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Analytical study and numerical experiments for degenerate scale problems in boundary element method using degenerate kernels and circulants

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Abstract

For a potential problem, the boundary integral equation approach has been shown to yield a nonunique solution when the geometry is equal to a degenerate scale. In this paper, the degenerate scale problem in boundary element method (BEM) is analytically studied using the degenerate kernels and circulants. For the circular domain problem, the singular problem of the degenerate scale with radius one can be overcome by using the hypersingular formulation instead of the singular formulation. A simple example is shown to demonstrate the failure using the singular integral equations. To deal with the problem with a degenerate scale, a constant term is added to the fundamental solution to obtain the unique solution and another numerical example with an annular region is also considered.

Keywords: Boundary elements; Degenerate scale; Circulant; Degenerate kernel

1. Introduction

The boundary element method (BEM) has received much attention not only from academic research groups [1] but also from the engineering community [2] in recent decades. The crack degeneracy in BEM was discussed by Cruse [1,2]. Chen and Hong [3] termed the crack surface 'degenerate boundary'. It is well known that rigid body motion test or so called use of simple solution can be employed to check the singular matrices for the strongly singular and hypersingular kernels for problems without degenerate boundaries, respectively. In this case, the singularity occurs physically and mathematically. The nontrivial solution for the singular matrix is found to be a rigid body term for the interior Neumann problem. However, in some special cases, the influence matrix of the weakly singular kernel may be singular for the Dirichlet problem [4] when geometry is special. The nonunique solution is not physically realizable but results from the zero eigenvalue of the influence matrix in integral formulation. For example, the unit circle case has been noted by Petrosky [5] and by Jaswon and Symm [6]. The special geometry which results in a nonunique solution for a potential problem is called degenerate scale. For several specific boundary conditions, some studies for potential problems [4,7,8] and plane elasticity problems [8–11] have been done. The difficulties due to nonuniqueness of solutions were overcome by the necessary and sufficient boundary integral formulation [10] and boundary contour method [11]. However, the boundary conditions in their cases are either the Dirichlet or mixed type and must be constant along the circular boundary. Fictitious BEM also has the degenerate scale problem when selecting a special fictitious boundary, and has been proven to yield nonunique solutions for two-dimensional potential problems [12]. Also, the degenerate scale of multiply-connected domain problems was discussed by Tomlinson et al. [13]. Nevertheless, no general proof has been done for the degenerate scale problem.

In this paper, many aspects, general boundary conditions, exterior problems and multiply-connected, are all considered. The degenerate scale in BEM will be studied analytically and numerical experiments will be performed. Degenerate kernels and circulant matrices are employed to find the eigenvalues for the influence matrices analytically in a discrete system for a circular problem. The singularity pattern distributed along a ring boundary resulting in a zero field will be determined when the ring boundary is a degenerate scale. An annular region is also considered and the possible degenerate scales are investigated. Also, the role of hypersingular formulation is examined for the degenerate scale problems in both simply-connected and multiply-connected problems.

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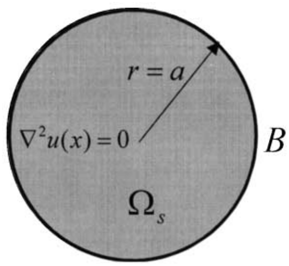


Fig. 1. Potential problem for a circular region.

2. Mathematical analysis of the degenerate scale problems for a circular problem

The governing equation for potential problem is Laplace equation

$$\nabla^2 u(x) = 0, \qquad x \text{ in } \Omega_s \tag{1}$$

where ∇^2 is the Laplace operator, Ω_s is the simply-connected domain of the problem. Here, we consider the problem with a circular region of radius a as shown in Fig. 1. The boundary condition is the Dirichlet type, $u = \bar{u}$, where $\bar{u}$ is the specified boundary data. Based on the dual boundary integral equation formulation for the potential problems [3], we have

$$\alpha u(x) = \text{C.P.V.} \int_{B(s)} T(s,x)u(s)\,\mathrm{d}B(s) - \text{R.P.V.} \int_{B(s)} U(s,x)t(s)\,\mathrm{d}B(s) \tag{2}$$

$$\alpha t(x) = \text{H.P.V.} \int_{B(s)} M(s,x)u(s)\,\mathrm{d}B(s) - \text{C.P.V.} \int_{B(s)} L(s,x)t(s)\,\mathrm{d}B(s) \tag{3}$$

where the kernel functions, $U(s,x) = \ln r$, $T(s,x) = \partial U(s,x)/\partial n_s$, $L(s,x) = \partial U(s,x)/\partial n_x$, $M(s,x) = \partial T(s,x)/\partial n_x$ and $r = |x - s|$, is the distance between x and s, R.P.V., C.P.V. and H.P.V. denote the Reimann principle value, Cauchy principle value and Hadamard principle value, $t(s) = \partial u(s)/\partial n_s$, and α depends on the collocation point ($\alpha = 2\pi$ for an interior point, $\alpha = \pi$ for a smooth boundary point, $\alpha = 0$ for an exterior point).

