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Derivation of stiffness and flexibility for rods and beams by using dual integral equations

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Abstract

In this paper, the dual boundary integral formulation is used to determine the stiffness and flexibility matrices for rods and beams by using the direct and indirect methods. Since any two boundary integral equations can be chosen for the beam problem, six options by choosing two from the four equations in dual formulation can be considered. It is found that only two options, either displacement-slope (single- and double-layer) or displacement-moment (single- and triple-layer) formulations in the direct (indirect) method can yield the stiffness matrix except the degenerate scale and a special fundamental solution. Not only rigid body mode in physics but also spurious mode in numerical implementation are found in the formulation by using SVD updating term and document, respectively. Both the rigid body mode and spurious mode can be extracted out from the right and left unitary vectors of the influence matrices by using SVD.

Keywords: Dual boundary integral equations; Stiffness; Flexibility; Rigid body mode; Singular value decomposition

1. Introduction

Concept of stiffness and flexibility in mechanics of material is well-known for undergraduate students [1]. For graduate students, they revisited the stiffness and flexibility matrices in the finite element course [2]. Rigid body modes occur for free–free bodies in physics as well as spurious modes appear for degenerate scales in numerical implementation [3]. Felippa et al. [4] constructed the free–free flexibility matrices by using the generalized inverse of stiffness through the concept of finite element method (FEM). Besides, Dumont [5] also studied the stiffness by using generalized inverse of matrices through variational boundary element formulation. A note to construct the relationship of the stiffness matrix between the FEM and boundary element method (BEM) was published by Q2 Pozrikidis in 2006 [6]. An unified formulation to derive the stiffness and flexibility matrices is not trivial and is the main concern of the present paper.

Applications of direct and indirect BEMs to solve rod and beam problems were reported in the textbook of Banerjee and Butterfield [7] and Hartmann [8]. For the free Q3 vibration and forced flexural vibrations of beams are numerically studied by using the direct BEM [9]. The flexural–torsion coupling vibration problem of Euler–Bernoulli beams of arbitrarily shaped cross section was also solved by BEM [10]. Shearing stresses of two-material curved beams were solved by using integral equations [11]. The beam problems subject to the transverse shear loading were investigated by using two-dimensional fundamental solution [12,13]. However, only conventional BEM instead of dual BEM was used in the direct method. Besides, only single- and double-layer potentials were adopted instead of higher-order layer potentials in the indirect method. Here, we will complete the possible alternatives to solve rod and beam problems.

DBIEs were developed by Hong and Chen [14] for 2-D and 3-D elasticity problems. This formulation can be employed to formulate the one-dimensional problem of rod and beam. Since DBIEs provide more equations than the conventional one, we may wonder the role of additional equations in mathematical aspects. Regarding to the role of dual formulation in computational mechanics, readers can consult with the review article [15].

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In the recent years, SVD technique has been applied to solve problems of continuum mechanics [16], fictitious-frequency problems [17], and spurious eigenvalues [18]. Based on these successful experiences, SVD updating technique will be employed to study the mathematical structure of the influence matrices derived by using dual formulation.

In this paper, rank deficiency for the influence matrices is also our concern. The rigid body mode and spurious mode in the dual formulation will be examined through SVD technique. The relation between zero singular values of updating matrices (updating terms and updating document) and nontrivial modes (rigid body mode and spurious mode) will be constructed. Both the rod and beam structures are considered as illustrative examples.

2. Dual boundary integral formulation for rod problems

Let us consider the rod problem as shown in Fig. 1(a) and (b). The governing equation for a rod is

$$\frac{d^2u(x)}{dx^2} = 0, \quad x \in D, \tag{1}$$

where $u(x)$ is the axial displacement of the rod, D is the domain between of $0<x<L$. By introducing the auxiliary system of the fundamental solution, we have

$$\frac{\partial^2 U(x,s)}{\partial x^2} = \delta(x-s), \quad -\infty<x<\infty, \tag{2}$$

where δ is the Dirac-delta function, x is the field point, and s is the source point. For simplicity, the fundamental solution is selected as

$$U(x,s) = \tfrac{1}{2}|x-s| \tag{3}$$

and can be expressed in terms of degenerate kernel in Table 1 as

$$U(x,s) = \begin{cases} \frac{1}{2}(x-s), & x>s, \\ \frac{1}{2}(s-x), & x<s. \end{cases} \tag{4}$$

By multiplying the auxiliary system in Eq. (3) with respect to the governing equation and integrating by parts, we have the boundary integral equation as

$$u(s) = \left[u(x)\frac{\partial U(x,s)}{\partial x} - u'(x)U(x,s) \right] \Bigg|_{x=0}^{x=L}. \tag{5}$$

