ELSEVIER

PII: S0955-7997(97)00032-5

Engineering Analysis with Boundary Elements **20** (1997) 25–33

0955-7997/97/$17.00

Analytical derivations for one-dimensional eigenproblems using dual boundary element method and multiple reciprocity method

J.T. Chen* & F.C. Wong

Department of Harbor and River Engineering, Taiwan Ocean University, Keelung, Taiwan

In this paper, MRM (multiple reciprocity method) and DBEM (dual boundary element method) are combined to solve one-dimensional eigenproblems. It is found that the hypersingular equation (LM method) for MRM provides an additional constraint to determine the eigenfunctions in case of failure using the singular equation (UT) method only. Also, augmented eigenvalues can be deleted by employing dual equations. The hypersingular equation of DBEM (LM equation) plays an important role in determining the eigenvalues and eigenmodes. Four numerical examples are given to verify the validity of the present formulation. © 1997 Elsevier Science Limited.

Key words: MRM, DBEM, eigenproblems.

1 INTRODUCTION

The dual boundary element method (BEM) has been applied in boundary value problems with degenerate boundary[1–5], corner problem[6], exterior problem[7] and error estimation for adaptive mesh generation[8]. By combining the conventional BEM and hypersingular integral equation, many problems can be solved more directly and efficiently. The roles of hypersingularity in BEM is shown in Fig. 1.

For a Helmholtz equation, the complex fundamental solution has been employed to solve the eigenproblems[9]. To avoid the computation in the domain of complex number, the multiple reciprocity method (MRM) has been employed to solve the Helmholtz problem[10–12]. In this algorithm, the Helmholtz equation is treated as a Poisson equation with an external source. Therefore, the fundamental solution of the Laplace equation is considered. However, the domain integral is present owing to the integration of the external source. MRM can transform this domain integral into boundary integrals iteratively such that the domain cell is not needed. In the literature, the conventional singular integral equation (UT equation) has been used only in MRM[10]. The role of the hypersingular integral equation (LM equation) in MRM is not clear to the author's knowledge.

In this paper, the role of the hypersingular integral equation for one-dimensional eigenproblems using MRM is demonstrated. The augmented eigenvalues and failure in determining the eigenmodes using the conventional MRM will be discussed analytically, and such difficulties will be solved by using the hypersingular integral equation. Finally, four example problems will be solved analytically using DBEM and MRM.

*Corresponding author.

2 PROBLEM STATEMENT AND ANALYTICAL DERIVATIONS

Consider a one-dimensional eigenvalue with the following governing equation

$$\frac{d^2u(x)}{dx^2} + \lambda u(x) = 0 \qquad (1)$$

where λ and $u(x)$ denote the eigenvalue and eigenmode respectively.

Four cases of boundary conditions are considered as follows:

Case 1, $u(0) = 0$ and $u(1) = 0$ (Dirichlet B.C.)
Case 2, $t(0) = 0$ and $t(1) = 0$ (Neumann B.C.)
Case 3, $u(0) = 0$ and $t(1) = 0$ (Robin B.C.)
Case 4, $u(1) = 0$ and $t(0) = 0$ (Robin B.C.).

where

$$t(x_0) = \frac{du(x)}{dx}\Big|_{x=x_0}$$

Consider an auxilliary system with a fundamental solution satisfying

$$\frac{d^2U(x,s)}{dx^2} = \delta(x-s) \tag{2}$$

where $U(x,s)$ is a fundamental solution as

$$U(x,s) = \frac{1}{2}|x-s| = U^{(0)}(x,s) \tag{3}$$

By employing Green's third identity, we have

$$\int_\Omega (u(x)\nabla^2 U(x,s) - U(x,s)\nabla^2 U(x))\, d\Omega = \int_\Gamma (u(x)\frac{\partial U(x,s)}{\partial n} - U(x,s)\frac{\partial U}{\partial n})\, d\Gamma \tag{4}$$

where Ω and Γ denote the domain and boundary respectively, and n is the normal vector on the boundary.

For simplicity, the one-dimensional case of unitary length is considered. Eqn (4) is reduced to

$$\int_0^1 \nabla^2 U^{(0)}(x,s)u(x)\, dx = \int_0^1 U^{(0)}(x,s)\nabla^2 u(x)\, dx + [u(x)\frac{dU^{(0)}(x,s)}{dx} - U^{(0)}(x,s)\frac{du(x)}{dx}]|_{x=0}^{x=1}$$

