
$\frac{\partial^4 U^{(j+1)}(x,s)}{\partial x^3 \partial s}$	$-\frac{1}{2}\frac{r^{(4j+3)}}{(4j+3)!}(j \geq 0)$	$\frac{1}{2}\frac{r^{(4j+3)}}{(4j+3)!}(j \geq 0)$	$\frac{\partial^6 U^{(j+1)}(x,s)}{\partial x^3 \partial s^3}$	$-\frac{1}{2}\frac{r^{(4j+1)}}{(4j+1)!}(j \geq 0)$	$\frac{1}{2}\frac{r^{(4j+1)}}{(4j+1)!}(j \geq 0)$

$$u''(s) = \Bigg\{u(x)\frac{\partial^5 U^{(0)}(x,s)}{\partial x^3 \partial s^2} - \frac{du(x)}{dx}\frac{\partial^4 U^{(0)}(x,s)}{\partial x^2 \partial s^2} + \frac{d^2u(x)}{dx^2}\frac{\partial^3 U^{(0)}(x,s)}{\partial x \partial s^2} - \frac{d^3u(x)}{dx^3}\frac{\partial^2 U^{(0)}(x,s)}{\partial s^2} + \sum_{j=0}^{N}\left[b^{(j)}\frac{\partial^5 U^{(j+1)}(x,s)}{\partial x^3 \partial s^2} - \frac{db^{(j)}}{dx}\frac{\partial^4 U^{(j+1)}(x,s)}{\partial x^2 \partial s^2} + \frac{d^2b^{(j)}}{dx^2}\frac{\partial^3 U^{(j+1)}(x,s)}{\partial x \partial s^2} - \frac{d^3b^{(j)}}{dx^3}\frac{\partial^2 U^{(j+1)}(x,s)}{\partial s^2}\right]\Bigg\}\Bigg|_{x=0}^{x=1} + R''_{N+1} \quad (13)$$

$$u'''(s) = \Bigg\{u(x)\frac{\partial^6 U^{(0)}(x,s)}{\partial x^3 \partial s^3} - \frac{du(x)}{dx}\frac{\partial^5 U^{(0)}(x,s)}{\partial x^2 \partial s^3} + \frac{d^2u(x)}{dx^2}\frac{\partial^4 U^{(0)}(x,s)}{\partial x \partial s^3} - \frac{d^3u(x)}{dx^3}\frac{\partial^3 U^{(0)}(x,s)}{\partial s^3} + \sum_{j=0}^{N}\left[b^{(j)}\frac{\partial^6 U^{(j+1)}(x,s)}{\partial x^3 \partial s^3} - \frac{db^{(j)}}{dx}\frac{\partial^5 U^{(j+1)}(x,s)}{\partial x^2 \partial s^3} + \frac{d^2b^{(j)}}{dx^2}\frac{\partial^4 U^{(j+1)}(x,s)}{\partial x \partial s^3} - \frac{d^3b^{(j)}}{dx^3}\frac{\partial^3 U^{(j+1)}(x,s)}{\partial s^3}\right]\Bigg\}\Bigg|_{x=0}^{x=1} + R'''_{N+1} \quad (14)$$

where R'_{N+1}, R''_{N+1} and R'''_{N+1} are the first, second, and third derivatives of R_{N+1} with respect to s, and the explicit forms for the kernel functions are shown in Table 1 and defined as

$$U_x^{(j+1)}(x,s) = \frac{\partial\{U^{(j+1)}(x,s)\}}{\partial x} \quad (15)$$

$$U_{xx}^{(j+1)}(x,s) = \frac{\partial^2\{U^{(j+1)}(x,s)\}}{\partial x^2} \quad (16)$$

$$U_{xxx}^{(j+1)}(x,s) = \frac{\partial^3\{U^{(j+1)}(x,s)\}}{\partial x^3} \quad (17)$$

$$U_s^{(j+1)}(x,s) = \frac{\partial\{U^{(j+1)}(x,s)\}}{\partial s} \quad (18)$$

$$U_{xs}^{(j+1)}(x,s) = \frac{\partial^2\{U^{(j+1)}(x,s)\}}{\partial x \partial s} \quad (19)$$

$$U_{xxs}^{(j+1)}(x,s) = \frac{\partial^3\{U^{(j+1)}(x,s)\}}{\partial x^2 \partial s} \tag{20}$$

$$U_{xxxs}^{(j+1)}(x,s) = \frac{\partial^4\{U^{(j+1)}(x,s)\}}{\partial x^3 \partial s} \tag{21}$$

$$U_{ss}^{(j+1)}(x,s) = \frac{\partial^2\{U^{(j+1)}(x,s)\}}{\partial s^2} \tag{22}$$

$$U_{xss}^{(j+1)}(x,s) = \frac{\partial^3\{U^{(j+1)}(x,s)\}}{\partial x \partial s^2} \tag{23}$$

$$U_{xxss}^{(j+1)}(x,s) = \frac{\partial^4\{U^{(j+1)}(x,s)\}}{\partial x^2 \partial s^2} \tag{24}$$

$$U_{xxxss}^{(j+1)}(x,s) = \frac{\partial^5\{U^{(j+1)}(x,s)\}}{\partial x^3 \partial s^2} \tag{25}$$

$$U_{sss}^{(j+1)}(x,s) = \frac{\partial^3\{U^{(j+1)}(x,s)\}}{\partial s^3} \tag{26}$$

$$U_{xsss}^{(j+1)}(x,s) = \frac{\partial^4\{U^{(j+1)}(x,s)\}}{\partial x \partial s^3} \tag{27}$$

$$U_{xxsss}^{(j+1)}(x,s) = \frac{\partial^5\{U^{(j+1)}(x,s)\}}{\partial x^2 \partial s^3} \tag{28}$$

$$U_{xxxsss}^{(j+1)}(x,s) = \frac{\partial^6\{U^{(j+1)}(x,s)\}}{\partial x^3 \partial s^3} \tag{29}$$

Eqs. (11) and (12) with Eqs. (13) and (14) comprise the dual equations for MRM. The first two are called the essential integral equations, the last two, the natural integral equations. The terms of essential integral equations and natural integral equations are named according to the field qunatities concerned, equations with primary field quantities, $u(s)$ and $u'(s)$, are called the essential integral equations and in the same way equations with secondary field quantities, $u''(s)$ and $u'''(s)$, are called the natural integral equations.

By moving the field point close to the boundary, the dual BEM can be derived as follows:

$$U_{xxx}^{(0)}\boldsymbol{u} - U_{xx}^{(0)}\boldsymbol{u}_x + U_{x}^{(0)}\boldsymbol{u}_{xx} - U^{(0)}\boldsymbol{u}_{xxx} = \sum_{i=1}^{N}(U_{xxx}^{(i)}(\lambda)\boldsymbol{u} - U_{xx}^{(i)}(\lambda)\boldsymbol{u}_x + U_{x}^{(i)}(\lambda)\boldsymbol{u}_{xx} - U^{(i)}(\lambda)\boldsymbol{u}_{xxx}) \tag{30}$$

$$U_{xxxs}^{(0)}\boldsymbol{u} - U_{xxs}^{(0)}\boldsymbol{u}_x + U_{xs}^{(0)}\boldsymbol{u}_{xx} - U_{s}^{(0)}\boldsymbol{u}_{xxx} = \sum_{i=1}^{N}(U_{xxxs}^{(i)}(\lambda)\boldsymbol{u} - U_{xxs}^{(i)}(\lambda)\boldsymbol{u}_x + U_{xs}^{(i)}(\lambda)\boldsymbol{u}_{xx} - U_{s}^{(i)}(\lambda)\boldsymbol{u}_{xxx}) \tag{31}$$

$$U_{xxxss}^{(0)}\boldsymbol{u} - U_{xxss}^{(0)}\boldsymbol{u}_x + U_{xss}^{(0)}\boldsymbol{u}_{xx} - U_{ss}^{(0)}\boldsymbol{u}_{xxx} = \sum_{i=1}^{N}(U_{xxxss}^{(i)}(\lambda)\boldsymbol{u} - U_{xxss}^{(i)}(\lambda)\boldsymbol{u}_x + U_{xss}^{(i)}(\lambda)\boldsymbol{u}_{xx} - U_{ss}^{(i)}(\lambda)\boldsymbol{u}_{xxx}) \tag{32}$$

$$U_{xxxsss}^{(0)}\boldsymbol{u} - U_{xxsss}^{(0)}\boldsymbol{u}_x + U_{xsss}^{(0)}\boldsymbol{u}_{xx} - U_{sss}^{(0)}\boldsymbol{u}_{xxx} = \sum_{i=1}^{N}(U_{xxxsss}^{(i)}(\lambda)\boldsymbol{u} - U_{xxsss}^{(i)}(\lambda)\boldsymbol{u}_x + U_{xsss}^{(i)}(\lambda)\boldsymbol{u}_{xx} - U_{sss}^{(i)}(\lambda)\boldsymbol{u}_{xxx}) \tag{33}$$

where $\boldsymbol{u}$, $\boldsymbol{u}_x$, $\boldsymbol{u}_{xx}$ and $\boldsymbol{u}_{xxx}$ are the column vectors of the boundary data.

The explict forms of the two groups of equations, the essential integral equation constructed by means of Eqs. (30) and (31) and the natural integral equation constructed by means of Eqs. (32) and (33), can be found to be