For a problem with a circular domain, the circular boundary can be discretized into $2N$ constant elements with equal arc length. The linear algebraic dual equations can be obtained as shown below

$$[U]_{2N\times 2N}\{t\}_{2N\times 1} = [T]_{2N\times 2N}\{u\}_{2N\times 1} \tag{4}$$

$$[L]_{2N\times 2N}\{t\}_{2N\times 1} = [M]_{2N\times 2N}\{u\}_{2N\times 1} \tag{5}$$

where $[U]$, $[T]$, $[L]$ and $[M]$ are the four influence matrices, $\{u\}$ and $\{t\}$ are the boundary data for the primary and the secondary fields, respectively. Based on the circular symmetry, the influence matrices for the discrete system are found to be circulants with the following forms [14–17]

$$[U] = \begin{bmatrix} u_0 & u_1 & u_2 & \cdots & u_{2N-1} \\ u_{2N-1} & u_0 & u_1 & \cdots & u_{2N-2} \\ u_{2N-2} & u_{2N-1} & u_0 & \cdots & u_{2N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_1 & u_2 & u_3 & \cdots & u_0 \end{bmatrix} \tag{6}$$

$$[T] = \begin{bmatrix} t_0 & t_1 & t_2 & \cdots & t_{2N-1} \\ t_{2N-1} & t_0 & t_1 & \cdots & t_{2N-2} \\ t_{2N-2} & t_{2N-1} & t_0 & \cdots & t_{2N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_1 & t_2 & t_3 & \cdots & t_0 \end{bmatrix} \tag{7}$$

$$[L] = \begin{bmatrix} l_0 & l_1 & l_2 & \cdots & l_{2N-1} \\ l_{2N-1} & l_0 & l_1 & \cdots & l_{2N-2} \\ l_{2N-2} & l_{2N-1} & l_0 & \cdots & l_{2N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_1 & l_2 & l_3 & \cdots & l_0 \end{bmatrix} \tag{8}$$

$$[M] = \begin{bmatrix} m_0 & m_1 & m_2 & \cdots & m_{2N-1} \\ m_{2N-1} & m_0 & m_1 & \cdots & m_{2N-2} \\ m_{2N-2} & m_{2N-1} & m_0 & \cdots & m_{2N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_1 & m_2 & m_3 & \cdots & m_0 \end{bmatrix} \tag{9}$$

where the elements u_m, t_m, l_m, and m_m, will be elaborated on later. The four matrices in Eqs. (6)–(9) have only $N + 1$ different elements since rotation symmetry for the circulant is found. Based on the separable properties for the kernels, the kernel functions in the dual BEM can be expanded into degenerate forms as shown below [16–18]

$$U(s,x) = \begin{cases} U^{\mathrm{i}}(R,\theta;\rho,\phi) = \ln R - \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{\rho}{R}\right)^m \cos(m(\theta-\phi)), & R > \rho \\ U^{\mathrm{e}}(R,\theta;\rho,\phi) = \ln \rho - \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{R}{\rho}\right)^m \cos(m(\theta-\phi)), & \rho > R \end{cases} \tag{10}$$

$$T(s,x) = \begin{cases} T^{\mathrm{i}}(R,\theta;\rho,\phi) = \frac{1}{R} + \sum_{m=1}^{\infty} \frac{\rho^m}{R^{m+1}}\cos(m(\theta-\phi)), & R > \rho \\ T^{\mathrm{e}}(R,\theta;\rho,\phi) = -\sum_{m=1}^{\infty} \frac{R^{m-1}}{\rho^m}\cos(m(\theta-\phi)), & \rho > R \end{cases} \tag{11}$$

$$L(s,x) = \begin{cases} L^{\mathrm{i}}(R,\theta;\rho,\phi) = -\sum_{m=1}^{\infty} \frac{\rho^{m-1}}{R^m}\cos(m(\theta-\phi)), & R>\rho \\ L^{\mathrm{e}}(R,\theta;\rho,\phi) = \frac{1}{\rho} + \sum_{m=1}^{\infty} \frac{R^m}{\rho^{m+1}}\cos(m(\theta-\phi)), & \rho>R \end{cases} \tag{12}$$

$$M(s,x) = \begin{cases} M^{\mathrm{i}}(R,\theta;\rho,\phi) = \sum_{m=1}^{\infty} \frac{m\rho^{m-1}}{R^{m+1}}\cos(m(\theta-\phi)), & R>\rho \\ M^{\mathrm{e}}(R,\theta;\rho,\phi) = \sum_{m=1}^{\infty} \frac{mR^{m-1}}{\rho^{m+1}}\cos(m(\theta-\phi)), & \rho>R \end{cases} \tag{13}$$

where ‘i’ and ‘e’ denotes the interior domain ($R > \rho$) and the exterior domain ($R < \rho$), $x = (\rho, \phi)$ and $s = (R, \theta)$ in the polar coordinate as shown in Fig. 2. It must be noted that the superscripts of the ‘i’ and ‘e’ kernels are used for the exterior and interior problems to avoid the source terms, respectively. Then, we have the influence coefficients in the four matrices as shown below:

$$u_m = \int_{\left(m-\frac{1}{2}\right)\Delta\theta}^{\left(m+\frac{1}{2}\right)\Delta\theta} U^{\mathrm{e}}(R,\theta;\rho,\phi)\rho\,\mathrm{d}\theta \approx U^{\mathrm{e}}(R,\theta_m;\rho,\phi)\rho\Delta\theta,$$