By transforming the domain intergal term on the right-hand side of the equal sign in eqn (5), we have

$$\begin{aligned} D^{(0)} &= \int_0^1 U^{(0)}(x,s)\nabla^2 u(x)\, dx \\ &= \int_0^1 \nabla^2 U^{(1)}(x,s) b^{(0)}\, dx \\ &= \int_0^1 U^{(1)}(x,s)\nabla^2 b^{(0)}\, dx + [b^{(0)}\frac{dU^{(1)}(x,s)}{dx} - U^{(1)}(x,s)\frac{db^{(0)}}{dx}]|_{x=0}^{x=1} \end{aligned} \tag{6}$$

where

$$\nabla^2 U^{(1)}(x,s) = U^{(0)}(x,s)$$

$$b^0 = \nabla^2 u(x) = -\lambda u(x)$$

By transforming the domain integral term on the right-hand side of the equal sign in eqn (6), we have

$$\begin{aligned} D^{(1)} &= \int_0^1 U^{(1)}(x,s)\nabla^2 b^{(0)}\, dx \\ &= \int_0^1 \nabla^2 U^{(2)}(x,s) b^{(1)}\, dx \\ &= \int_0^1 U^{(2)}(x,s)\nabla^2 b^{(1)}\, dx + [b^{(1)}\frac{dU^{(2)}(x,s)}{dx} - U^{(2)}(x,s)\frac{db^{(1)}}{dx}]|_{x=0}^{x=1} \end{aligned} \tag{7}$$

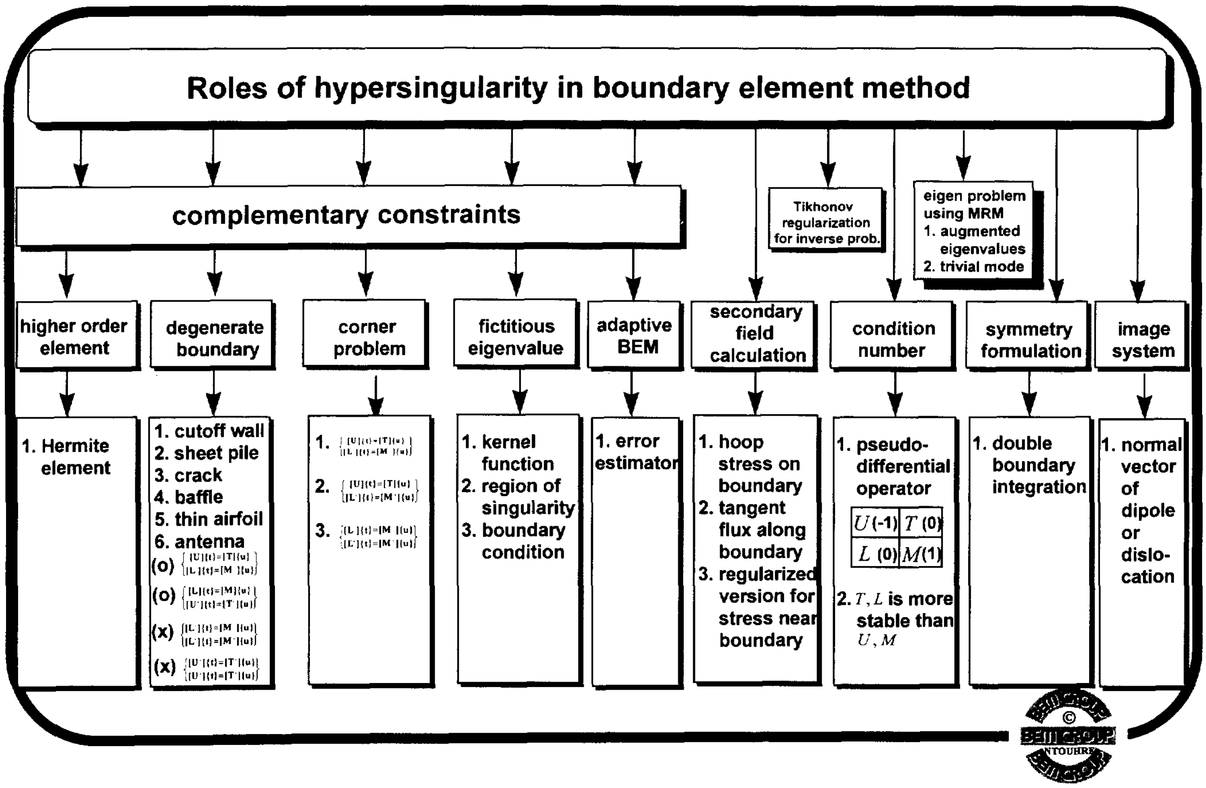

Fig. 1. The role of hypersingularity in the boundary element method.

where

$$\nabla^2 U^{(2)}(x,s) = U^{(1)}(x,s)$$

$$b^{(1)} = \nabla^2 b^{(0)} = -\lambda(\nabla^2 u(x)) = (-\lambda)^2 u(x)$$

Repeating the above process, many boundary terms will appear except one remainder of the domain integral as follows:

$$D^{(0)} = \sum_{j=0}^{N} [b^{(j)} \frac{\mathrm{d}U^{(j+1)}(x,s)}{\mathrm{d}x} - U^{(j+1)}(x,s)\frac{\mathrm{d}b^{(j)}}{\mathrm{d}x}]|_{x=0}^{x=1} + R_{N+1} \quad (8)$$

where R_{N+1} is a remainder term, and the body source term and remainder term are found to be