$$\begin{bmatrix} 1+U_{xxx}^{(0)}(0,0^+) & -U_{xxx}^{(0)}(1,0^+) & -U_{xx}^{(0)}(0,0^+) & U_{xx}^{(0)}(1,0^+) \\ U_{xxx}^{(0)}(0,1^-) & 1-U_{xxx}^{(0)}(1,1^-) & -U_{xx}^{(0)}(0,1^-) & U_{xx}^{(0)}(1,1^-) \\ U_{xxxs}^{(0)}(0,0^+) & -U_{xxxs}^{(0)}(1,0^+) & 1-U_{xxs}^{(0)}(0,0^+) & U_{xxs}^{(0)}(1,0^+) \\ U_{xxxs}^{(0)}(0,1^-) & -U_{xxxs}^{(0)}(1,1^-) & -U_{xxs}^{(0)}(0,1^-) & 1+U_{xxs}^{(0)}(1,1^-) \end{bmatrix} \begin{Bmatrix} u(0) \\ u(1) \\ u_x(0) \\ u_x(1) \end{Bmatrix}$$
$$+\begin{bmatrix} U_{x}^{(0)}(0,0^+) & -U_{x}^{(0)}(1,0^+) & -U^{(0)}(0,0^+) & U^{(0)}(1,0^+) \\ U_{x}^{(0)}(0,1^-) & -U_{x}^{(0)}(1,1^-) & -U^{(0)}(0,1^-) & U^{(0)}(1,1^-) \\ U_{xs}^{(0)}(0,0^+) & -U_{xs}^{(0)}(1,0^+) & -U_{s}^{(0)}(0,0^+) & U_{s}^{(0)}(1,0^+) \\ U_{xs}^{(0)}(0,1^-) & -U_{xs}^{(0)}(1,1^-) & -U_{s}^{(0)}(0,1^-) & U_{s}^{(0)}(1,1^-) \end{bmatrix} \begin{Bmatrix} u_{xx}(0) \\ u_{xx}(1) \\ u_{xxx}(0) \\ u_{xxx}(1) \end{Bmatrix}$$
$$=\sum_{j=0}^{N}(\lambda)^{j+1}\begin{bmatrix} -U_{xxx}^{(j+1)}(0,0^+) & U_{xxx}^{(j+1)}(1,0^+) & U_{xx}^{(j+1)}(0,0^+) & -U_{xx}^{(j+1)}(1,0^+) \\ -U_{xxx}^{(j+1)}(0,1^-) & U_{xxx}^{(j+1)}(1,1^-) & U_{xx}^{(j+1)}(0,1^-) & -U_{xx}^{(j+1)}(1,1^-) \\ -U_{xxxs}^{(j+1)}(0,0^+) & U_{xxxs}^{(j+1)}(1,0^+) & U_{xxs}^{(j+1)}(0,0^+) & -U_{xxs}^{(j+1)}(1,0^+) \\ -U_{xxxs}^{(j+1)}(0,1^-) & U_{xxxs}^{(j+1)}(1,1^-) & U_{xxs}^{(j+1)}(0,1^-) & -U_{xxs}^{(j+1)}(1,1^-) \end{bmatrix} \begin{Bmatrix} u(0) \\ u(1) \\ u_x(0) \\ u_x(1) \end{Bmatrix}$$
$$+\sum_{j=0}^{N}(\lambda)^{j+1}\begin{bmatrix} -U_{x}^{(j+1)}(0,0^+) & U_{x}^{(j+1)}(1,0^+) & U^{(j+1)}(0,0^+) & -U^{(j+1)}(1,0^+) \\ -U_{x}^{(j+1)}(0,1^-) & U_{x}^{(j+1)}(1,1^-) & U^{(j+1)}(0,1^-) & -U^{(j+1)}(1,1^-) \\ -U_{xs}^{(j+1)}(0,0^+) & U_{xs}^{(j+1)}(1,0^+) & U_{s}^{(j+1)}(0,0^+) & -U_{s}^{(j+1)}(1,0^+) \\ -U_{xs}^{(j+1)}(0,1^-) & U_{xs}^{(j+1)}(1,1^-) & U_{s}^{(j+1)}(0,1^-) & -U_{s}^{(j+1)}(1,1^-) \end{bmatrix} \begin{Bmatrix} u_{xx}(0) \\ u_{xx}(1) \\ u_{xxx}(0) \\ u_{xxx}(1) \end{Bmatrix} \tag{34}$$

and

$$\begin{bmatrix} U_{xxxss}^{(0)}(0,0^+) & -U_{xxxss}^{(0)}(1,0^+) & -U_{xxss}^{(0)}(0,0^+) & U_{xxss}^{(0)}(1,0^+) \\ U_{xxxss}^{(0)}(0,1^-) & -U_{xxxss}^{(0)}(1,1^-) & -U_{xxss}^{(0)}(0,1^-) & U_{xxss}^{(0)}(1,1^-) \\ U_{xxxsss}^{(0)}(0,0^+) & -U_{xxxsss}^{(0)}(1,0^+) & -U_{xxsss}^{(0)}(0,0^+) & U_{xxsss}^{(0)}(1,0^+) \\ U_{xxxsss}^{(0)}(0,1^-) & -U_{xxxsss}^{(0)}(1,1^-) & -U_{xxsss}^{(0)}(0,1^-) & U_{xxsss}^{(0)}(1,1^-) \end{bmatrix} \begin{Bmatrix} u(0) \\ u(1) \\ u_x(0) \\ u_x(1) \end{Bmatrix}$$
$$+\begin{bmatrix} 1+U_{xss}^{(0)}(0,0^+) & -U_{xss}^{(0)}(1,0^+) & -U_{ss}^{(0)}(0,0^+) & U_{ss}^{(0)}(1,0^+) \\ U_{xss}^{(0)}(0,1^-) & 1-U_{xss}^{(0)}(1,1^-) & -U_{ss}^{(0)}(0,1^-) & U_{ss}^{(0)}(1,1^-) \\ U_{xsss}^{(0)}(0,0^+) & -U_{xsss}^{(0)}(1,0^+) & 1-U_{sss}^{(0)}(0,0^+) & U_{sss}^{(0)}(1,0^+) \\ U_{xsss}^{(0)}(0,1^-) & -U_{xsss}^{(0)}(1,1^-) & -U_{sss}^{(0)}(0,1^-) & 1+U_{sss}^{(0)}(1,1^-) \end{bmatrix} \begin{Bmatrix} u_{xx}(0) \\ u_{xx}(1) \\ u_{xxx}(0) \\ u_{xxx}(1) \end{Bmatrix}$$
$$=\sum_{j=0}^{N}(\lambda)^{j+1}\begin{bmatrix} -U_{xxxss}^{(j+1)}(0,0^+) & U_{xxxss}^{(j+1)}(1,0^+) & U_{xxss}^{(j+1)}(0,0^+) & -U_{xxss}^{(j+1)}(1,0^+) \\ -U_{xxxss}^{(j+1)}(0,1^-) & U_{xxxss}^{(j+1)}(1,1^-) & U_{xss}^{(j+1)}(0,1^-) & -U_{xxss}^{(j+1)}(1,1^-) \\ -U_{xxxsss}^{(j+1)}(0,0^+) & U_{xxxsss}^{(j+1)}(1,0^+) & U_{xxsss}^{(j+1)}(0,0^+) & -U_{xxsss}^{(j+1)}(1,0^+) \\ -U_{xxxsss}^{(j+1)}(0,1^-) & U_{xxxsss}^{(j+1)}(1,1^-) & U_{xxsss}^{(j+1)}(0,1^-) & -U_{xxsss}^{(j+1)}(1,1^-) \end{bmatrix} \begin{Bmatrix} u(0) \\ u(1) \\ u_x(0) \\ u_x(1) \end{Bmatrix}$$
$$+\sum_{j=0}^{N}(\lambda)^{j+1}\begin{bmatrix} -U_{xss}^{(j+1)}(0,0^+) & U_{xss}^{(j+1)}(1,0^+) & U_{ss}^{(j+1)}(0,0^+) & -U_{ss}^{(j+1)}(1,0^+) \\ -U_{xss}^{(j+1)}(0,1^-) & U_{xss}^{(j+1)}(1,1^-) & U_{ss}^{(j+1)}(0,1^-) & -U_{ss}^{(j+1)}(1,1^-) \\ -U_{xsss}^{(j+1)}(0,0^+) & U_{xsss}^{(j+1)}(1,0^+) & U_{sss}^{(j+1)}(0,0^+) & -U_{sss}^{(j+1)}(1,0^+) \\ -U_{xsss}^{(j+1)}(0,1^-) & U_{xsss}^{(j+1)}(1,1^-) & U_{sss}^{(j+1)}(0,1^-) & -U_{sss}^{(j+1)}(1,1^-) \end{bmatrix} \begin{Bmatrix} u_{xx}(0) \\ u_{xx}(1) \\ u_{xxx}(0) \\ u_{xxx}(1) \end{Bmatrix} \tag{35}$$

According to Eqs. (34) and (35), it is found that only seven different series are present as shown below:

$$\sum_{j=0}^{N}\frac{(\lambda)^{j+1}}{(4j+1)!},\sum_{j=0}^{N}\frac{(\lambda)^{j+1}}{(4j+2)!},\sum_{j=0}^{N}\frac{(\lambda)^{j+1}}{(4j+3)!},\sum_{j=0}^{N}\frac{(\lambda)^{j+1}}{(4j+4)!},\sum_{j=0}^{N}\frac{(\lambda)^{j+1}}{(4j+5)!},\sum_{j=0}^{N}\frac{(\lambda)^{j+1}}{(4j+6)!}\text{ and }\sum_{j=0}^{N}\frac{(\lambda)^{j+1}}{(4j+7)!}$$

For clarity, we define

$$A(N)\equiv\sum_{j=0}^{N}\frac{(\lambda)^{j+1}}{(4j+1)!} \tag{36}$$

$$B(N)\equiv\sum_{j=0}^{N}\frac{(\lambda)^{j+1}}{(4j+2)!} \tag{37}$$

$$C(N)\equiv\sum_{j=0}^{N}\frac{(\lambda)^{j+1}}{(4j+3)!} \tag{38}$$

$$D(N)\equiv 1+\sum_{j=0}^{N}\frac{(\lambda)^{j+1}}{(4j+4)!} \tag{39}$$

$$E(N)\equiv 1+\sum_{j=0}^{N}\frac{(\lambda)^{j+1}}{(4j+5)!} \tag{40}$$

$$F(N)\equiv 1+2\sum_{j=0}^{N}\frac{(\lambda)^{j+1}}{(4j+6)!} \tag{41}$$

$$G(N)\equiv 1+6\sum_{j=0}^{N}\frac{(\lambda)^{j+1}}{(4j+7)!} \tag{42}$$

When N approaches infinity, the following equations are obtained:

$$a\equiv\lim_{N\to\infty}A(N)=\frac{(\lambda)^{\frac{3}{4}}(\sinh\sqrt[4]{\lambda}+\sin\sqrt[4]{\lambda})}{2} \tag{43}$$

$$b\equiv\lim_{N\to\infty}B(N)=\frac{(\lambda)^{\frac{1}{2}}(\cosh\sqrt[4]{\lambda}-\cos\sqrt[4]{\lambda})}{2} \tag{44}$$

$$c\equiv\lim_{N\to\infty}C(N)=\frac{(\lambda)^{\frac{1}{4}}(\sinh\sqrt[4]{\lambda}-\sin\sqrt[4]{\lambda})}{2} \tag{45}$$

$$d\equiv\lim_{N\to\infty}D(N)=\frac{(\cosh\sqrt[4]{\lambda}+\cos\sqrt[4]{\lambda})}{2} \tag{46}$$

$$e\equiv\lim_{N\to\infty}E(N)=\frac{(\sinh\sqrt[4]{\lambda}+\sin\sqrt[4]{\lambda})}{2(\lambda)^{\frac{1}{4}}} \tag{47}$$

$$f\equiv\lim_{N\to\infty}F(N)=\frac{(\cosh\sqrt[4]{\lambda}-\cos\sqrt[4]{\lambda})}{(\lambda)^{\frac{1}{2}}} \tag{48}$$

$$g\equiv\lim_{N\to\infty}G(N)=\frac{3(\sinh\sqrt[4]{\lambda}-\sin\sqrt[4]{\lambda})}{(\lambda)^{\frac{3}{4}}} \tag{49}$$