$$m = 0,1,2,\ldots 2N-1 \tag{14}$$

$$t_m = \int_{\left(m-\frac{1}{2}\right)\Delta\theta}^{\left(m+\frac{1}{2}\right)\Delta\theta} T^{\mathrm{e}}(R,\theta;\rho,\phi)\rho\,\mathrm{d}\theta \approx T^{\mathrm{e}}(R,\theta_m;\rho,\phi)\rho\Delta\theta,$$

$$m = 0,1,2,\ldots 2N-1 \tag{15}$$

$$l_m = \int_{\left(m-\frac{1}{2}\right)\Delta\theta}^{\left(m+\frac{1}{2}\right)\Delta\theta} L^{\mathrm{e}}(R,\theta;\rho,\phi)\rho\,\mathrm{d}\theta \approx L^{\mathrm{e}}(R,\theta_m;\rho,\phi)\rho\Delta\theta,$$

$$m = 0,1,2,\ldots 2N-1 \tag{16}$$

$$m_m = \int_{\left(m-\frac{1}{2}\right)\Delta\theta}^{\left(m+\frac{1}{2}\right)\Delta\theta} M^{\mathrm{e}}(R,\theta;\rho,\phi)\rho\,\mathrm{d}\theta \approx M^{\mathrm{e}}(R,\theta_m;\rho,\phi)\rho\Delta\theta,$$

$$m = 0,1,2,\ldots 2N-1 \tag{17}$$

where $\Delta\theta = 2\pi/2N$ and $\theta_m = m\Delta\theta$. By introducing the following bases for circulants, I, $C_{2N}^1, C_{2N}^2, \ldots, C_{2N}^{2N-1}$, we can expand matrix $[U]$ into

$$[U] = u_0\mathrm{I} + u_1 C_{2N}^1 + u_2 C_{2N}^2 + \cdots + u_{2N-1} C_{2N}^{2N-1} \tag{18}$$

where

$$C_{2N}^1 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}_{2N\times 2N} \tag{19}$$

Based on the similar properties for the matrices of $[U]$ and $[C_{2N}]$, we have

$$\lambda_l^{[U]} = u_0 + u_1\alpha_1 + u_2\alpha_l^2 + \cdots + u_{2N-1}\alpha_l^{2N-1},$$
$$l = 0, \pm 1, \pm 2, \ldots, \pm N-1, N \tag{20}$$

where $\lambda_l^{[U]}$ and α_l are the lth eigenvalues for $[U]$ and $[C_{2N}]$, respectively. It is easily found that the eigenvalues and eigenvectors for the circulants $[C_{2N}]$, are the roots for $z^{2N} = 1$ as shown below

$$\alpha_n = e^{\mathrm{i}(2\pi n/2N)}, \mathrm{n} = 0, \pm 1, \pm 2, \ldots, \pm \mathrm{N}-1, \mathrm{N}, \text{ or}$$
$$\mathrm{n} = 0, 1, 2, \ldots, 2\mathrm{N}-1 \tag{21}$$

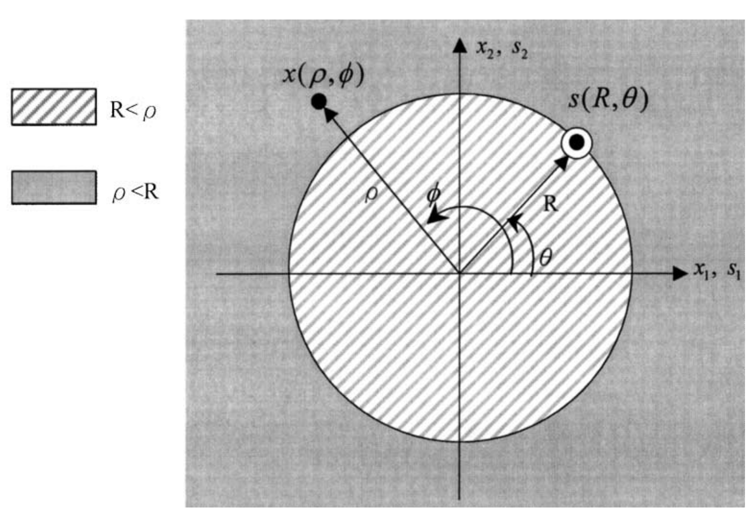


Fig. 2. Symbols for degenerate kernels.

$$\{\phi_n\} = \left\{ \begin{array}{c} 1 \\ \alpha_n \\ \alpha_n^2 \\ \vdots \\ \alpha_n^{2N-1} \end{array} \right\}, \tag{22}$$

respectively. Substituting Eq. (21) into Eq. (20), we have

$$\begin{aligned} \lambda_l^{[U]} &= \sum_{m=0}^{2N-1} u_m(\alpha_l)^m = \sum_{m=0}^{2N-1} u_m\, \mathrm{e}^{\mathrm{i}ml(2\pi/2N)} \\ &= \sum_{m=0}^{2N-1} U^{\mathrm{e}}(a,\theta;a,0)a\Delta\theta\, \mathrm{e}^{\mathrm{i}ml\Delta\theta} \\ &= a\sum_{m=0}^{2N-1} U^{\mathrm{e}}(a,\theta;a,0)\, \mathrm{e}^{\mathrm{i}ml\Delta\theta}\Delta\theta \end{aligned} \tag{23}$$