$$b^{(j)}(x) = (-\lambda)^{(j+1)} u(x)$$

$$\frac{\mathrm{d}b^{(j)}(x)}{\mathrm{d}x} = (-\lambda)^{(j+1)} u'(x)$$

$$R_{N+1} = \int_0^1 U^{(N+1)}(x,s)\nabla^2 b^{(N)}\,\mathrm{d}x$$

The primary and secodary field for $u(s)$ and $t(s)$ can be expressed as

$$u(s) = \{u(x)T^{(0)}(x,s) - U^{(0)}(x,s)t(x) + \sum_{j=0}^{N} [b^{(j)}T^{(j+1)}(x,s) - U^{(j+1)}(x,s)\frac{\mathrm{d}b^{(j)}}{\mathrm{d}x}]\}|_{x=0}^{x=1} + R_{N+1} \quad (9)$$

$$t(s) = \{u(x)M^{(0)}(x,s) - L^{(0)}(x,s)t(x) + \sum_{j=0}^{N} [b^{(j)}M^{(j+1)}(x,s) - L^{(j+1)}(x,s)\frac{\mathrm{d}b^{(j)}}{\mathrm{d}x}]\}|_{x=0}^{x=1} + R'_{N+1} \quad (10)$$

where R'_{N+1} is the derivative of R_{N+1} with respect to s, and the explicit forms for the kernel functions are shown in Table 1 and are defined as

$$T^{(j+1)}(x,s) = \frac{\partial\{U^{(j+1)}(x,s)\}}{\partial x}$$

$$L^{(j+1)}(x,s) = \frac{\partial\{U^{(j+1)}(x,s)\}}{\partial s}$$

$$M^{(j+1)}(x,s) = \frac{\partial\{U^{(j+1)}(x,s)\}}{\delta x \partial s}$$

Eqns (9) and eqn (10) construct the dual equations for the MRM.

By moving the field point close to the boundary, the dual BEM is derived as follows:

$$T_0 \underset{\sim}{u} - U_0 \underset{\sim}{t} = \sum_{i=1}^{N} (T_i(\lambda)\underset{\sim}{u} - U_i(\lambda)\underset{\sim}{t}) \quad (11)$$

and

$$M_0 \underset{\sim}{u} - L_0 \underset{\sim}{t} = \sum_{i=1}^{N} (M_i(\lambda)\underset{\sim}{u} - L_i(\lambda)\underset{\sim}{t}) \quad (12)$$

The explicit forms of eqn (11) and eqn (12) can be found to be

$$\begin{bmatrix} 1+T^{(0)}(0,0^+) & -T^{(0)}(1,0^+) \\ T^{(0)}(0,1^-) & 1-T^{(0)}(1,1^-) \end{bmatrix} \begin{Bmatrix} u(0) \\ u(1) \end{Bmatrix} - \begin{bmatrix} U^{(0)}(0,0^+) & -U^{(0)}(1,0^+) \\ U^{(0)}(0,1^-) & -U^{(0)}(1,1^-) \end{bmatrix} \begin{Bmatrix} t(0) \\ t(1) \end{Bmatrix}$$
$$= \sum_{j=0}^{N} \begin{bmatrix} -(-\lambda)^{j+1}T^{(j+1)}(0,0^+) & (-\lambda)^{j+1}T^{(j+1)}(1,0^+) \\ -(-\lambda)^{j+1}T^{(j+1)}(0,1^-) & (-\lambda)^{j+1}T^{(j+1)}(1,1^-) \end{bmatrix} \begin{Bmatrix} u(0) \\ u(1) \end{Bmatrix}$$
$$-\sum_{j=0}^{N} \begin{bmatrix} -(-\lambda)^{j+1}U^{(j+1)}(0,0^+) & (-\lambda)^{j+1}U^{(j+1)}(1,0^+) \\ -(-\lambda)^{j+1}U^{(j+1)}(0,1^-) & (-\lambda)^{j+1}U^{(j+1)}(1,1^-) \end{bmatrix} \begin{Bmatrix} t(0) \\ t(1) \end{Bmatrix} \quad (13)$$

Table 1. Explicit forms for the kernel functions

$U^{(j+1)}(x,s)$		$T^{(j+1)}(x,s)$		$L^{(j+1)}(x,s)$		$M^{(j+1)}(x,s)$	
$x > s$	$x < s$	$x > s$	$x < s$	$x > s$	$x < s$	$x > s$	$x < s$
$\frac{1}{2}\frac{\lvert r\rvert^{2j+3}}{(2j+3)!}$		$\frac{1}{2}\frac{\lvert r\rvert^{2j+2}}{(2j+2)!}$	$-\frac{1}{2}\frac{\lvert r\rvert^{2j+2}}{(2j+2)!}$	$-\frac{1}{2}\frac{\lvert r\rvert^{2j+2}}{(2j+2)!}$	$\frac{1}{2}\frac{\lvert r\rvert^{2j+2}}{(2j+2)!}$	$-\frac{1}{2}\frac{\lvert r\rvert^{2j+1}}{(2j+1)!}$ $j \geq 0$; 0 $j = -1$	