It is interesting to find that the four terms are present as follows: $\sinh\sqrt[4]{\lambda}$, $\sin\sqrt[4]{\lambda}$, $\cosh\sqrt[4]{\lambda}$ and $\cos\sqrt[4]{\lambda}$. Substituting the values of the kernel functions shown in Table 1 into Eqs. (30)–(33), we have

$$\begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & 0 & \frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{-1}{2} \\ 0 & 0 & \frac{-1}{2} & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} u(0) \\ u(1) \\ u_x(0) \\ u_x(1) \end{Bmatrix} + \begin{bmatrix} 0 & \frac{-1}{4} & 0 & \frac{1}{12} \\ \frac{-1}{4} & 0 & \frac{-1}{12} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{-1}{4} \\ \frac{-1}{2} & 0 & \frac{-1}{4} & 0 \end{bmatrix} \begin{Bmatrix} u_{xx}(0) \\ u_{xx}(1) \\ u_{xxx}(0) \\ u_{xxx}(1) \end{Bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{1}{2}(D-1) & 0 & \frac{-1}{2}(E-1) \\ \frac{1}{2}(D-1) & 0 & \frac{1}{2}(E-1) & 0 \\ 0 & \frac{-1}{2}C & 0 & \frac{1}{2}(D-1) \\ \frac{1}{2}C & 0 & \frac{1}{2}(D-1) & 0 \end{bmatrix} \begin{Bmatrix} u(0) \\ u(1) \\ u_x(0) \\ u_{xx}(1) \end{Bmatrix}$$

$$+ \begin{bmatrix} 0 & \frac{1}{4}(F-1) & 0 & \frac{-1}{12}(G-1) \\ \frac{1}{4}(F-1) & 0 & \frac{1}{12}(G-1) & 0 \\ 0 & \frac{-1}{2}(E-1) & 0 & \frac{1}{4}(F-1) \\ \frac{1}{2}(E-1) & 0 & \frac{1}{4}(F-1) & 0 \end{bmatrix} \begin{Bmatrix} u_{xx}(0) \\ u_{xx}(1) \\ u_{xxx}(0) \\ u_{xxx}(1) \end{Bmatrix} \tag{50}$$

and

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u(0) \\ u(1) \\ u_x(0) \\ u_x(1) \end{Bmatrix} + \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & 0 & \frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{-1}{2} \\ 0 & 0 & \frac{-1}{2} & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} u_{xx}(0) \\ u_{xx}(1) \\ u_{xxx}(0) \\ u_{xxx}(1) \end{Bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{1}{2}B & 0 & \frac{-1}{2}C \\ \frac{1}{2}B & 0 & \frac{1}{2}C & 0 \\ 0 & \frac{-1}{2}A & 0 & \frac{1}{2}B \\ \frac{1}{2}A & 0 & \frac{1}{2}B & 0 \end{bmatrix} \begin{Bmatrix} u(0) \\ u(1) \\ u_x(0) \\ u_x(1) \end{Bmatrix} + \begin{bmatrix} 0 & \frac{1}{2}(D-1) & 0 & \frac{-1}{2}(E-1) \\ \frac{1}{2}(D-1) & 0 & \frac{1}{2}(E-1) & 0 \\ 0 & \frac{-1}{2}C & 0 & \frac{1}{2}(D-1) \\ \frac{1}{2}C & 0 & \frac{1}{2}(D-1) & 0 \end{bmatrix} \begin{Bmatrix} u_{xx}(0) \\ u_{xx}(1) \\ u_{xxx}(0) \\ u_{xxx}(1) \end{Bmatrix} \tag{51}$$

3. Solving the eigenproblem by means of the essential and natural integral equations

For simplicity, a simply supported beam is studied in the following. After substituting the boundary conditions of the simply supported beam into Eqs. (50) and (51), the dual BEM has the following essential integral and natural integral equations:

essential integral equation

$$\begin{bmatrix} 0 & \frac{1}{2}E & 0 & \frac{1}{12}G \\ \frac{-1}{2}E & 0 & \frac{-1}{12}G & 0 \\ \frac{1}{2} & \frac{-1}{2}D & 0 & \frac{-1}{4}F \\ \frac{-1}{2}D & \frac{1}{2} & \frac{-1}{4}F & 0 \end{bmatrix} \begin{Bmatrix} u'(0) \\ u'(1) \\ u'''(0) \\ u'''(1) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{52}$$

natural integral equation

$$\begin{bmatrix} 0 & \frac{1}{2}C & 0 & \frac{1}{2}E \\ \frac{-1}{2}C & 0 & \frac{-1}{2}E & 0 \\ 0 & \frac{-1}{2}B & \frac{1}{2} & \frac{-1}{2}D \\ \frac{-1}{2}B & 0 & \frac{-1}{2}D & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} u'(0) \\ u'(1) \\ u'''(0) \\ u'''(1) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{53}$$

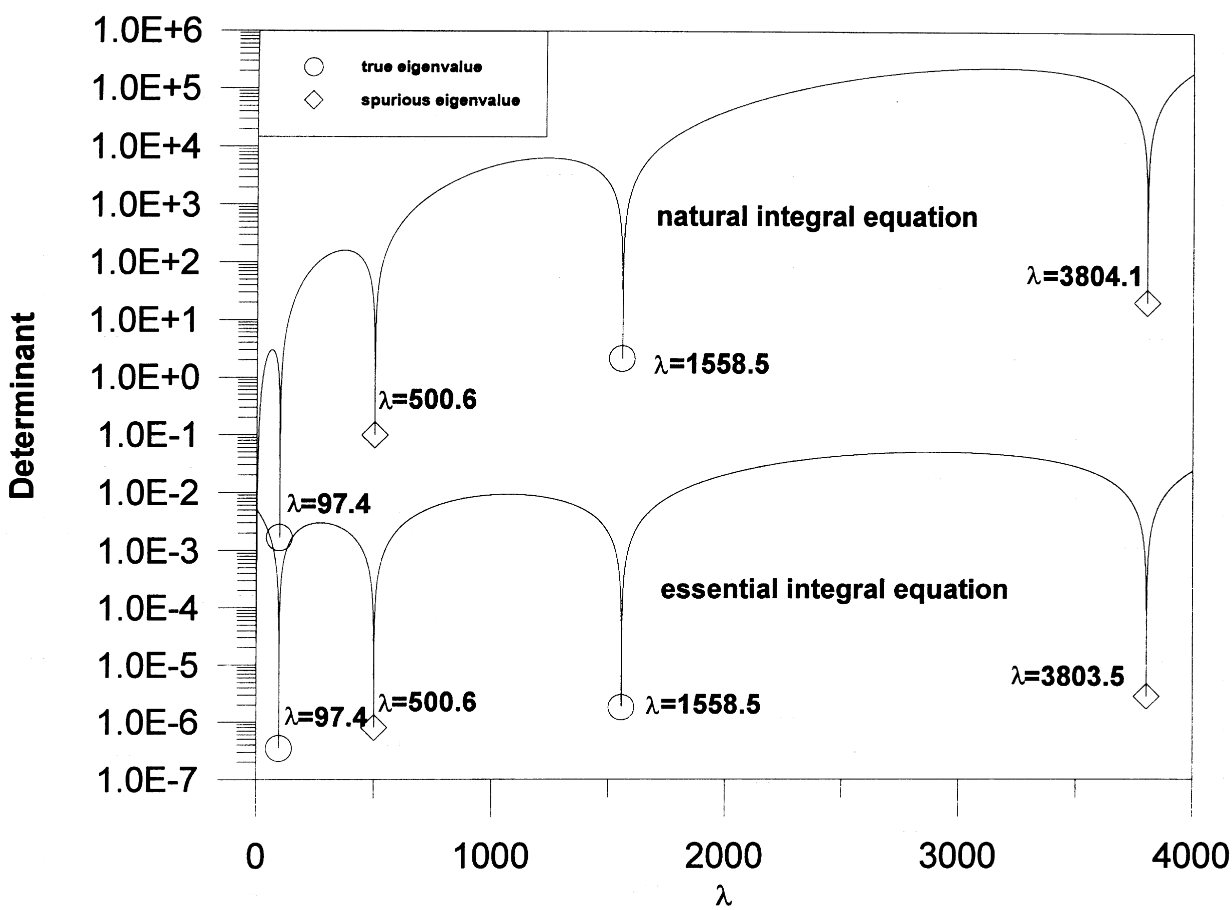

Fig. 1. Direct search of eigenvalues using the essential and natural integral equations for the simply supported beam case.

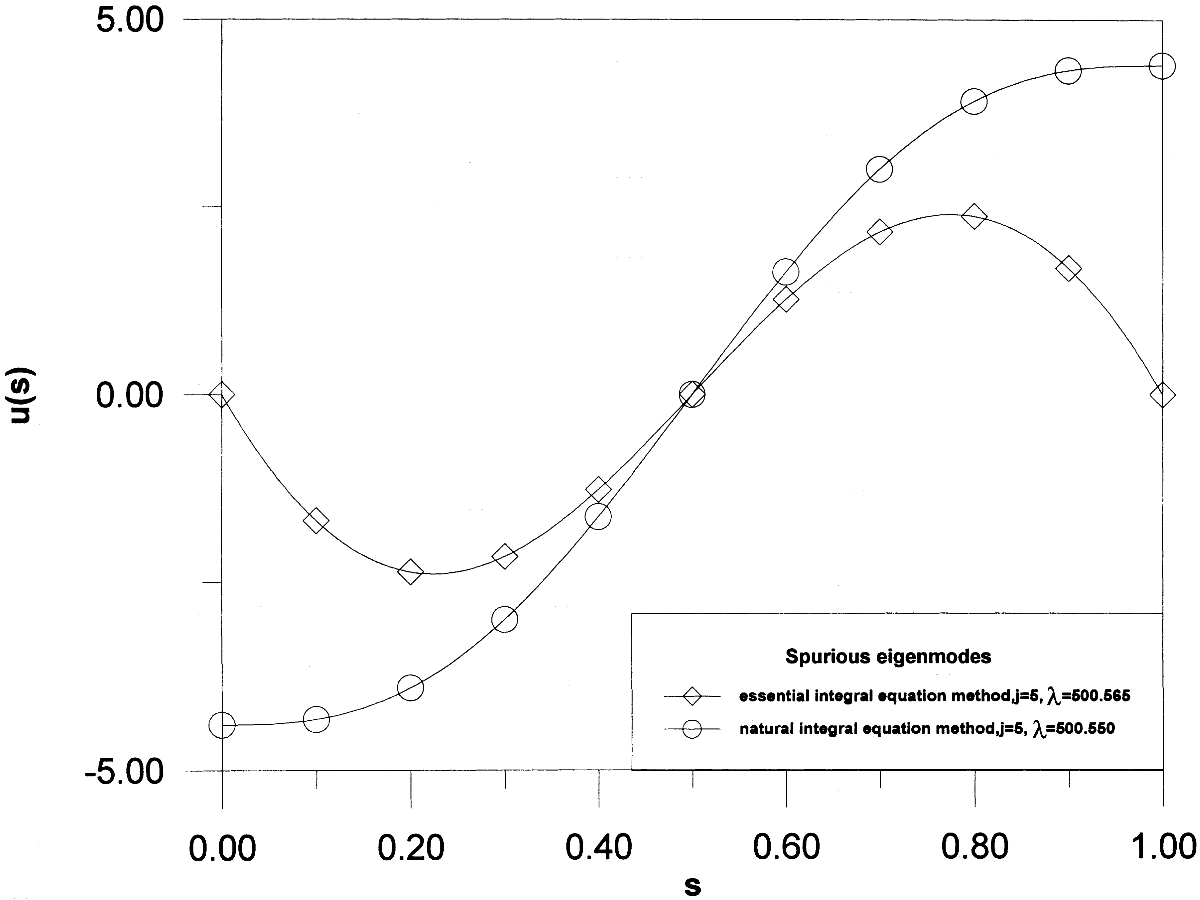

Fig. 2. The first spurious eigenmodes determined using the essential and natural integral equations for the simply supported beam case.