When N approaches infinity, the Riemann sum in Eq. (23) can be transformed to the following integral:

$$\lambda_l^{[U]} = a\int_0^{2\pi} U^{\mathrm{e}}(a,\theta;a,0)\, \mathrm{e}^{\mathrm{i}l\theta}\, \mathrm{d}\theta \tag{24}$$

By substituting U^{e} kernel of Eq. (10) into Eq. (24) for the interior problem to avoid the source term, we have

$$\begin{aligned} \lambda_l^{[U]} &= a\int_0^{2\pi}\left(\ln a - \sum_{m=1}^{\infty}\frac{1}{m}\cos(m\theta)\right)\mathrm{e}^{\mathrm{i}l\theta}\,\mathrm{d}\theta \\ &= \begin{cases} 2\pi a\ln a, & l=0 \\ -\pi\dfrac{a}{|l|}, & l=\pm1,\pm2,\dots,\pm(N-1),N \end{cases} \end{aligned} \tag{25}$$

Similarly, we can obtain the eigenvalues for the other influence matrices

$$\lambda_l^{[T]} = \begin{cases} 0, & l=0 \\ -\pi & l=\pm1,\pm2,\dots,\pm(N-1),N \end{cases} \tag{26}$$

$$\lambda_l^{[L]} = \begin{cases} 2\pi, & l=0 \\ \pi & l=\pm1,\pm2,\dots,\pm(N-1),N \end{cases} \tag{27}$$

$$\lambda_l^{[M]} = \begin{cases} 0, & l=0 \\ \pi\dfrac{|l|}{a}, & l=\pm1,\pm2,\dots,\pm(N-1),N \end{cases} \tag{28}$$

In the eigen spectrum, one of eigenvalues of the matrix $[U]$ is zero when the radius a is equal to one if we use Eq. (4) for the interior Dirichlet problem. In this case, the solution is nonunique when a is this specific magnitude which is equal to its degenerate scale. Instead of using Eq. (4), we can employ Eq. (5) to avoid the zero eigenvalue because all the eigenvalues of the influence matrix $[L]$ are never zero from Eq. (27) no matter what the value of radius, a, is.

Similarly, we can avoid the zero eigenvalue for the influence matrices of exterior problem as shown below

$$\lambda_l^{[U]} = \begin{cases} 2\pi a\ln a, & l=0 \\ -\pi\dfrac{a}{|l|}, & l=\pm1,\pm2,\dots,\pm(N-1),N \end{cases} \tag{29}$$

$$\lambda_l^{[T]} = \begin{cases} 2\pi, & l=0 \\ \pi, & l=\pm1,\pm2,\dots,\pm(N-1),N \end{cases} \tag{30}$$

$$\lambda_l^{[L]} = \begin{cases} 0, & l=0 \\ -\pi, & l=\pm1,\pm2,\dots,\pm(N-1),N \end{cases} \tag{31}$$

$$\lambda_l^{[M]} = \begin{cases} 0, & l=0 \\ \pi\dfrac{|l|}{a}, & l=\pm1,\pm2,\dots,\pm(N-1),N \end{cases} \tag{32}$$

For the exterior problem, one of eigenvalues of the matrix $[U]$ is zero when the radius a is equal to one if we use Eq. (4) for the Dirichlet problem. Similarly, the solution is nonunique when a is a specific magnitude which is equal to its degenerate scale. Instead of using Eq. (4), we expected to employ Eq. (5) to obtain the accurate solution. However, in Eq. (31), one eigenvalue of the influence matrix $[L]$ is zero. Eq. (5) will fail no matter what the value of radius, a, is.

3. Mathematical analysis of the degenerate scale problems for an annular problem

An annular domain composed of two concentric circles (Fig. 3) is to be studied here with the following governing equation:

$$\nabla^2 u(x) = 0, \qquad x \text{ in } \Omega_m \tag{33}$$

where Ω_{m} is the multiply-connected domain and the well-posed boundary conditions may be

$$u(x) = u_1 \text{ or } t(x) = t_1 \qquad x \text{ on } B_1(r=r_1) \tag{34}$$

$$u(x) = u_2 \text{ or } t(x) = t_2 \qquad x \text{ on } B_2(r=r_2) \tag{35}$$

where B_1 and B_2 are the boundaries of the interior and exterior circles, respectively. By collocating the point x on

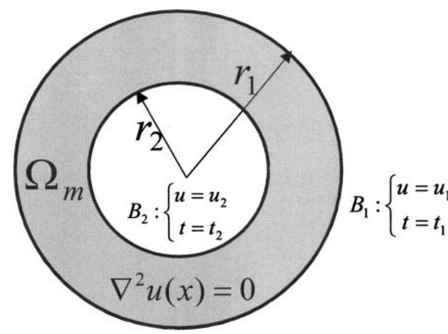


Fig. 3. Potential problem for an annular region.