and

$$\begin{bmatrix} M^{(0)}(0,0^+) & -M^{(0)}(1,0^+) \\ M^{(0)}(0,1^-) & -M^{(0)}(1,1^-) \end{bmatrix} \begin{Bmatrix} u(0) \\ u(1) \end{Bmatrix} - \begin{bmatrix} -1+L^{(0)}(0,0^+) & -L^{(0)}(1,0^+) \\ L^{(0)}(0,1^-) & -(1+L^{(0)}(1,1^-)) \end{bmatrix} \begin{Bmatrix} t(0) \\ t(1) \end{Bmatrix}$$
$$= \sum_{j=0}^{N} \begin{bmatrix} -(-\lambda)^{j+1}M^{(j+1)}(0,0^+) & (-\lambda)^{j+1}M^{(j+1)}(1,0^+) \\ -(-\lambda)^{j+1}M^{(j+1)}(0,1^-) & (-\lambda)^{j+1}M^{(j+1)}(1,1^-) \end{bmatrix} \begin{Bmatrix} u(0) \\ u(1) \end{Bmatrix}$$
$$- \sum_{j=0}^{N} \begin{bmatrix} -(-\lambda)^{j+1}L^{(j+1)}(0,0^+) & (-\lambda)^{j+1}L^{(j+1)}(1,0^+) \\ -(-\lambda)^{j+1}L^{(j+1)}(0,1^-) & (-\lambda)^{j+1}L^{(j+1)}(1,1^-) \end{bmatrix} \begin{Bmatrix} t(0) \\ t(1) \end{Bmatrix} \tag{14}$$

where $\underset{\sim}{u}$ and $\underset{\sim}{t}$ are column vectors of the boundary data.

Substituting the values of the kernel functions shown in Table 1 into eqn (11) and eqn (12), we have

$$\begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} u(0) \\ u(1) \end{Bmatrix} - \begin{bmatrix} 0 & \frac{-1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \begin{Bmatrix} t(0) \\ t(1) \end{Bmatrix}$$
$$= \begin{bmatrix} 0 & \sum_{j=0}^{N} \frac{1}{2}\frac{(-\lambda)^{j+1}}{(2j+2)!} \\ \sum_{j=0}^{N} \frac{1}{2}\frac{(-\lambda)^{j+1}}{(2j+2)!} & 0 \end{bmatrix} \begin{Bmatrix} u(0) \\ u(1) \end{Bmatrix} - \begin{bmatrix} 0 & \sum_{j=0}^{N} \frac{1}{2}\frac{(-\lambda)^{j+1}}{(2j+3)!} \\ -\sum_{j=0}^{N} \frac{1}{2}\frac{(-\lambda)^{j+1}}{(2j+3)!} & 0 \end{bmatrix} \begin{Bmatrix} t(0) \\ t(1) \end{Bmatrix} \tag{15}$$

and

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} u(0) \\ u(1) \end{Bmatrix} - \begin{bmatrix} \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{bmatrix} \begin{Bmatrix} t(0) \\ t(1) \end{Bmatrix}$$
$$= \begin{bmatrix} 0 & -\sum_{j=0}^{N} \frac{1}{2}\frac{(-\lambda)^{j+1}}{(2j+1)!} \\ \sum_{j=0}^{N} \frac{1}{2}\frac{(-\lambda)^{j+1}}{(2j+1)!} & 0 \end{bmatrix} \begin{Bmatrix} u(0) \\ u(1) \end{Bmatrix} - \begin{bmatrix} 0 & -\sum_{j=0}^{N} \frac{1}{2}\frac{(-\lambda)^{j+1}}{(2j+2)!} \\ -\sum_{j=0}^{N} \frac{1}{2}\frac{(-\lambda)^{j+1}}{(2j+2)!} & 0 \end{bmatrix} \begin{Bmatrix} t(0) \\ t(1) \end{Bmatrix} \tag{16}$$

After substituting the four cases of boundary conditions, the dual BEM has the dual matrices of UT and LM equations.

For Case 1 with B.C. $u(0) = 0$ and $u(1) = 0$, we haveUT equation

$$\begin{bmatrix} 0 & \frac{-1}{2}\left(1+\sum_{j=0}^{N} \frac{(-\lambda)^{j+1}}{(2j+3)!}\right) \\ \frac{1}{2}\left(1+\sum_{j=0}^{N} \frac{(-\lambda)^{j+1}}{(2j+3)!}\right) & 0 \end{bmatrix} \begin{Bmatrix} t(0) \\ t(1) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{17}$$

LM equation

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2}\left(1+\sum_{j=0}^{N} \frac{(-\lambda)^{j+1}}{(2j+2)!}\right) \\ -\frac{1}{2}\left(1+\sum_{j=0}^{N} \frac{(-\lambda)^{j+1}}{(2j+2)!}\right) & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} t(0) \\ t(1) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{18}$$