To solve the eigenproblem of Eqs. (52) and (53), the direct search method [8] can be employed to find the eigenvalues. The result obtained using the direct search method is shown in Fig. 1 for both the essential and natural integral equation methods. It is found that both methods find close eigenvalues numerically. Several methods, e.g. the Newton–Raphson method, can be adopted for solving the nonlinear eigenequation as we encounter here; however, for knowing the full spectrum response the direct search method is used in this research. Analytically speaking, eigenequations derived from both methods are the same. However, some of these eigenvalues are spurious. To filter out the spurious eigenvalues, the essential and natural integral equations should be used together in a way similar to the method used in [8]. After finding the eigenvalue, no matter whether it is true or spurious, one can obtain the corresponding boundary eigenvector. Then, the eigenmode can be determined by substituting boundary data into Eq. (11). Comparing the eigenmodes obtained by these two methods, one can find that the true eigenvalue will result in the same eigenmodes for both methods, but that the spurious eigenvalue will result in different eighenmodes as shown in Fig. 2. In other words, for the spurious eigenvalue, the boundary eigenvector obtained by the essential integral equation cannot satisfy the constraints of eigenequations derived by the natural integral equation and vise versa (although the spurious eigenvalues obtained by these two methods are very close). That is to say, only the null vector is satisfied.

Although only the coefficient matrix for a simply supported beam has been examined, the coefficient matrix for other cases can be found in Table 2.

For cases (1), (2) and (3), the essential and natural integral equations can successfully find the eigenvalues shown in Tables 3–5, respectively, and the spurious eigenvalues can be filtered out by the algorithm as stated above. Further, it is found that both methods obtain the same spurious eigenvalues. An explaination for this will be given analytically in the next section and rechecked by means of numerical experiments. The first two modes obtained by both methods are illustrated in Figs. 3–5. In all these figures, the normalized eigenmode is plotted; i.e. the absolute value of the maximum displacement response is set to be 1. It is seen that the numerical results show good agreement with the analytical solutions.

However, it is found that neither the essential nor the natural integral equation can successfully find the boundary eigenvectors for the fixed-fixed supported beam (case 4) although they can find true eigenvalues as shown in Table 6. It can easily be seen that after substitution of boundary conditions into the equations, there are only four boundary unknowns, but there are

Table 2
Coefficient matrices for the essential and natural equations

	Essential integral equation	Natural integral equation
Simply supported beam	$\begin{bmatrix} 0 & \frac{1}{2}E & 0 & \frac{1}{12}G \\ -\frac{1}{2}E & 0 & -\frac{1}{12}G & 0 \\ \frac{1}{2} & -\frac{1}{2}D & 0 & -\frac{1}{4}F \\ -\frac{1}{2}D & \frac{1}{2} & -\frac{1}{4}F & 0 \end{bmatrix} \begin{Bmatrix} u'(0) \\ u'(1) \\ u''(0) \\ u''(1) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$	$\begin{bmatrix} 0 & \frac{1}{2}C & 0 & \frac{1}{2}E \\ -\frac{1}{2}C & 0 & \frac{-1}{2}E & 0 \\ 0 & -\frac{1}{2}B & \frac{1}{2} & -\frac{1}{2}D \\ -\frac{1}{2}B & 0 & -\frac{1}{2}D & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} u'(0) \\ u'(1) \\ u''(0) \\ u''(1) \end{Bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$
Cantilever beam	$\begin{bmatrix} \frac{1}{2}D & -\frac{1}{2}E & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{4}F & \frac{1}{12}G \\ -\frac{1}{2}C & \frac{1}{2}D & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2}E & \frac{1}{4}F \end{bmatrix} \begin{Bmatrix} u(1) \\ u'(1) \\ u''(0) \\ u'''(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$	$\begin{bmatrix} \frac{1}{2}B & -\frac{1}{2}C & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2}D & \frac{1}{2}E \\ -\frac{1}{2}A & \frac{1}{2}B & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2}C & \frac{1}{2}D \end{bmatrix} \begin{Bmatrix} u(1) \\ u'(1) \\ u''(0) \\ u'''(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$
Fixed-roller suppported beam	$\begin{bmatrix} -\frac{1}{2}E & 0 & 0 & -\frac{1}{12}G \\ 0 & \frac{1}{4}F & \frac{1}{12}G & 0 \\ \frac{1}{2}D & 0 & 0 & \frac{1}{4}F \\ -\frac{1}{2} & \frac{1}{2}E & \frac{1}{4}F & 0 \end{bmatrix} \begin{Bmatrix} u'(1) \\ u'(0) \\ u'''(0) \\ u'''(1) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$	$\begin{bmatrix} -\frac{1}{2}C & -\frac{1}{2} & 0 & -\frac{1}{2}E \\ 0 & \frac{1}{2}D & \frac{1}{2}E & 0 \\ \frac{1}{2}B & 0 & -\frac{1}{2} & \frac{1}{2}D \\ 0 & \frac{1}{2}C & \frac{1}{2}D & -\frac{1}{2} \end{bmatrix} \begin{Bmatrix} u'(1) \\ u'(0) \\ u'''(0) \\ u'''(1) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$
Fixed-fixed supported beam	$\begin{bmatrix} 0 & \frac{1}{4}F & 0 & -\frac{1}{12}G \\ \frac{1}{4}F & 0 & \frac{1}{12}G & 0 \\ 0 & -\frac{1}{2}E & 0 & \frac{1}{4}F \\ \frac{1}{2}E & 0 & -\frac{1}{4}F & 0 \end{bmatrix} \begin{Bmatrix} u''(0) \\ u''(1) \\ u'''(0) \\ u'''(1) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$	$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2}D & 0 & -\frac{1}{2}E \\ \frac{1}{2}D & -\frac{1}{2} & \frac{1}{2}E & 0 \\ 0 & -\frac{1}{2}C & -\frac{1}{2} & \frac{1}{2}D \\ \frac{1}{2}C & 0 & \frac{1}{2}D & -\frac{1}{2} \end{bmatrix} \begin{Bmatrix} u''(0) \\ u''(1) \\ u'''(0) \\ u'''(1) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$

eight equations in all. Therefore, there is no preferred choice of equations for solving this eigenproblem. Dividing the equations into two groups, essential and natural equations, is merely a natural division in the derivation stage. Theoretically, any four equations among these eight equations can be adopted, provided that the rank of the leading coefficient matrix is sufficient to obtain the boundary eigenvector. (The rank should be 3 for the eigenvalue λ.) The essential and natural integral equations fail to find the boundary eigenvectors since the rank of both leading coefficient matrices from the two integral equations is only 2 for the obtained eigenvalues. However, other equations can find the eigenvalues and boundary eigenvectors at the same time as shown in Table 6. The first two eigenmodes obtained by the appropriate equations in Table 6 are illustrated in Fig. 6. Again, good agreement between the numerical result and analytical solution is obtained. For the reader's reference, the analytical results for the eigenvalues, boundary eigenvectors and eigensolutions for all four cases are shown in Tables 7–10.

4. Analytical derivations for the eigenequation obtained using the dual MRM

According to Eqs. (43)–(49) for the case of $N \to \infty$, Eqs. (52) and (53) for a simply supported beam can be expressed as follows.

Table 3
Eigensolutions for the simply supported beam case

Analytical solution	First mode		Spurious mode	Second mode		Spurious mode
	Eigenvalues	Boundary mode	Eigenvalues	Eigenvalues	Boundary mode	Eigenvalues
	97.409	See Table 7	N.A.	1558.545	See Table 7	N.A.
Essential integral equation						
$N=1$	97.565	—	N.A.	1341.535	—	N.A.
$N=2$	97.405	—	501.445	1439.015	—	2530.745
$N=3$	97.405	—	500.565	1556.145	—	3568.315
$N=4$	97.405	—	500.565	1558.535	—	3797.745
$N=5$	97.405	$\begin{Bmatrix} u'(0) \\ u'(1) \\ u'''(0) \\ u'''(1) \end{Bmatrix} = \begin{Bmatrix} 7.12\times10^{-2} \\ -7.13\times10^{-2} \\ -7.03\times10^{-1} \\ 7.04\times10^{-1} \end{Bmatrix}$	500.565	1558.545	$\begin{Bmatrix} u'(0) \\ u'(1) \\ u'''(0) \\ u'''(1) \end{Bmatrix} = \begin{Bmatrix} -1.79\times10^{-2} \\ -1.79\times10^{-2} \\ 7.08\times10^{-1} \\ 7.06\times10^{-1} \end{Bmatrix}$	3803.525
True eigen equation: $\sin(\sqrt[4]{\lambda_n})=0$						
Spurious eigen equation: $\sinh(\sqrt[4]{\lambda_n})\{1-\cosh(\sqrt[4]{\lambda_n})\cos(\sqrt[4]{\lambda_n})\}=0$						
Natural integral equation						
$N=1$	95.650	—	N.A.	N.A.	—	N.A.
$N=2$	97.450	—	490.750	N.A.	—	N.A.
$N=3$	97.450	—	500.450	1558.750	—	N.A.
$N=4$	97.450	—	500.550	1558.750	—	3858.451
$N=5$	97.450	$\begin{Bmatrix} u'(0) \\ u'(1) \\ u'''(0) \\ u'''(1) \end{Bmatrix} = \begin{Bmatrix} 7.13\times10^{-2} \\ -7.13\times10^{-2} \\ -7.03\times10^{-1} \\ 7.04\times10^{-1} \end{Bmatrix}$	500.550	1558.550	$\begin{Bmatrix} u'(0) \\ u'(1) \\ u'''(0) \\ u'''(1) \end{Bmatrix} = \begin{Bmatrix} -1.79\times10^{-2} \\ -1.79\times10^{-2} \\ 7.08\times10^{-1} \\ 7.06\times10^{-1} \end{Bmatrix}$	3804.150
True eigen equation: $\sin(\sqrt[4]{\lambda_n})=0$						
Spurious eigen equation: $\sinh(\sqrt[4]{\lambda_n})\{1-\cosh(\sqrt[4]{\lambda_n})\cos(\sqrt[4]{\lambda_n})\}=0$						