the point (x_1 on r_1^+) and the point (x_2 on r_2^-), the dual boundary integral equations can be derived as shown below

$$\int_{B_1} T(s,x_1)u_1(s)\,\mathrm{d}B(s) - \int_{B_1} U(s,x_1)t_1(s)\,\mathrm{d}B(s) + \int_{B_2} T(s,x_1)u_2(s)\,\mathrm{d}B(s) - \int_{B_2} U(s,x_1)t_2(s)\,\mathrm{d}B(s) = 0, \quad x_1 \to r_1^+ \tag{36}$$

$$\int_{B_1} T(s,x_2)u_1(s)\,\mathrm{d}B(s) - \int_{B_1} U(s,x_2)t_1(s)\,\mathrm{d}B(s) + \int_{B_2} T(s,x_2)u_2(s)\,\mathrm{d}B(s) - \int_{B_2} U(s,x_2)t_2(s)\,\mathrm{d}B(s) = 0, \quad x_2 \to r_2^- \tag{37}$$

and

$$\int_{B_1} M(s,x_1)u_1(s)\,\mathrm{d}B(s) - \int_{B_1} L(s,x_1)t_1(s)\,\mathrm{d}B(s) + \int_{B_2} M(s,x_1)u_2(s)\,\mathrm{d}B(s) - \int_{B_2} L(s,x_1)t_2(s)\,\mathrm{d}B(s) = 0, \quad x_1 \to r_1^+ \tag{38}$$

$$\int_{B_1} M(s,x_2)u_1(s)\,\mathrm{d}B(s) - \int_{B_1} L(s,x_2)t_1(s)\,\mathrm{d}B(s) + \int_{B_2} M(s,x_2)u_2(s)\,\mathrm{d}B(s) - \int_{B_2} L(s,x_2)t_2(s)\,\mathrm{d}B(s) = 0, \quad x_2 \to r_2^+. \tag{39}$$

Discretizing the interior (B_2) and exterior (B_1) circles into $2N$ constant elements, we can transform Eq. (36)–(39) into matrix forms

$$\begin{bmatrix} T_{11} & U_{11} & T_{12} & U_{12} \\ T_{21} & U_{21} & T_{22} & U_{22} \end{bmatrix} \begin{Bmatrix} u_1 \\ t_1 \\ u_2 \\ t_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{40}$$

and

$$\begin{bmatrix} M_{11} & L_{11} & M_{12} & L_{12} \\ M_{21} & L_{21} & M_{22} & L_{22} \end{bmatrix} \begin{Bmatrix} u_1 \\ t_1 \\ u_2 \\ t_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{41}$$

where the first subscript 'i' in U_{ij}, T_{ij}, L_{ij} or M_{ij} denotes the position of collocation point (1 for B_1, and 2 for B_2), the second subscript 'j' identifies the boundary data, u_j or t_j. Similarly, we can obtain the eigenvalues for the influence submatrices, respectively. The results are shown in Table 1.

When the boundary condition of the potential problem is the Dirichlet type ($u_1 = \bar{u}_1$ and $u_2 = \bar{u}_2$ are specified), Eq. (40) and Eq. (41) reduce to

$$\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \end{Bmatrix} = -\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{Bmatrix} \tag{42}$$

and

$$\begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \end{Bmatrix} = -\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{Bmatrix}. \tag{43}$$

In order to obtain the determinant for the assembled matrices, we can decompose the submatrices U_{ij} and L_{ij} circulants into

$$[U_{ij}] = [R][\bar{U}_{ij}][R]^{-1} \tag{44}$$

Table 1
Eigenvalues for $[U_{ij}]$, $[T_{ij}]$, $[L_{ij}]$ and $[M_{ij}]$ in an annular problem ($U = \ln r$)

	$l = 0$	$l = \pm 1, \pm 2, \pm 3, \ldots, \pm N-1, N$
$\lambda_l^{[T_{11}]}$	0	$-\pi$
$\lambda_l^{[U_{11}]}$	$-2\pi r_1 \ln r_1$	$\pi \frac{r_1}{\lvert l\rvert}$
$\lambda_l^{[T_{12}]}$	0	$-\pi\left(\frac{r_2}{r_1}\right)^{\lvert l\rvert}$
$\lambda_l^{[U_{12}]}$	$-2\pi r_2 \ln r_1$	$\pi \frac{r_2}{\lvert l\rvert}\left(\frac{r_2}{r_1}\right)^{\lvert l\rvert}$
$\lambda_l^{[T_{21}]}$	2π	$\pi\left(\frac{r_2}{r_1}\right)^{\lvert l\rvert}$
$\lambda_l^{[U_{21}]}$	$-2\pi r_1 \ln r_1$	$\pi \frac{r_1}{\lvert l\rvert}\left(\frac{r_2}{r_1}\right)^{\lvert l\rvert}$
$\lambda_l^{[T_{22}]}$	2π	π
$\lambda_l^{[U_{22}]}$	$-2\pi r_2 \ln r_2$	$\pi \frac{r_2}{\lvert l\rvert}$
$\lambda_l^{[M_{11}]}$	0	$\pi \frac{\lvert l\rvert}{r_1}$
$\lambda_l^{[L_{11}]}$	2π	$-\pi$
$\lambda_l^{[M_{12}]}$	0	$\pi \frac{\lvert l\rvert}{r_1}\left(\frac{r_2}{r_1}\right)^{\lvert l\rvert}$
$\lambda_l^{[L_{12}]}$	$2\pi(r_2/r_1)$	$-\pi\left(\frac{r_2}{r_1}\right)^{\lvert l\rvert+1}$
$\lambda_l^{[M_{21}]}$	0	$\pi \frac{\lvert l\rvert}{r_2}\left(\frac{r_2}{r_1}\right)^{\lvert l\rvert}$
$\lambda_l^{[L_{21}]}$	0	$\pi\left(\frac{r_2}{r_1}\right)^{\lvert l\rvert-1}$
$\lambda_l^{[M_{22}]}$	0	$\pi \frac{\lvert l\rvert}{r_2}$
$\lambda_l^{[L_{22}]}$	0	π