For Case 2 with B.C. $t(0) = 0$ and $t(1) = 0$, we have UT equation

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2}\left(1+\sum_{j=0}^{N} \frac{(-\lambda)^{j+1}}{(2j+2)!}\right) \\ \frac{-1}{2}\left(1+\sum_{j=0}^{N} \frac{(-\lambda)^{j+1}}{(2j+2)!}\right) & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} u(0) \\ u(1) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{19}$$

LM equation

$$\begin{bmatrix} 0 & \frac{1}{2}\left(1+\sum_{j=0}^{N} \frac{(-\lambda)^{j+1}}{(2j+1)!}\right) \\ -\frac{1}{2}\sum_{j=0}^{N} \frac{(-\lambda)^{j+1}}{(2j+1)!} & 0 \end{bmatrix} \begin{Bmatrix} u(0) \\ u(1) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{20}$$

For Case 3 with B.C. $u(0) = 0$ and $t(1) = 0$, we have

UT equation

$$\begin{bmatrix} 0 & \frac{1}{2}(1+\sum_{j=0}^{N}\frac{(-\lambda)^{j+1}}{(2j+2)!}) \\ \frac{1}{2}(1+\sum_{j=0}^{N}\frac{(-\lambda)^{j+1}}{(2j+3)!}) & \frac{-1}{2} \end{bmatrix} \begin{Bmatrix} t(0) \\ u(1) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{21}$$

LM equation

$$\begin{bmatrix} -\frac{1}{2} & \frac{-1}{2}\sum_{j=0}^{N}\frac{(-\lambda)^{j+1}}{(2j+1)!} \\ \frac{1}{2}(1+\sum_{j=0}^{N}\frac{(-\lambda)^{j+1}}{(2j+2)!}) & 0 \end{bmatrix} \begin{Bmatrix} t(0) \\ u(1) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{22}$$

For Case 4 with B.C. $u(1) = 0$ and $t(0) = 0$, we have

UT equation

$$\begin{bmatrix} \frac{-1}{2} & \frac{-1}{2}(1+\sum_{j=0}^{N}\frac{(-\lambda)^{j+1}}{(2j+3)!}) \\ \frac{1}{2}(1+\sum_{j=0}^{N}\frac{(-\lambda)^{j+1}}{(2j+2)!}) & 0 \end{bmatrix} \begin{Bmatrix} u(0) \\ t(1) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{23}$$

LM equation

$$\begin{bmatrix} 0 & -\frac{1}{2}(1+\sum_{j=0}^{N}\frac{(-\lambda)^{j+1}}{(2j+2)!}) \\ -\frac{1}{2}(1+\sum_{j=0}^{N}\frac{(-\lambda)^{j+1}}{(2j+1)!}) & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} u(0) \\ t(1) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{24}$$

3 ANALYTICAL SOLUTIONS BY DBEM AND MRM

According to eqns (17)–(24), it is found that only three terms involving series are present as shown below:

$$\sum_{j=0}^{N}\frac{(-\lambda)^{j+1}}{(2j+3)!},\ \sum_{j=0}^{N}\frac{(-\lambda)^{j+1}}{(2j+2)!},\ \sum_{j=0}^{N}\frac{(-\lambda)^{j+1}}{(2j+1)!} \tag{25}$$

The series forms in eqn (25) can be found to have a closed form as follows:

$$1+\sum_{j=0}^{\infty}\frac{(-\lambda)^{j+1}}{(2j+3)!} = \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} \tag{26}$$

$$1+\sum_{j=0}^{\infty}\frac{(-\lambda)^{j+1}}{(2j+2)!} = \cos\sqrt{\lambda} \tag{27}$$

$$\sum_{j=0}^{\infty}\frac{(-\lambda)^{j+1}}{(2j+1)!} = -\sqrt{\lambda}\sin\sqrt{\lambda} \tag{28}$$

Therefore, eqns (17)–(24) can be expressed as follows:

Case 1.UT equation

$$\begin{bmatrix} 0 & \frac{-1}{2}(\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}) \\ \frac{1}{2}(\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}) & 0 \end{bmatrix} \begin{Bmatrix} t(0) \\ t(1) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{29}$$

LM equation

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2}\cos\sqrt{\lambda} \\ -\frac{1}{2}\cos\sqrt{\lambda} & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} t(0) \\ t(1) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{30}$$

Case 2.UT equation

$$\begin{bmatrix} \frac{1}{2} & \frac{-1}{2}\cos\sqrt{\lambda} \\ \frac{-1}{2}\cos\sqrt{\lambda} & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} u(0) \\ u(1) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{31}$$

LM equation

$$\begin{bmatrix} 0 & -\frac{1}{2}\sqrt{\lambda}\sin\sqrt{\lambda} \\ \frac{1}{2}\sqrt{\lambda}\sin\sqrt{\lambda} & 0 \end{bmatrix} \begin{Bmatrix} u(0) \\ u(1) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{32}$$