By differentiating with respect to the source point s, in Eq. (5), the dual boundary integral equations are shown below:

$$u(s) = [T(x,s)u(x) - U(x,s)t(x)]\big|_{x=0}^{x=L}, \tag{6}$$

$$t(s) = [M(x,s)u(x) - L(x,s)t(x)]\big|_{x=0}^{x=L}, \tag{7}$$

where $t(s) = du(s)/ds$, and the kernels are defined as

$$T(x,s) = \frac{\partial U(x,s)}{\partial n_x}, \tag{8}$$

$$L(x,s) = \frac{\partial U(x,s)}{\partial n_s}, \tag{9}$$

$$M(x,s) = \frac{\partial U(x,s)}{\partial n_x \partial n_s}. \tag{10}$$

By approaching s to 0^+ and L^- into Eq. (6), we have

$$\left[\frac{1}{2}u(0) - \frac{1}{2}u(L)\right] = -\frac{L}{2}t(L), \tag{11}$$

$$\left[\frac{1}{2}u(L) - \frac{1}{2}u(L)\right] = \frac{L}{2}t(0). \tag{12}$$

By assembling Eqs. (11) and (12) into matrix form, we have

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} u(0) \\ u(L) \end{bmatrix} = L \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} t(0) \\ t(L) \end{bmatrix}. \tag{13}$$

By approaching s to 0^+ and L^- into Eq. (7), we have

$$\tfrac{1}{2}t(0) - \tfrac{1}{2}t(L) = 0, \tag{14}$$

$$-\tfrac{1}{2}t(0) + \tfrac{1}{2}t(L) = 0. \tag{15}$$

Similarly, Eqs. (14) and (15) can be written as

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u(0) \\ u(L) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} t(0) \\ t(L) \end{bmatrix}. \tag{16}$$

Eqs. (13) and (16) are denoted as

$$[A]\begin{bmatrix} u(0) \\ u(L) \end{bmatrix} = [B]\begin{bmatrix} t(0) \\ t(L) \end{bmatrix}, \tag{17}$$

where $[A]$ and $[B]$ are arranged in Table 2. Also, the ranks of influence matrices are calculated.

Table 1
Degenerate kernels for rod problems

Domain	Kernels			
	$U(x,s)$	$T(x,s)$	$L(x,s)$	$M(x,s)$
$x>s$	$\frac{1}{2}(x-s)$	$\frac{1}{2}$	$-\frac{1}{2}$	0
$x<s$	$\frac{1}{2}(s-x)$	$-\frac{1}{2}$	$\frac{1}{2}$	0

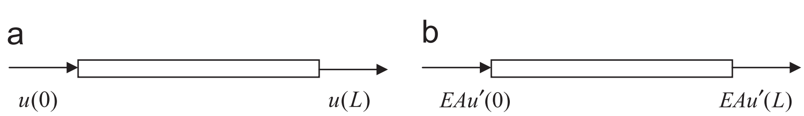

Fig. 1. (a) Generalized displacement and (b) generalized force.

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Table 2
Stiffness matrix for rod problems using dual BEM

Equation	[A]	[B]	$[K_B]$	$[K_F]$
(11) and (12)	$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} Rank(A)=1$	$\begin{bmatrix} 0 & -\frac{1}{2}L \\ \frac{1}{2}L & 0 \end{bmatrix} Rank(B)=2$	$\frac{1}{L}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} Rank(K_B)=1$	$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} Rank(K_F)=1$
(14) and (15)	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} Rank(A)=0$	$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} Rank(B)=1$	NA	NA

2.1. The stiffness matrix of rods

We utilize the simple structure in Fig. 2 to define the notations of generalized displacement and generalized force to connect the FEM notations.

For the degree of freedom (DOF) of generalized displacements, we have

$$\begin{bmatrix} u(0) \\ u(L) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_L \end{bmatrix} = T_{ru} \begin{bmatrix} u_0 \\ u_L \end{bmatrix}. \tag{18}$$

Also, we can obtain

$$\begin{bmatrix} t(0) \\ t(L) \end{bmatrix} = \frac{1}{EA} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_0 \\ p_L \end{bmatrix} = T_{rt} \begin{bmatrix} p_0 \\ p_L \end{bmatrix} \tag{19}$$

for the DOF of generalized forces.

By substituting Eqs. (18) and (19) into Eq. (17), we have

$$[A]T_{ru} \begin{bmatrix} u_0 \\ u_L \end{bmatrix} = [B]T_{rt} \begin{bmatrix} p_0 \\ p_L \end{bmatrix} \tag{20}$$

in which

$$[A]T_{ru} = [A_T], \tag{21}$$

$$[B]T_{rt} = [B_T]. \tag{22}$$

The relation between generalized displacement and generalized force is shown below

$$[A_T] \begin{bmatrix} u_0 \\ u_L \end{bmatrix} = [B_T] \begin{bmatrix} p_0 \\ p_L \end{bmatrix}. \tag{23}$$

The stiffness matrices are defined as

$$\begin{bmatrix} t(0) \\ t(L) \end{bmatrix} = [K_B] \begin{bmatrix} u(0) \\ u(L) \end{bmatrix}, \tag{24}$$

$$\begin{bmatrix} p_0 \\ p_L \end{bmatrix} = [K_F] \begin{bmatrix} u_0 \\ u_L \end{bmatrix}. \tag{25}$$

It is found that Eqs. (14) and (15) fail in constructing the stiffness matrix due to the rank deficiency of [B], and the stiffness matrix can be expressed as the same form of that derived by FEM as shown in Table 2.