Table 2
Degenerate scale for potential problems with an annular region using the singular integral formulation ($U = \ln r$)

B.C.	Degenerate scale
$u_1 = \bar{u}_1$, $u_2 = \bar{u}_2$	$r_1 = 1$
$u_1 = \bar{u}_1$, $t_2 = \bar{t}_2$	$r_1 = 1$
$u_2 = \bar{u}_2$, $t_1 = \bar{t}_1$	$r_1 = 1$

$$[L_{ij}] = [R][\bar{L}_{ij}][R]^{-1} \tag{45}$$

where R is an orthogonal matrix composed by $\{\phi_n\}$ [19]. The elements in the diagonal matrices $[\bar{U}_{ij}]$ and $[\bar{L}_{ij}]$ are the eigenvalues of the $[U_{ij}]$ and $[L_{ij}]$. Then the determinant of the assembled matrix

$$\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$$

is

$$\det\left\|\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}\right\| = \det\begin{vmatrix} \bar{U}_{11} & \bar{U}_{12} \\ \bar{U}_{21} & \bar{U}_{22} \end{vmatrix} \cdot \det|[R]^{-1}[R]|$$

$$= \det\begin{vmatrix} \bar{U}_{11} & \bar{U}_{12} \\ \bar{U}_{21} & \bar{U}_{22} \end{vmatrix} = \prod_{l=0}^{2N-1} \det\begin{vmatrix} \lambda_l^{[U_{11}]} & \lambda_l^{[U_{12}]} \\ \lambda_l^{[U_{21}]} & \lambda_l^{[U_{22}]} \end{vmatrix} \tag{46}$$

Similarly, we have

$$\det\left\|\begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}\right\| = \prod_{l=0}^{2N-1} \det\begin{vmatrix} \lambda_l^{[L_{11}]} & \lambda_l^{[L_{12}]} \\ \lambda_l^{[L_{21}]} & \lambda_l^{[L_{22}]} \end{vmatrix} \tag{47}$$

where $\lambda_l^{[A]}$ is the lth eigenvalue for matrix $[A]$. For the above Dirichlet problem, the determinant of the influence matrix is zero as $r_1 = 1$ since $\lambda_0^{[U_{11}]}\lambda_0^{[U_{22}]} - \lambda_0^{[U_{12}]}\lambda_0^{[U_{21}]} = 0$ in Eq. (46) if singular equation is used. Obviously, it yields a non-unique solution when r_1 is a specific magnitude which is equal to the degenerate scale. Table 2 shows the degenerate scale for the problems with different boundary conditions of the mixed type $u_1 = \bar{u}_1$, $t_2 = \bar{t}_2$ or $t_1 = \bar{t}_1$, $u_2 = \bar{u}_2$. According to the eigenspectrum in Eq. (25)–(28), we can avoid the degenerate scale problem and obtain the accurate solution by using the hypersingular equation Eq. (3) for the interior problem. However, it still yields a non-unique solution no matter what the radius r_1 is chosen for multiply-connected problem. It is surprising that Eq. (3) fails to solve the potential problems in an annular domain no matter what the boundary condition is the Dirichlet or mixed, and no matter what the radiuses, r_1 and r_2, are.

To deal with the degenerate scale problem, we can also superimpose a constant term c in the U kernel [20]. The corresponding eigenvalues for the submatrices are shown in Table 3. Thus, the degenerate scale using the modified kernel moves to e^{-c} (Table 4) instead of 1 using the original kernel.

Based on the zero eigenvalue check, we can determine the degenerate scale in the above sections. Now we will propose an alternative way to find the special geometry. If we superimpose the single-layer potential $\phi(s)$ along the boundary B, we have

$$u(x) = \int_B U(s,x)\Phi(s)\,dB(s) \tag{48}$$

$$t(x) = \int_B L(s,x)\Phi(s)\,dB(s) \tag{49}$$

Table 3
Eigenvalues for $[U_{ij}]$, $[T_{ij}]$, $[L_{ij}]$ and $[M_{ij}]$ in an annular problem ($U = \ln r + c$)