Case 3.UT equation

$$\begin{bmatrix} 0 & \frac{1}{2}\cos\sqrt{\lambda} \\ \frac{1}{2}\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} & \frac{-1}{2} \end{bmatrix} \begin{Bmatrix} t(0) \\ u(1) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{33}$$

LM equation

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{\lambda}\sin\sqrt{\lambda} \\ \frac{1}{2}\cos\sqrt{\lambda} & 0 \end{bmatrix} \begin{Bmatrix} t(0) \\ u(1) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{34}$$

Table 2. Table 2 Eigensolutions for the Dirichlet problem (case 1)

Case 1	Dirichlet problem $u(0)=0, u(1)=0$	
	UT	LM
Eigenvalues (λ)	$(n\pi)^2$	
Eigenequations	$\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}=0$	(a) $\cos\sqrt{\lambda}+1=0$ (b) $\cos\sqrt{\lambda}-1=0$
Boundary eigenvector$\{t(0), t(1)\}$	x	$\{1,-1\}^{(a)}$ $\{1,1\}^{(b)}$
Reason for failure in determining eigenmodes	Null matrix	O.K.

Case 4.

UT equation

$$\begin{bmatrix} \frac{-1}{2} & \frac{-1}{2}\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} \\ \frac{1}{2}\cos\sqrt{\lambda} & 0 \end{bmatrix} \begin{Bmatrix} u(0) \\ t(1) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (35)$$

LM equation

$$\begin{bmatrix} 0 & -\frac{1}{2}\cos\sqrt{\lambda} \\ \frac{1}{2}\sqrt{\lambda}\sin\sqrt{\lambda} & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} u(0) \\ t(1) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (36)$$

By combining the UT and LM equations from eqns (29)–(36) for the four cases, the trivial solutions for the boundary eigenvectors can be dealt with when the UT or LM equation is only used for cases 1 and 2. Also, the augmented eigenvalues and eigenequations can be deleted for cases 3 and 4. The analytical solutions for the eigenequations and eugenvalues are shown in Tables 2, 3, 4, and 5 for cases 1, 2, 3 and 4 respectively. After the eigenvalues and boundary eigenvectors are determined, the eigenmodes in eqn (9) can be derived as follows:

$$u(s)=\sum_{j=-1}^{N}(-\lambda)^{(j+1)}\{T^{(j+1)}(x,s)u(x) - U^{(j+1)}(x,s)t(x)\}|_{x=0}^{x=1}$$

$$=\sum_{j=-1}^{N}(-\lambda)^{(j+1)}\{T^{(j+1)}(1,s)u(1)-T^{(j+1)}(0,s)u(0)\}$$

$$-\sum_{j=-1}^{N}(-\lambda)^{(j+1)}\{U^{(j+1)}(1,s)t(1)-U^{(j+1)}(0,s)t(0)\}$$

$$=\sum_{j=-1}^{N}(-\lambda)^{(j+1)}\{\frac{1}{2}\frac{(1-s)^{2j+2}}{(2j+2)!}u(1) -(\frac{-1}{2})\frac{s^{2j+2}}{(2j+2)!}u(0)\}$$

$$-\sum_{j=-1}^{N}(-\lambda)^{(j+1)}\{\frac{1}{2}\frac{(1-s)^{2j+3}}{(2j+3)!}t(1) -(\frac{1}{2})\frac{s^{2j+3}}{(2j+3)!}t(0)\}$$

$$=\frac{1}{2}\{(1-\lambda\frac{(1-s)^2}{2!}+\lambda^2\frac{(1-s)^4}{4!}+...)u(1) +(1-\lambda\frac{s^2}{2!}+\lambda^2\frac{s^4}{4!}+...)u(0)\}$$

$$-\frac{1}{2}\{((1-s)-\lambda\frac{(1-s)^3}{3!}+\lambda^2\frac{(1-s)^5}{5!}+...)t(1) -(s-\lambda\frac{s^3}{3!}+\lambda^2\frac{s^5}{5!}+...)t(0)\}$$

$$=\frac{1}{2}\{\cos(\sqrt{\lambda}(1-s))u(1)+\cos(\sqrt{\lambda}s)u(0)\} -\frac{1}{2}\{\frac{\sin(\sqrt{\lambda}(1-s))}{\sqrt{\lambda}}t(1)-\frac{\sin(\sqrt{s})}{\sqrt{\lambda}}t(0)\} \quad (37)$$

By substituting the homogeneous boundary conditions in the four cases, where their corresponding eigenvalues and corresponding boundary eigenvectors are shown in Tables 2–5, we can obtain the exact form of the eigenmodes as shown in Table 6.