Essential integral equation:

$$\begin{bmatrix} 0 & \frac{1}{2}e & 0 & \frac{1}{12}g \\ \frac{-1}{2}e & 0 & \frac{-1}{12}g & 0 \\ \frac{1}{2} & \frac{-1}{2}d & 0 & \frac{-1}{4}f \\ \frac{-1}{2}d & \frac{1}{2} & \frac{-1}{4}f & 0 \end{bmatrix} \begin{Bmatrix} u'(0) \\ u'(1) \\ u'''(0) \\ u'''(1) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{54}$$

In order to obtain a nontrivial solution for the boundary eigenvector, the determinant should be zero. Therefore, we have the eigenequation

$$-\frac{1}{32(\lambda)^{3/2}}\sin\sqrt[4]{\lambda}\sinh\sqrt[4]{\lambda}(1-\cosh\sqrt[4]{\lambda}\cos\sqrt[4]{\lambda})=0 \tag{55}$$

Table 4
Eigensolutions for the cantilever beam case

Analytical solution	First mode		Second mode		Spurious mode	Spurious mode
	Eigenvalues	Boundary mode	Eigenvalues	Boundary mode	Eigenvalues	Eigenvalues
	12.360	See Table 8	485.481	See Table 8	N.A.	N.A.
Essential integral equation						
$N=1$	12.365	—		—	N.A.	N.A.
$N=2$	12.365	—	483.715	—	500.755	N.A.
$N=3$	12.365	—	485.505	—	500.565	3716.855
$N=4$	12.365	—	485.515	—	500.565	3802.135
$N=5$	12.365	$\begin{Bmatrix} u(1) \\ u'(1) \\ u''(0) \\ u'''(0) \end{Bmatrix} = \begin{Bmatrix} -1.61\times10^{-1} \\ -2.21\times10^{-1} \\ -5.65\times10^{-1} \\ 7.78\times10^{-1} \end{Bmatrix}$	485.515	$\begin{Bmatrix} u(1) \\ u'(1) \\ u''(0) \\ u'''(0) \end{Bmatrix} = \begin{Bmatrix} 9.11\times10^{-3} \\ 4.361\times10^{-2} \\ -2.05\times10^{-1} \\ 9.78\times10^{-1} \end{Bmatrix}$	500.565	3803.565
True eigen equation: $\cosh(\sqrt[4]{\lambda_n})\cos(\sqrt[4]{\lambda_n}) + 1 = 0$						
Spurious eigen equation: $\cosh(\sqrt[4]{\lambda_n})\cos(\sqrt[4]{\lambda_n}) - 1 = 0$						
Natural integral equation						
$N=1$	12.365	—		—	N.A.	N.A.
$N=2$	12.365	—	483.715	—	521.075	N.A.
$N=3$	12.365	—	485.505	—	500.755	2883.855
$N=4$	12.365	—	485.515	—	500.565	3716.825
$N=5$	12.365	$\begin{Bmatrix} u(1) \\ u'(1) \\ u''(0) \\ u'''(0) \end{Bmatrix} = \begin{Bmatrix} -1.61\times10^{-1} \\ -2.21\times10^{-1} \\ -5.65\times10^{-1} \\ 7.78\times10^{-1} \end{Bmatrix}$	485.515	$\begin{Bmatrix} u(1) \\ u'(1) \\ u''(0) \\ u'''(0) \end{Bmatrix} = \begin{Bmatrix} 8.97\times10^{-3} \\ 4.29\times10^{-2} \\ -2.05\times10^{-1} \\ 9.78\times10^{-1} \end{Bmatrix}$	500.565	3802.095
True eigen equation: $\cosh(\sqrt[4]{\lambda_n})\cos(\sqrt[4]{\lambda_n}) + 1 = 0$						
Spurious eigen equation: $\cosh(\sqrt[4]{\lambda_n})\cos(\sqrt[4]{\lambda_n}) - 1 = 0$						

Natural integral equation:

$$\begin{bmatrix} 0 & \frac{1}{2}c & 0 & \frac{1}{2}e \\ \frac{-1}{2}c & 0 & \frac{-1}{2}e & 0 \\ 0 & \frac{-1}{2}b & \frac{1}{2} & \frac{-1}{2}d \\ \frac{-1}{2}b & 0 & \frac{-1}{2}d & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} u'(0) \\ u'(1) \\ u'''(0) \\ u'''(1) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{56}$$

In the same way, we have

$$-\frac{(\lambda)^{1/2}}{2}\sin\sqrt[4]{\lambda}\sinh\sqrt[4]{\lambda}(1 - \cosh\sqrt[4]{\lambda}\cos\sqrt[4]{\lambda}) = 0 \tag{57}$$

The eigenequation for the simply supported beam should be $\sin\sqrt[4]{\lambda} = 0$. Comparing this with the one we derive from the dual MRM, the eigenequation derived from the dual MRM has supurious eigenvalues resulting from $(1 - \cosh\sqrt[4]{\lambda}\cos\sqrt[4]{\lambda}) = 0$ as found in either Eq. (55) or Eq. (57). It is interesting to find that both the essential and natural integral equations obtain the same spurious eigenvalues because they have the same spurious eigenequation. The reason why the spurious eigenvalues occur has been explained by Yeih et al. [15]. Further, only the analytical derivation for a simply supported beam is illustrated here. The

Table 5
Eigensolutions for the fixed-roller supported beam case

Analytical solution	First mode		Spurious mode	Second mode		Spurious mode
	Eigenvalues	Boundary mode	Eigenvalues	Eigenvalues	Boundary mode	Eigenvalues
	237.721	See Table 9	N.A.	2496.487	See Table 9	N.A.
Essential integral equation						
$N=1$	241.050	—	521.080	N.A.	—	N.A.
$N=2$	237.750	—	500.760	N.A.	—	N.A.
$N=3$	237.730	—	500.570	2473.925	—	3716.860
$N=4$	237.730	—	500.570	2496.240	—	3802.140
$N=5$	237.730	$\begin{Bmatrix} u'(1) \\ u''(0) \\ u'''(0) \\ u'''(1) \end{Bmatrix} = \begin{Bmatrix} 3.80\times10^{-2} \\ -2.05\times10^{-1} \\ 8.06\times10^{-1} \\ -5.54\times10^{-1} \end{Bmatrix}$	500.570	2496.500	$\begin{Bmatrix} u'(1) \\ u'(0) \\ u'''(0) \\ u'''(1) \end{Bmatrix} = \begin{Bmatrix} 1.15\times10^{-2} \\ 1.15\times10^{-1} \\ -8.10\times10^{-1} \\ 5.75\times10^{-1} \end{Bmatrix}$	3803.570
True eigen equation: $\cos(\sqrt[4]{\lambda_n})\sinh(\sqrt[4]{\lambda_n}) - \cosh(\sqrt[4]{\lambda_n})\sin(\sqrt[4]{\lambda_n}) = 0$						
Spurious eigen equation: $\cosh(\sqrt[4]{\lambda_n})\cos(\sqrt[4]{\lambda_n}) - 1 = 0$						
Natural integral equation						
$N=1$	219.450	—	N.A.	N.A.	—	N.A.
$N=2$	237.450	—	498.580	N.A.	—	N.A.
$N=3$	237.730	—	500.560	N.A.	—	3792.960
$N=4$	237.730	—	500.565	2499.24	—	3816.940
$N=5$	237.730	$\begin{Bmatrix} u'(1) \\ u''(0) \\ u'''(0) \\ u'''(1) \end{Bmatrix} = \begin{Bmatrix} 3.80\times10^{-2} \\ -2.05\times10^{-1} \\ 8.06\times10^{-1} \\ -5.54\times10^{-1} \end{Bmatrix}$	500.570	2496.51	$\begin{Bmatrix} u'(1) \\ u'(0) \\ u'''(0) \\ u'''(1) \end{Bmatrix} = \begin{Bmatrix} 1.15\times10^{-2} \\ 1.15\times10^{-1} \\ -8.10\times10^{-1} \\ 5.75\times10^{-1} \end{Bmatrix}$	3803.680
True eigen equation: $\cos(\sqrt[4]{\lambda_n})\sinh(\sqrt[4]{\lambda_n}) - \cosh(\sqrt[4]{\lambda_n})\sin(\sqrt[4]{\lambda_n}) = 0$						
Spurious eigen equation: $\cosh(\sqrt[4]{\lambda_n})\cos(\sqrt[4]{\lambda_n}) - 1 = 0$						

corresponding eigenequations and spurious eigenequations are shown in Tables 3–6. It should be noted here that no spurious eigenvalue problem occurs in the fixed-fixed supported beam case.

After obtaining the boundary eigenvector, the representation for the displacement can be expressed as follows:

$$u(s) = [\,H_1(s) \quad H_2(s) \quad H_3(s) \quad H_4(s) \quad H_5(s) \quad H_6(s) \quad H_7(s) \quad H_8(s)\,]\begin{Bmatrix} u(0) \\ u(1) \\ u'(0) \\ u'(1) \\ u''(0) \\ u''(1) \\ u'''(0) \\ u'''(1) \end{Bmatrix} \tag{58}$$

where

$$H_1(s) \equiv \frac{1}{2}\left(\frac{\cosh\sqrt[4]{\lambda}(-s) + \cos\sqrt[4]{\lambda}(-s)}{2} - 1\right) \tag{59}$$