	$l = 0$	$l = \pm1, \pm2, \pm3, \ldots, \pm N-1, N$
$\lambda_l^{[T_{11}]}$	0	$-\pi$
$\lambda_l^{[U_{11}]}$	$-2\pi r_1(\ln r_1 + c)$	$\pi\frac{r_1}{\lvert l\rvert}$
$\lambda_l^{[T_{12}]}$	0	$-\pi\left(\frac{r_2}{r_1}\right)^{\lvert l\rvert}$
$\lambda_l^{[U_{12}]}$	$-2\pi r_2(\ln r_1 + c)$	$\pi\frac{r_2}{\lvert l\rvert}\left(\frac{r_2}{r_1}\right)^{\lvert l\rvert}$
$\lambda_l^{[T_{21}]}$	2π	$\pi\left(\frac{r_2}{r_1}\right)^{\lvert l\rvert}$
$\lambda_l^{[U_{21}]}$	$-2\pi r_1(\ln r_1 + c)$	$\pi\frac{r_1}{\lvert l\rvert}\left(\frac{r_2}{r_1}\right)^{\lvert l\rvert}$
$\lambda_l^{[T_{22}]}$	2π	π
$\lambda_l^{[U_{22}]}$	$-2\pi r_2(\ln r_2 + c)$	$\pi\frac{r_2}{\lvert l\rvert}$
$\lambda_l^{[M_{11}]}$	0	$\pi\frac{\lvert l\rvert}{r_1}$
$\lambda_l^{[L_{11}]}$	2π	$-\pi$
$\lambda_l^{[M_{12}]}$	0	$\pi\frac{\lvert l\rvert}{r_1}\left(\frac{r_2}{r_1}\right)^{\lvert l\rvert}$
$\lambda_l^{[L_{12}]}$	$2\pi(r_2/r_1)$	$-\pi\left(\frac{r_2}{r_1}\right)^{\lvert l\rvert+1}$
$\lambda_l^{[M_{21}]}$	0	$\pi\frac{\lvert l\rvert}{r_2}\left(\frac{r_2}{r_1}\right)^{\lvert l\rvert}$
$\lambda_l^{[L_{21}]}$	0	$\pi\left(\frac{r_2}{r_1}\right)^{\lvert l\rvert-1}$
$\lambda_l^{[M_{22}]}$	0	$\pi\frac{\lvert l\rvert}{r_2}$
$\lambda_l^{[L_{22}]}$	0	π

Table 4
Degenerate scales for potential problems with an annular region using the singular formulation ($U = \ln r + c$)

B.C.	Degenerate scale
$u_1 = \bar{u}_1$, $u_2 = \bar{u}_2$	$r_1 = e^{-c}$
$u_1 = \bar{u}_1$, $t_2 = \bar{t}_2$	$r_1 = e^{-c}$
$u_2 = \bar{u}_2$, $t_1 = \bar{t}_1$	$r_1 = e^{-c}$

where the single-layer density $\Phi(s)$ can be expressed by series form

$$\Phi(s) = \sum_{j=0}^{N} c_j \phi_j(s) \tag{50}$$

Similarly, we have

$$u(x) = \int_B T(s,x)\Psi(s)\, \mathrm{d}B(s) \tag{51}$$

$$t(x) = \int_B M(s,x)\Psi(s)\, \mathrm{d}B(s) \tag{52}$$

where the double-layer potential $\Psi(s)$ can be represented by series form

$$\Psi(s) = \sum_{j=0}^{N} d_j \psi_j(s) \tag{53}$$

The primary potential across the boundary is continuous in Eq. (48) and discontinuous in Eq. (51). The secondary field across the boundary is discontinuous in Eq. (49) and continuous in Eq. (52). We can obtain the primary and secondary fields as shown in Table 5. We find that the field in the interior domain resulting from the U kernel for ϕ_0 distribution is equal to zero everywhere when the radius ρ is equal to one. Therefore, the strength of this singularity distribution, c_0, cannot be determined. Then we observe that $\rho = 1$ is the degenerate scale in the circular or the annular problem. Similarly, we can obtain the other cases of degenerate scales as shown in Tables 6 and 7.

4. Numerical examples for circular and annular problems

4.1. Circular problem

We consider the interior potential problem subjected to a circular domain (Fig. 1) with the mixed type condition as follows

$$t(r,\theta) = \cos(\theta), \qquad r = a, \ -\pi < \theta < \frac{1}{2}\pi, \ (r,\theta) \in B_1 \tag{54}$$

$$u(r,\theta) = a\cos(\theta), \qquad r = a, \ \frac{1}{2}\pi < \theta < \pi, \ (r,\theta) \in B_2 \tag{55}$$

Table 5
The field responses in the domain for singularity distribution ϕ_n or ψ_n along the boundary B

Singularity distributed on the boundary B	Field response across the boundary B	Field variable	Related kernel	Domain of interest	$n = 0$ $\phi_n(s)$ or $\psi_n(s)$	$n = 1,2,\ldots,N$ $\phi_n(s)$ or $\psi_n(s)$
ϕ_n	u: continuous	u	U	D^i	$-\rho \ln \rho$	$\frac{\rho}{2n}\left(\frac{R}{\rho}\right)^n \cos n\theta$
				D^e	$-\rho \ln R$	$\frac{\rho}{2n}\left(\frac{\rho}{R}\right)^n \cos n\theta$
	t: discontinuous	t	L	D^i	0	$\frac{1}{2}\left(\frac{R}{\rho}\right)^{n-1} \cos n\theta$
				D^e	$-\frac{\rho}{R}$	$-\frac{1}{2}\left(\frac{\rho}{R}\right)^{n+1} \cos n\theta$
ψ_n	u: discontinuous	u	T	D^i	-1	$-\frac{1}{2}\left(\frac{R}{\rho}\right)^n \cos n\theta$
				D^e	0	$\frac{1}{2}\left(\frac{\rho}{R}\right)^n \cos n\theta$
	t: continuous	t	M	D^i	0	$-\frac{n}{2R}\left(\frac{R}{\rho}\right)^n \cos n\theta$
				D^e	0	$-\frac{n}{2R}\left(\frac{\rho}{R}\right)^n \cos n\theta$