Table 3. Eigensolutions for the Neumann problem (case 2)

Case 2	Neumann problem $t(0)=0, t(1)=0$	
	UT	LM
Eigenvalues (λ)	$(n\pi)^2$	
Eigenequations	(a) $\cos\sqrt{\lambda}+1=0$ (b) $\cos\sqrt{\lambda}-1=0$	$-\sqrt{\lambda}\sin\sqrt{\lambda}=0$
Boundary eigenvector$\{u(0), u(1)\}$	\{1,-1\}$^{(a)}$--\{1,1\}$^{(b)}$	x
Reason for failure in determining eigenmodes	O.K.	Null matrix

Table 4. Eigensolutions for the Robin problem (case 3)

Case 3		Robin (mixed) problem $u(0) = 0, t(1) = 0$	
		UT	LM
Eigenvalues (λ)		$(\frac{(2n-1)\pi}{2})^2$	
Eigenequations	Correct	$\cos\sqrt{\lambda}=0$	$\cos\sqrt{\lambda}=0$
	Augmented	$\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}=0$	$-\sqrt{\lambda}\sin\sqrt{\lambda}=0$
Boundary eigenvector $\{u(2), t(0)\}$		$\{(-1)^{n+1}, \frac{(2n-1)\pi}{2}\}$	$\{(-1)^{n+1}, \frac{(2n-1)\pi}{2}\}$

Table 5. Eigensolutions for the Robin problem (case 4)

Case 4		Robin (mixed) problem $u(1) = 0, t(0) = 0$	
		UT	LM
Eigenvalues (λ)		$(\frac{(2n-1)\pi}{2})^2$	
Eigenequations	Correct	$\cos\sqrt{\lambda}=0$	$\cos\sqrt{\lambda}=0$
	Augmented	$\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}=0$	$-\sqrt{\lambda}\sin\sqrt{\lambda}=0$
Boundary eigenvector $\{u(0), t(1)\}$		$\{1,(-1)^n \frac{(2n-1)\pi}{2}\}$	$\{1,(-1)^n \frac{(2n-1)\pi}{2}\}$

4 RESULTS AND DISCUSSIONS

1. Tables 2–5 show the analytical solutions for the eigenvalues and eigenmodes. It is found that the UT equation alone cannot determine the eigenmodes for case 1 with the Dirichelet problem as shown in Table 2. However, the LM equation alone also cannot determine the eigenmodes for case 2 with the Neumann problem as shown in Table 3.
2. For mixed type problems of cases 3 and 4, augmented eigenvalues are present. After combining the UT and LM equations, the corresponding eigenmodes are found to be trivial for the augmented eigenvalues. The analytical solutions are shown in Tables 4 and 5.
3. After combining the UT and LM equations, the eigenvalues and boundary eigenmodes can be determined easily. The eigensolutions for the four cases are summarized in Table 6. After substituting the eigenvalues into eqn (9), the eigenmodes are as shown in Figs 2–5 for cases 1–4 respectively.
4. Although only four boundary data are treated as the degrees of freedom to describe the problems,

Table 6. Eigensolutions for the four cases

Case λn	1 $(n\pi)^2$	2 $(n\pi)^2$	3 $((2n-1)\pi/2)^2$	4 $((2n-1)\pi/2)^2$
Boundary mode $\begin{Bmatrix} u(0) \\ u(1) \\ t(0) \\ t(1) \end{Bmatrix}$	$\begin{Bmatrix} 0^* \\ 0^* \\ 1 \\ (-1)^n \end{Bmatrix}$	$\begin{Bmatrix} 1 \\ (-1)^n \\ 0^* \\ 0^* \end{Bmatrix}$	$\begin{Bmatrix} 0^* \\ (-1)^{n+1} \\ \frac{(2n-1)\pi}{2} \\ 0^* \end{Bmatrix}$	$\begin{Bmatrix} 1 \\ 0^* \\ 0^* \\ (-1)^n\frac{(2n-1)\pi}{2} \end{Bmatrix}$
$u_n(x)$	$\sin(n\pi x)$	$\cos(n\pi x)$	$\sin((2n-1)\pi x/2)$	$\cos((2n-1)\pi x/2)$

*The given B.S.s.

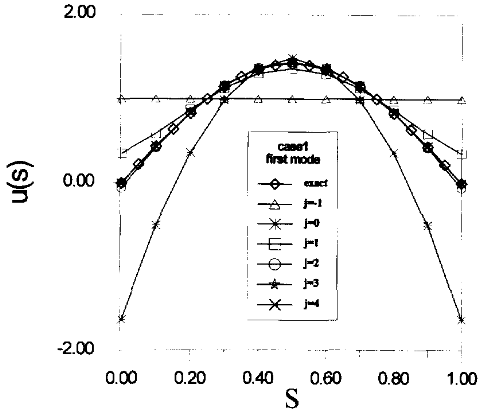

Fig. 2. Eigenmode for case 1.

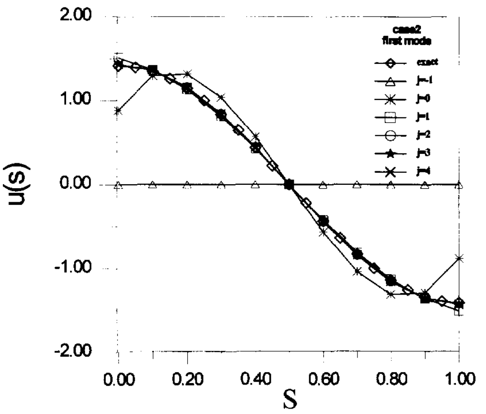

Fig. 3. Eigenmode for case 2.