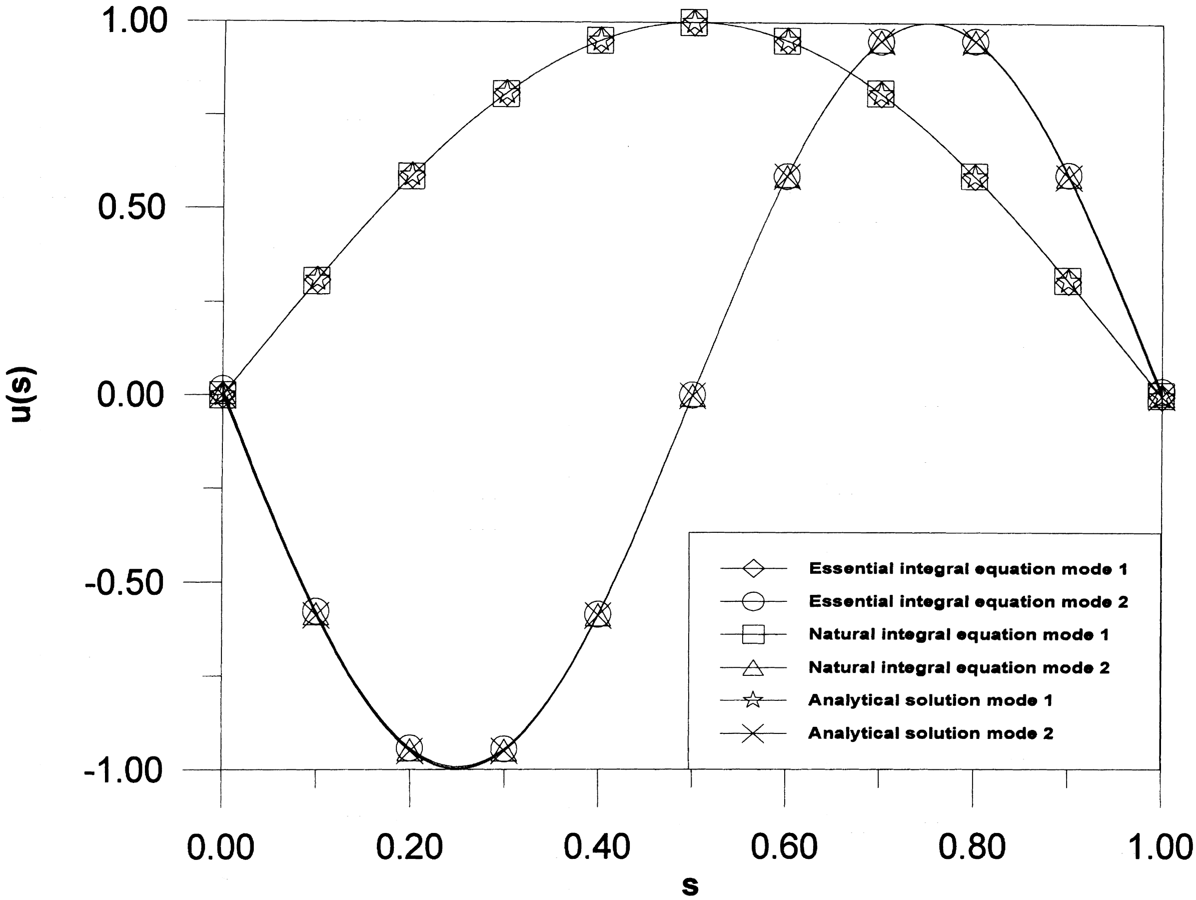

Fig. 3. The first two eigenmodes determined using the essential and natural integral equations for the simply supported beam case.

$$H_2(s) \equiv \frac{1}{2}\left(\frac{\cosh\sqrt[4]{\lambda}(1-s) + \cos\sqrt[4]{\lambda}(1-s)}{2} - 1\right) \tag{60}$$

$$H_3(s) \equiv \frac{-1}{2}\left(\frac{\sinh\sqrt[4]{\lambda}(-s) + \sin\sqrt[4]{\lambda}(-s)}{2(\lambda^{1/4})} - 1\right) \tag{61}$$

$$H_4(s) \equiv \frac{-1}{2}\left(\frac{\sinh\sqrt[4]{\lambda}(1-s) + \sin\sqrt[4]{\lambda}(1-s)}{2(\lambda^{1/4})} - 1\right) \tag{62}$$

$$H_5(s) \equiv \frac{1}{4}\left(\frac{\cosh\sqrt[4]{\lambda}(-s) - \cos\sqrt[4]{\lambda}(-s)}{\lambda^{1/2}} - 1\right) \tag{63}$$

$$H_6(s) \equiv \frac{1}{4}\left(\frac{\cosh\sqrt[4]{\lambda}(1-s) - \cos\sqrt[4]{\lambda}(1-s)}{\lambda^{1/2}} - 1\right) \tag{64}$$

$$H_7(s) \equiv \frac{-1}{12}\left(\frac{3(\sinh\sqrt[4]{\lambda}(-s) + \sin\sqrt[4]{\lambda}(-s))}{\lambda^{3/4}} - 1\right) \tag{65}$$

$$H_8(s) \equiv \frac{-1}{12}\left(\frac{3(\sinh\sqrt[4]{\lambda}(1-s) + \sin\sqrt[4]{\lambda}(1-s))}{\lambda^{3/4}} - 1\right) \tag{66}$$

It is interesting to find that only five independent functions are provided according to Eqs. (59)–(66).

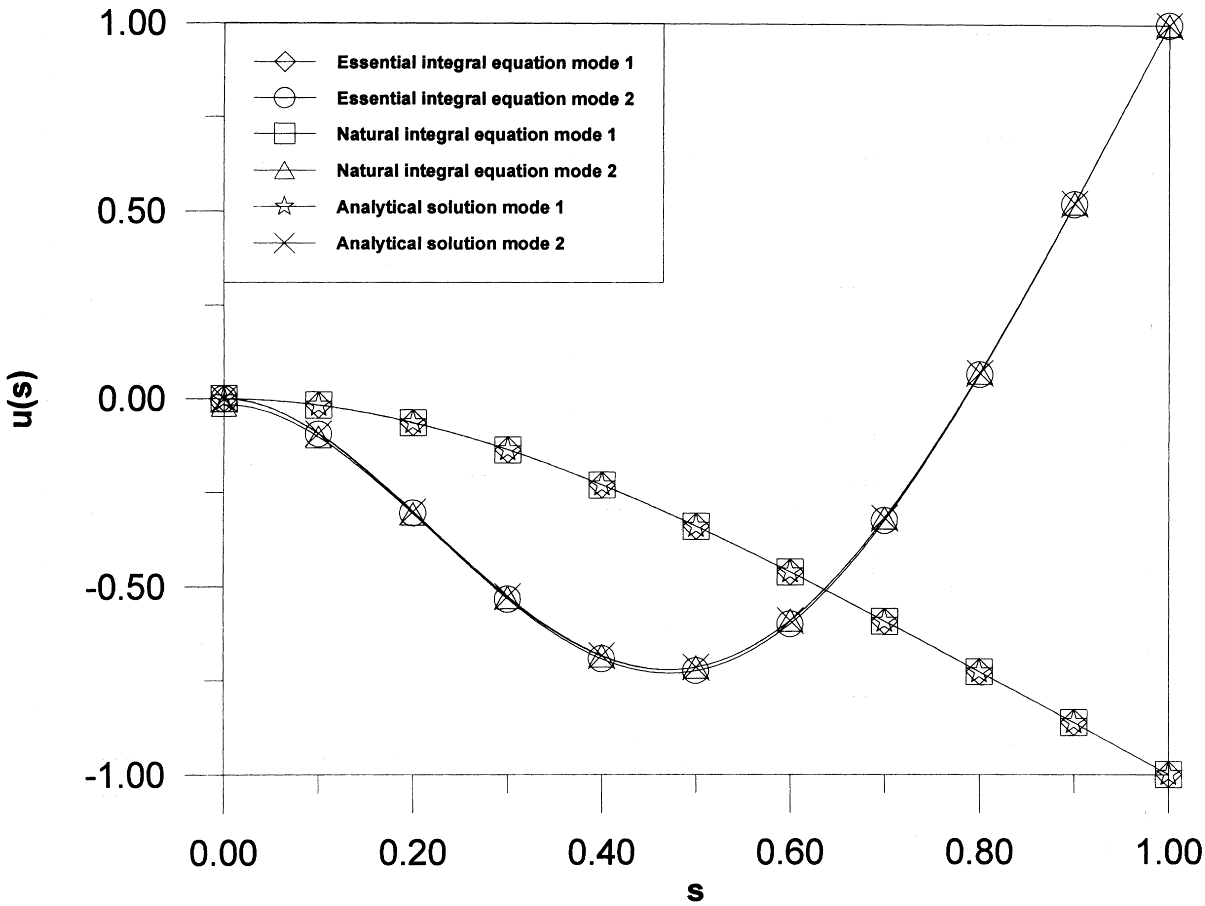

Fig. 4. The first two eigenmodes determined using the essential and natural integral equations for the cantilever beam case.

5. Determination of the eigenproblem using the SVD method

After substituting the homogeneous boundary conditions for both the essential and natural integral equations, the eigenproblem in general can be expressed as

$$[\mathbf{A}(\lambda)]_{4\times4}\mathbf{x}_{4\times1} = 0 \tag{67}$$

where $\mathbf{A}(\lambda)$ is the leading coefficient matrix function of λ, and $\mathbf{x}$ is the boundary eigenvector. When λ is equal to the eigenvalue, the determinant of $\mathbf{A}$ is zero, which means that the rank of $\mathbf{A}$ must be at most equal to 3 in order to have a nontrivial solution. We should remember that we have in total eight equations. Therefore, the boundary eigenvectors should satisfy all eight equations although we obtain the eigenvalues from four selected equations. It has been mentioned above that the spurious boundary eiegnvectors corresponding to the spurious eigenvalues cannot satisfy the remaining eigenequations. This means that the only eigensolution corresponding to the spurious eigenvalues is a null vector since the matrix of $\mathbf{A}_{8\times4}$ has a rank of 4. In another words, a selection of four equations which has a rank of 4 is possible. However, when we select a special group of equations, the rank of the system of equations may be reduced by one order, which may mislead us and cause us to believe it is a true eigenvalue. It is quite interesting to ask ourselves a question: can one determine the true eigenvalues using all eight equations without information from the eigenmodes? This question will be answered in this section.

To determine the boundary eigenvector $\mathbf{x}$, a standard procedure is to let one element in the vector of $\mathbf{x}$ be equal to one and reduce the equation to

$$[\overline{\mathbf{A}}]_{3\times3}\overline{\mathbf{x}}_{3\times1} = \mathbf{b}_{3\times1} \tag{68}$$

Then, the remaining components in the boundary eigenvector can be determined. However, the above-mentioned algorithm is true only when the rank of the leading coefficient matrix, $\mathbf{A}$, is equal to 3. When the rank of the leading coefficient matrix is lower than 3, the algorithm fails.