Table 6
The degenerate scales in the direct BEM

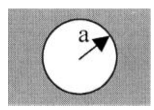
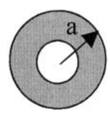
	Domain of interest	Singular integral equation (UT equation)	Hypersingular integral Equation (LM equation)
Simply-connected domain	Interior 	$a = 1$	None
	Exterior 	$a = 1$	a is arbitrary
Multiply-connected domain		$a = 1$	a is arbitrary

Table 7
The degenerate scales in the fictitious BEM

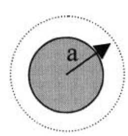
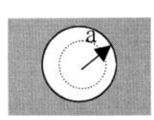
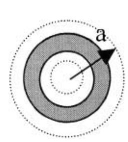
	Domain	Single layer potential method	Double layer potential method
Simply-connected	Interior 	$a = 1$	None
	Exterior 	$a = 1$	a is arbitrary
Multiply-connected		$a = 1$	a is arbitrary

Table 8
The results for the potential problems with a circular region

		u (error %) ($\theta = 0$)		
		Analytical solution	Singular integral formulation	Hypersingular integral formulation
$2N = 20$	$a = 1.012$ (degenerate scale)	1.000	0.667 (33.3%)	0.975 (2.5%)
	$a = 2$	2.000	2.015 (0.75%)	1.950 (2.5%)

In the BEM mesh, 10 elements are distributed uniformly on the B_1 boundary, and ten elements on the B_2. The analytical solutions are $u(r,\theta) = a\cos(\theta)$ and $t(r,\theta) = \cos(\theta)$. The numerical results are shown in Table 8. The degenerate scale occurs numerically at $a = 1.012$ instead of the value $a = 1$ analytically. It is found that the results of conventional BEM have a great error of 33% in case of degenerate scale. Nevertheless, the hypersingular formulation obtains the acceptable results within 3%.

4.2. Annular problem

Given an annular problem with the mixed boundary conditions as follows:

$$t_1(x) = \frac{100}{r_1 \ln(r_2/r_1)}, \qquad x \text{ on } B_1 \tag{56}$$

$$u_2(x) = 100, \qquad x \text{ on } B_2 \tag{57}$$

the analytical solution is [21]

$$u = u_1 + \frac{\ln(r/r_1)}{\ln(r_2/r_1)(u_2 - u_1)}, \qquad r_2 \le r \le r_1 \tag{58}$$

where $2r_2 = r_1$ and u_1 is the potential on B_1. The same number of elements on the internal and exterior circles are adopted. The results are listed in Table 9, where $2N$ denotes the number of elements. From Table 9, we find that the errors of u_1 and t_2 are very large in the case of degenerate scale ($r_1 = 1$) using the singular formulation. By using the hypersingular Eq. (3) to solve the potential problem with an annular region, it is found that no good results can be obtained even though the radius is not a degenerate scale ($r_1 = 2$). This indicates that hypersingular formulation fails for multiply-connected problems as predicting theoretically in Eq. (47). But we can obtain accurate results by adding a constant term, 10, in the fundamental solution using the fewer number of elements. Also, the results of NSBIE method in reference of Ref. [7] are compared with and the numerical instability can be dealt with by the proposed method.

5. Conclusions

In this paper, we have proved why the degenerate scale is embedded in the BEM formulation by two approaches. In the first approach, degenerate kernels and theory of circulants were employed. For the second method, a singularity distribution along a degenerate scale of boundary is found to have zero field. For the simply connected domain of a circular domain with the Dirichlet boundary condition, the radius of one is a degenerate case if the singular integral equation is used. To overcome the problem, hypersingular equation can be adopted. In case of the exterior problem, the radius of one is a degenerate case if the singular integral equation is used. No matter what the radius is, it will be the degenerate case if the hypersingular integral equation is

Table 9
Numerical results for the potential problems with an annular region

	r_1	Analytical		Eq. (2) ($U = \ln r$)		Eq. (2) ($U = \ln r + 10$)		Eq. (3) ($U = \ln r$)		Eq. (3) ($U = \ln r + 10$)		NSBIE [4]	
		u_1	t_2	u_1	t_2	u_1	t_2	u_1	t_2	u_1	t_2	u_1	t_2
$2N = 16$	1.0	0	288.54	26.30	220.61	3.80	289.08	407805300	171.78	−4947840000	1113.980	−3.94	289.54
	2.0	0	144.27	1.10	148.64	3.81	144.52	955828500	−0.94	1295667000	186.015	−2.49	142.27
$2N = 48$	1.0	0	288.54	19.10	234.12	0.32	288.55	−867836400	834.962	−867835600	834.962	−0.52	289.53
	2.0	0	144.27	0.10	144.61	0.32	144.31	5757387000	−1371.00	5757387000	−1371.00	−0.52	144.27

used. For the multiply-connected domain with an annular region, outer radius of one results in a nonunique solution if the singular formulation is employed, no matter what the inner radius is for any types of boundary conditions. The hypersingular formulation fails for the multiply-connected problems no matter what the inner or outer radius is. The hypersingular equation was successfully applied to solve the degenerate scale problem of interior problem as shown in the illustrative example. Also, another example has been shown that we can deal with the degenerate scale problem and obtain the accurate solution by adding a constant in the fundamental solution.

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