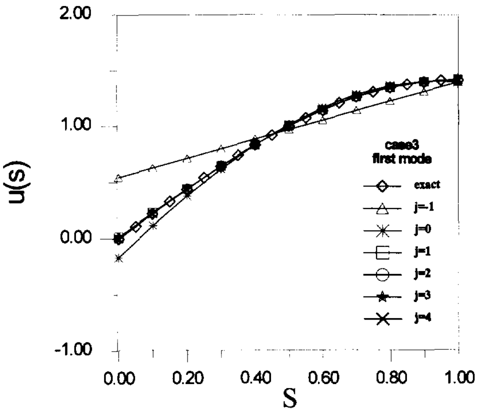

Fig. 4. Eigenmode for case 3.

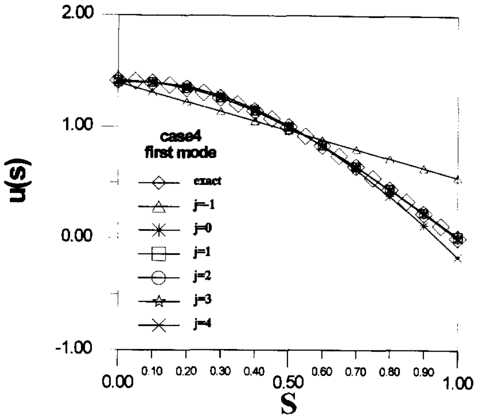

Fig. 5. Eigenmode for case 4.

analytical solutions can be derived by combining DBEM and MRM.

5 CONCLUDING REMARKS

In this paper, we have constructed dual equations for MRM to solve the one-dimensional eigenproblem analytically. It has been found that the hypersingular equation of the dual equations plays an important role in determining the eigensolutions. After combining this equation, the augmented eigenvalues and the failure in determining the eigenmodes by employing the conventional MRM only are solved. Further reasearch is being conducted on the dual approach with regard to acoustic problems with incomplete partitions using DBEM and MRM. Also, a complete MRM is being investigated[13].

ACKNOWLEDGEMENT

Financial support from the National Science Council, Grant No. NSC-86-2211-E-019-006, for National Taiwan Ocean University is gratefully acknowledged.

REFERENCES

1. Hong, H.-K. & Chen, J. T. Derivation of integral equations in elasticity. *J. Eng. Mech. Div. ASCE.*, 1988, **144**, 1028–1044.
2. Chen, J.T. and Hong, H.-K., *Boundary Element Method*, 2nd Ed., New World Press, Taipei, Taiwan, 1992 (in Chinese).
3. Chen, J.T., On a dual integral representation and its applications to computational mechanics, Ph.D. Dissertation, Department of Civil Engineering, National Taiwan University, 1994.
4. Chen, J.T. & Hong, H.-K. Boundary element analysis and design in seepage flow problems with sheetpiles. *Finite Elements in Analysis and Design*, 1994, **17**, 1–20.
5. Chen, J.T. & Hong, H.-K. On the dual integral representation of boundary value problem in Laplace equation. *Boundary Element Abstracts*, 1993, **3**, 114–116.

6. Chen, J.T. & Hong, H.K. Dual boundary integral equations at a corner using contour approach around singularity. *Advances in Engineering Softwares*, 1994, **21**, 169–178.
7. Chen, J.T., Liang, M.T. & Yang, S.S. Dual boundary integral equations for exterior problems. *Engineering Analysis Boundary Elements,*, 1996, **16**, 333–340.
8. Liang, J.T., Chen, J.T. and Yang, S.S., Error estimation for boundary element method, *Computers and Structures*, 1996, in press.
9. Kamiya, N., Ando, E. & Nogae, K. A new complex-valued formulation and eigenvalue analysis of the Helmholtz equation by boundary element method. *Advances in Engineering Softwares*, 1996, **26**, 219–227.
10. Nowak, A.J. and Neves, A.C. eds., *The Multiple Reciprocity Boundary Element Method*, Comp. Publ., Southampton, 1994.
11. Nowak, A.J. & Brebbia, C.A. The multiple reciprocity method—a new approach for transforming BEM domain integrals to the boundary. *Engineering Analysis with Boundary Elements,*, 1989, **6**, 164–167.
12. Kamiya, N. & Andoh, E. A note on multiple reciprocity integral formulation for Helmholtz Equation. *Communications in Numerical Methods in Engineering,*, 1993, **9**, 9–13.
13. Yeih, W., Chen, J.T., Chen, K.H. & Wong, F.C., A study on the multiple reciprocity method and complex-valued formulation for the Helmoltz equation. *Advances in Engineering Softwares*, accepted for publication, 1997.