In the fixed-fixed supported beam case, it is found that the rank of the leading coefficient matrix in either the essential or natural equation is equal to 2. This means that the system of equations appearing in either the essential or the natural equation is

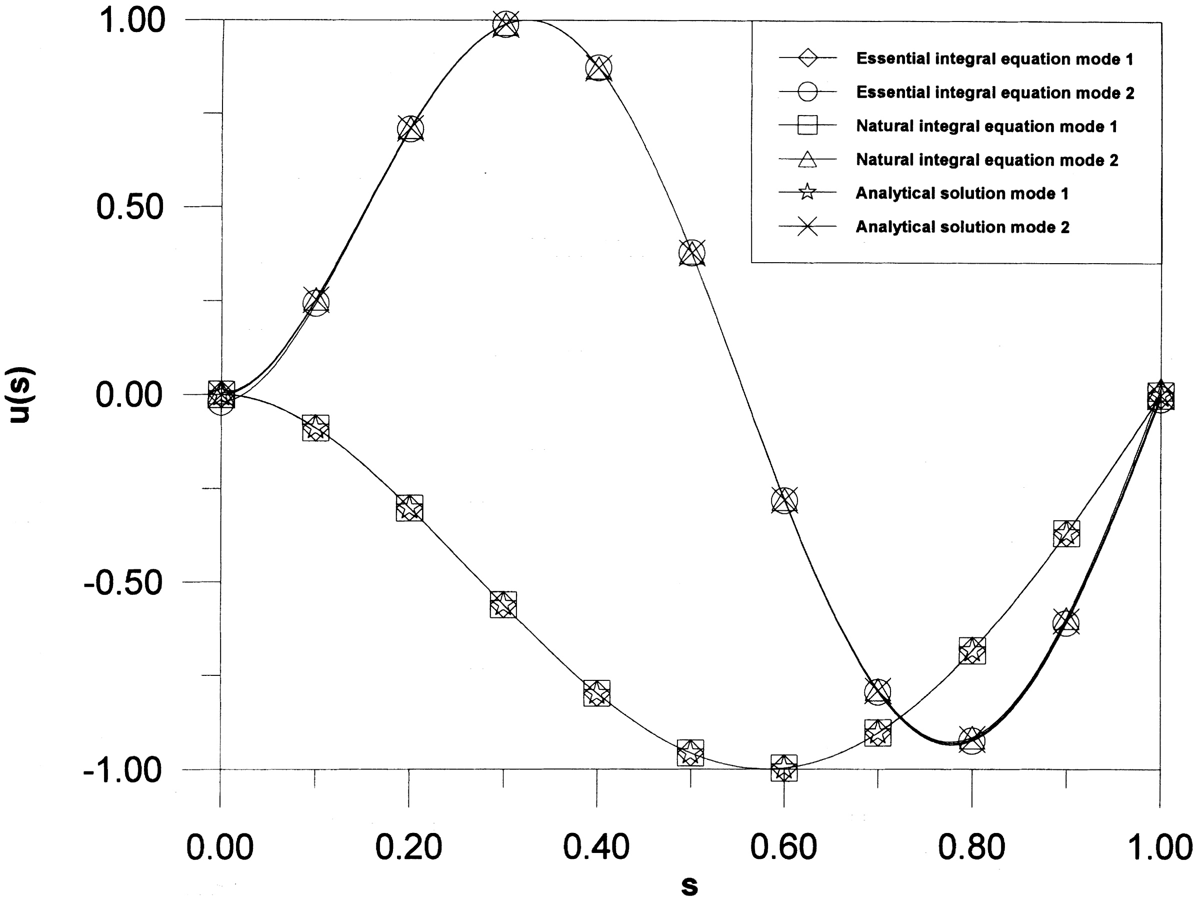

Fig. 5. The first two eigenmodes determined using the essential and natural integral equations for the fixed-roller supported beam case.

highly dependent, so that the boundary eigenvector can be chosen arbitrary with a nullity of 2. As mentioned above, the way in which we divide the equations into two groups, essential and natural equations, is merely a natural division in the derivation. In general, the boundary eigenvector should satisfy all eight equations. Therefore, we have more equations than unknowns in this framework.

As mentioned previously, the conventional method of finding the eigenvalues and corresponding boundary eigenvectors may encounter two difficulties: the spurious eigenvalue problem and the undeterminancy of boundary eigenvectors. Here, we propose the singular value decomposition method (SVD) to solve these two difficulties at the same time. A brief introduction to SVD is given next.

Consider a linear algebra probelm with more equations than unknowns:

$$[\mathbf{A}]_{m\times n}x_{n\times 1} = \mathbf{b}_{m\times 1}, \quad m > n \tag{69}$$

where m is the number of equations, n is the number of unknowns and $\mathbf{A}$ is the leading matrix, which can be decomposed into

$$[\mathbf{A}]_{m\times n} = \mathbf{U}_{m\times m}\mathbf{\Sigma}_{m\times n}\mathbf{V}^{*}_{n\times n} \tag{70}$$

Here, $\mathbf{U}$ is a left unitary matrix constructed by the left singular vectors, $\mathbf{\Sigma}$ is a diagonal matrix which has singular values $\sigma_1,\sigma_2,\ldots,\sigma_n$ allocated in the diagonal line as

$$\Sigma_1 = \begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_n \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}, \quad m > n \tag{71}$$

Table 6
Rank and eigensolutions for different selections of equations for the fixed-fixed supported beam case

Equations	Eigenvalues	Eigenmode	Rank	Equations	Eigenvalues	Eigenmode	Rank
1,2,3,4	497.895	$\begin{Bmatrix} -2.107\times10^{-1} \\ 0 \\ -9.776\times10^{-1} \\ 0 \end{Bmatrix}$	2	5,6,7,8	500.1	$\begin{Bmatrix} 2.036\times10^{-1} \\ 1.378\times10^{-2} \\ -9.794\times10^{-1} \\ 3.171\times10^{-2} \end{Bmatrix}$	2
1,2,5,6	500.565	$\begin{Bmatrix} 1.490\times10^{-1} \\ 1.490\times10^{-1} \\ -6.912\times10^{-1} \\ 6.912\times10^{-1} \end{Bmatrix}$	3	3,4,7,8	500.565	$\begin{Bmatrix} 1.488\times10^{-1} \\ 1.488\times10^{-1} \\ -6.913\times10^{-1} \\ 6.913\times10^{-1} \end{Bmatrix}$	3
1,2,7,8	500.015	$\begin{Bmatrix} 1.488\times10^{-1} \\ 1.488\times10^{-1} \\ -6.913\times10^{-1} \\ 6.913\times10^{-1} \end{Bmatrix}$	3	3,4,5,6	500.1	$\begin{Bmatrix} 1.488\times10^{-1} \\ 1.488\times10^{-1} \\ -6.913\times10^{-1} \\ 6.913\times10^{-1} \end{Bmatrix}$	3
1,4,5,6	500.565	$\begin{Bmatrix} 1.488\times10^{-1} \\ 1.487\times10^{-1} \\ -6.914\times10^{-1} \\ 6.911\times10^{-1} \end{Bmatrix}$	3	2,3,7,8	500.565	$\begin{Bmatrix} 1.488\times10^{-1} \\ 1.487\times10^{-1} \\ -6.913\times10^{-1} \\ 6.913\times10^{-1} \end{Bmatrix}$	3
1,4,5,8	500.575	$\begin{Bmatrix} 1.487\times10^{-1} \\ 1.488\times10^{-1} \\ -6.908\times10^{-1} \\ 6.918\times10^{-1} \end{Bmatrix}$	3	2,3,6,7	500.575	$\begin{Bmatrix} 1.488\times10^{-1} \\ 1.487\times10^{-1} \\ -6.918\times10^{-1} \\ 6.918\times10^{-1} \end{Bmatrix}$	3

Eqs. 1, 2, 3, 4 true eigen equation $\{\cosh(\sqrt[4]{\lambda_n})\cos(\sqrt[4]{\lambda_n}) - 1\}^2 = 0$; Eqs. 5, 6, 7, 8 true eigen equation $\{\cosh\sqrt[4]{\lambda_n}\cos\sqrt[4]{\lambda_n} - 1\}^2 = 0$.

in which $\sigma_1 \geq \sigma_2 \ldots \geq \sigma_n$ and $\mathbf{V}^*$ is the complex conjugate transpose of a right unitary matrix constructed by the right singular vectors. As we can see in Eq. (71), there are at most n nonzero singular values. This means that we can find at most n linear independent equations in the system of equations. If we have p zero singular values ($0 \leq p \leq n$), this means that the rank of the system of equations is equal to $n - p$. However, the singular value may be very close to zero numerically, resulting in rank deficiency. For a general eigenproblem as shown in this paper, the eigenvalues will cause the rank to be $n - 1$ (i.e. 3).

Determining the eigenvalues of the system of equations has now been transformed into finding the values of λ which make the rank of the leading coefficient matrix 3. This means that when $m = 8$, $n = 4$ and $\mathbf{b}_{4\times1} = \mathbf{0}$, the eigenvalues will make $p = 1$, such that the minimum singular value must be zero or very close to zero.

Let us take the simply supported beam as an example; the result of the direct search method based on the SVD method is plotted in Fig. 7. It is seen that the true eigenvalues cause the minimum singular value to be much lower than the other λ values. The more the eigenvalue approaches the true value, the more the minimum singular value approaches to be zero, in which case the rank of the matrix will be reduced to 3. However, this is not true for the spurious case.

To find the boundary eigenvector associated with the eigevalue, we can set one of the elements in the boundary eigenvector to be one and then reduce the equations into the form of Eq. (69), where $\mathbf{b}$ is now a nontrivial vector, $m = 8$ and $n = 3$.

Then, the pseudo-inverse matrix, $\mathbf{A}^+$ of $\mathbf{A}$, is expressed as

$$\mathbf{A}^+_{n\times m} = \mathbf{V}_{n\times n}\mathbf{\Sigma}^+_{n\times n}\mathbf{U}^*_{m\times m} \tag{72}$$

where $\mathbf{\Sigma}^+$ is constructed by taking the transpose of $\mathbf{\Sigma}$ and then replacing the diagonal singular value terms with its inverse,

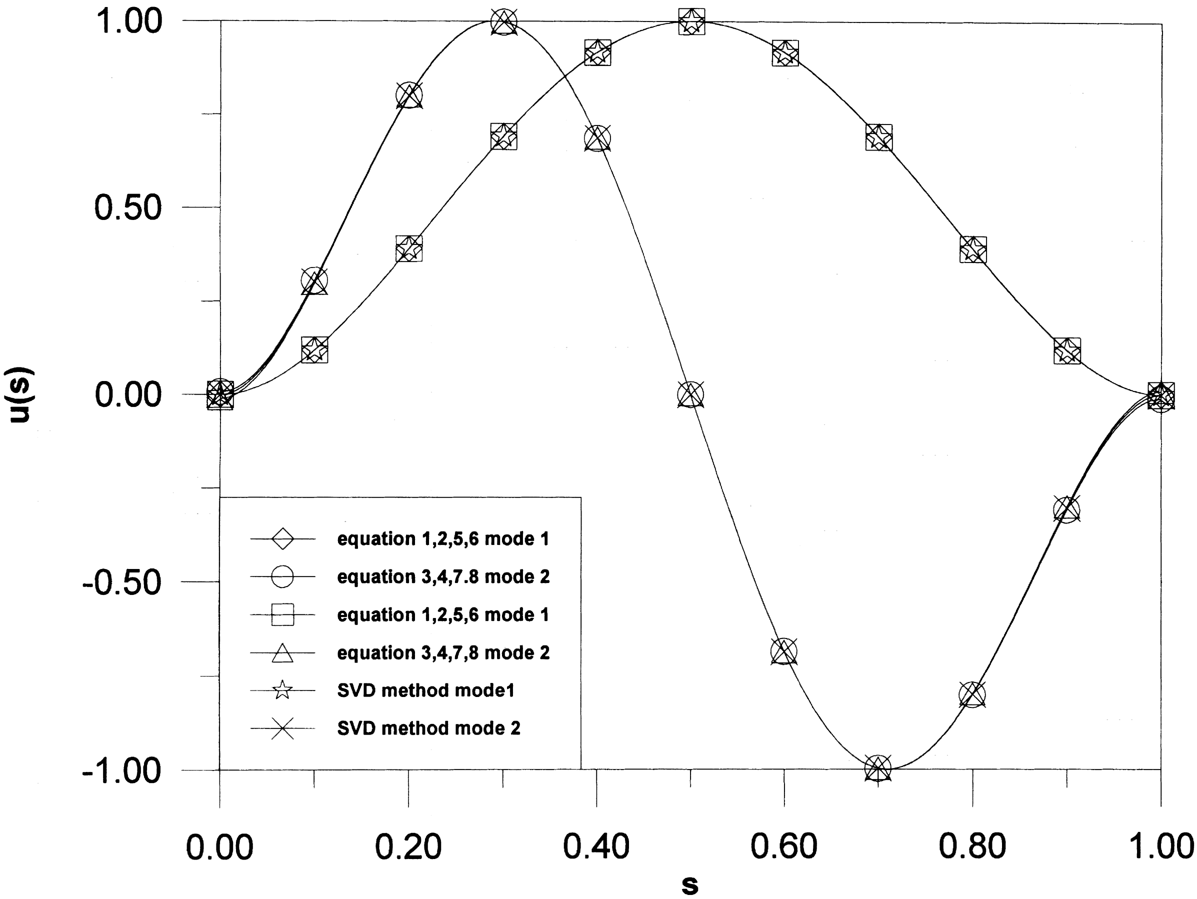

Fig. 6. The first two eigenmodes determined using appropriate selection of equations and the SVD method for the fixed-fixed supported beam case.

Table 7
Analytical solutions for eigenvalues, boundary modes and eigenmodes for the simply supported beam case (*the given B.C.s.)

Simply supported beam		
Boundary conditions	$u(0)=0,\ u(1)=0,\ u''(0)=0,\ u''(1)=0$	
λ_n	$\lambda_1 = 97.409$	$\lambda_2 = 1558.545$
Boundary mode		
$u(0)$	0^*	0^*
$u(1)$	0^*	0^*
$u'(0)$	π	2π
$u'(1)$	$-\pi$	2π
$u''(0)$	0^*	0^*
$u''(1)$	0^*	0^*
$u'''(0)$	$-\pi^3$	$-(2\pi)^3$
$u'''(1)$	π^3	$-(2\pi)^3$
Mode shape	$\sin(\sqrt[4]{\lambda_n}x)$	

Table 8
Analytical solutions for eigenvalues, boundary modes and eigenmodes for the cantilever beam case (* the given B.C.s)

Cantilever		
Boundary conditions	$u(0)=0,\ u'(0)=0,\ u''(1)=0,\ u'''(1)=0$	
λ_n	$\lambda_1 = 12.360$	$\lambda_2 = 485.481$
Boundary mode		
$u(0)$	0^*	0^*
$u(1)$	2	−2
$u'(0)$	0^*	0^*
$u'(1)$	2.754	−9.574
$u''(0)$	7.031	44.067
$u''(1)$	0^*	0^*
$u'''(0)$	−14.697	−210.679
$u'''(1)$	0^*	0^*
Mode shape	$\cosh(\sqrt[4]{\lambda_n}x) - \cos(\sqrt[4]{\lambda_n}x) - \sigma_i[\sinh(\sqrt[4]{\lambda_n}x) - \sin(\sqrt[4]{\lambda_n}x)]$, $\sigma_1 = 0.732$, $\sigma_2 = 1.018$	

Table 9
Analytical solutions for eigenvalues, boundary modes and eigenmodes for the fixed-roller supported beam case

Fixed-roller supported beam		
Boundary conditions	$u(0)=0$, iu(1) = 0, $u'(0)=0,\ u''(1)=0$	
λ_n	$\lambda_1 = 237.721$	$\lambda_2 = 2496.487$
Boundary mode		
$u(0)$	0^*	0^*
$u(1)$	0^*	0^*
$u'(0)$	0^*	0^*
$u'(1)$	−5.713	9.989
$u''(0)$	30.836	99.93
$u''(1)$	0^*	0^*
$u'''(0)$	−121.179	−706.367
$u'''(1)$	83.21	−499.825
Mode shape	$\cosh(\sqrt[4]{\lambda_n}x) - \cos(\sqrt[4]{\lambda_n}x) - \sigma_i[\sinh(\sqrt[4]{\lambda_n}x) - \sin(\sqrt[4]{\lambda_n}x)]$, $\sigma_1 = 1.001$, $\sigma_2 = 1.000$	

Table 10
Analytical solutions for eigenvalues, boundary modes and eigenmodes for the fixed-fixed beam case (* the given B.C.s)

Fixed-fixed supported beam		
Boundary conditions	$u(0)=0,\ u(1)=0,\ u'(0)=0,\ u'(1)=0$	
λ_n	$\lambda_1 = 500.564$	$\lambda_2 = 3803.537$
Boundary mode		
$u(0)$	0^*	0^*
$u(1)$	0^*	0^*
$u'(0)$	0^*	0^*
$u'(1)$	0^*	0^*
$u''(0)$	44.75	123.34
$u''(1)$	44.75	123.34
$u'''(0)$	−207.94	−969.55
$u'''(1)$	−207.94	−969.55
Mode shape	$\cosh(\sqrt[4]{\lambda_n}x) - \cos(\sqrt[4]{\lambda_n}x) - \sigma_i[\sinh(\sqrt[4]{\lambda_n}x) - \sin(\sqrt[4]{\lambda_n}x)]$, $\sigma_1 = 0.983$, $\sigma_2 = 1.001$	

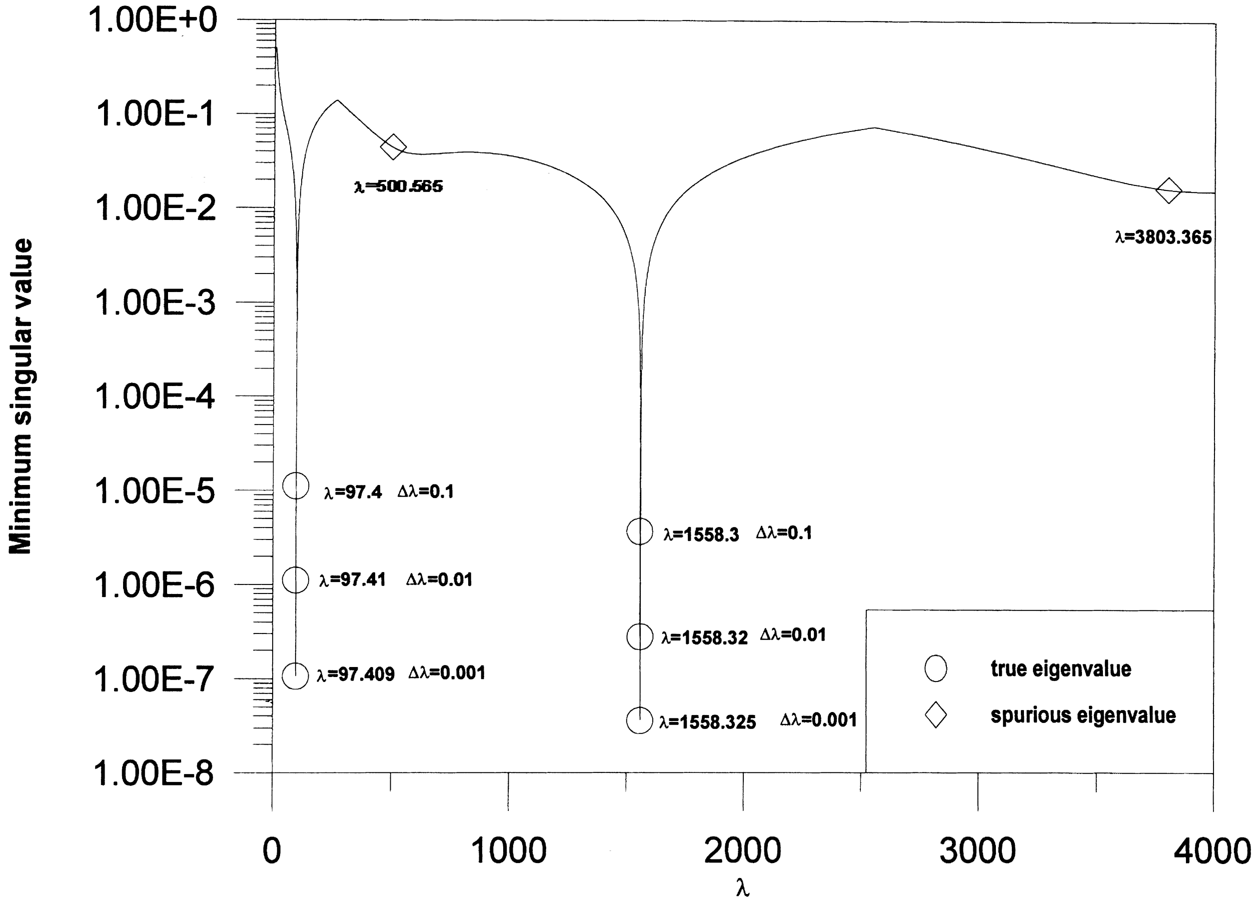

Fig. 7. Direct search of eigenvalues using SVD for the simply supported beam case.

expressed as

$$\boldsymbol{\Sigma}^{+} = \begin{bmatrix} \frac{1}{\sigma_1} & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{\sigma_n} & \cdots & 0 \end{bmatrix} \quad m > n \tag{73}$$

The above-mentioned SVD method has been proved to be equivalent to the least square error solution in determining the unknown vector when the number of equations is larger than the number of unknowns [16]. After introducing the SVD method, we do not need to worry about how to pick a specific group of equations such that the rank of the leading coefficient is sufficient to solve for the eigenvector. On the other hand, we can take all eight equations into account, which apparently causes the rank of the leading coefficient matrix to be equal to three. Thus, the eigenvector can be easily found in the sense of the least square error. The eigenmodes determined using the SVD method are the same as those obtained using the above-mentioned method since the boundary eigenvectors are almost the same; therefore, only the first two mode shapes for the fixed-fixed supported beam determined using the SVD method are illustrated in Fig. 6. It can be confirmed that the SVD method can solve the spurious eigenvalue problem and eliminate possible indeterminancy of the boundary eigenvector at the same time. For further details concerning the SVD method, please refer to Ref. [17].

6. Concluding remarks

In this paper, we have constructed the dual equations for MRM to find the natural frequencies and modes of a beam numerically and analytically. The role of the natural integral equation in the dual MRM has been examined and used to filter out spurious eigendata. Also, the spurious eigenequation has been analytically derived and found to be the same for both the essential and natural integral equation methods. The SVD technique can be employed to distinguish whether or not the eigenvalue is true. Further, the SVD method has been proposed to determine the true eigenvalues and the boundary eigenvectors, which requires no special selection of equations. Four examples with different boundary conditions have been used to show the validity of the present formulation. Although only a one-dimensional structure is studied in this paper, extension of the proposed method to higher dimensional structures has being investigated.

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