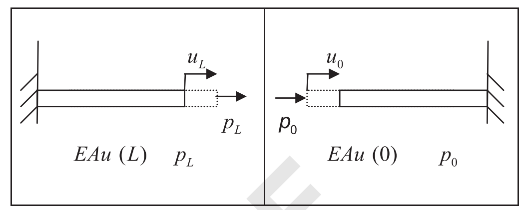

Fig. 2. Notations of generalized displacements and generalized forces in a simple structure. Q1

2.2. The flexibility matrix of rods

The flexibility matrix cannot be obtained, because the [A] matrix is singular in Table 2. We utilize the SVD technique to calculate $[A^{-1}]$ and try to get the flexibility matrix of the rod. By employing SVD technique, we have

$$[A] = [\Phi][\Sigma][\Psi]^{\mathrm{T}}, \tag{26}$$

where $[\Phi]$ and $[\Psi]$ are the right and left unitary matrices, and $[\Sigma]$ is a diagonal matrix composed of singular value. It is found that

$$[\Phi] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}, \tag{27}$$

$$[\Sigma] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \tag{28}$$

$$[\Psi] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}. \tag{29}$$

The matrix [A] can be expressed as

$$[A] = \sum_{i=1}^{r} [u_i][\sigma_i][v_i]^{\mathrm{T}}, \tag{30}$$

where σ_i is the singular value, $[u_i]$ and $[v_i]$ are the left and right unitary vectors, respectively. The inverse of the [A] matrix is

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Table 3
Flexibility matrix for rod problems using the dual BEM

Equation	[A]	[B]	$[F_B]$	$[F_F]$
(11), (12)	$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} Rank(A)=1$	$\begin{bmatrix} 0 & -\frac{1}{2}L \\ \frac{1}{2}L & 0 \end{bmatrix} Rank(B)=2$	$\frac{L}{4}\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} Rank(F_B)=1$	$\frac{L}{4EA}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} Rank(F_F)=1$
(14), (15)	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} Rank(A)=0$	$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} Rank(B)=1$	NA	NA

$$[A]^{-1} = \sum_{i=1}^{r}[v_i][\sigma_i]^{-1}[u_i]^{\mathrm{T}} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}_{2\times1} \begin{bmatrix} \frac{1}{1} \end{bmatrix}_{1\times1} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}_{1\times2} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix}. \tag{31}$$

The flexibility matrices are defined as

$$\begin{bmatrix} u(0) \\ u(L) \end{bmatrix} = [F_B]\begin{bmatrix} t(0) \\ t(L) \end{bmatrix}, \tag{32}$$

$$\begin{bmatrix} u_0 \\ u_L \end{bmatrix} = [F_F]\begin{bmatrix} p_0 \\ p_L \end{bmatrix}. \tag{33}$$

It is found that Eqs. (14) and (15) fail in constructing the flexibility matrix due to rank deficiency of [B], and the flexibility matrix can be expressed as the same form of that derived by FEM as shown in Table 3.

3. Dual boundary integral formulation for beam problems

Based on the successful experience of deriving the stiffness for a rod using BEM, we extend the one-dimensional Laplace equation to biharmonic equation for a beam. Let us consider the Euler beam problems as shown Fig. 3(a) and (b).

The governing equation for the Euler beam is

$$\frac{\mathrm{d}^4 u(x)}{\mathrm{d}x^4} = 0, \quad x \in D, \tag{34}$$

where L is the length of the beam, $u(x)$ is the lateral displacement, D is the domain between $0<x<L$. By introducing one auxiliary system of the fundamental solution

$$\frac{\partial^4 U(x,s)}{\partial x^4} = \delta(x-s), \quad -\infty<x<\infty, \tag{35}$$

where δ is the Dirac-delta function, x is field point, and s is the source point. For simplicity, the fundamental solution is selected as

$$U(x,s) = \frac{1}{12}|x-s|^3 \tag{36}$$

and can be expressed in terms of degenerate kernel as

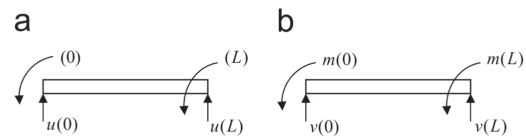

Fig. 3. (a) Generalized displacement DOF (u, θ) and (b) generalized force DOF (m, v).

$$U(x,s) = \begin{cases} \frac{1}{12}(x-s)^3, & x>s, \\ \frac{1}{12}(s-x)^3, & x<s. \end{cases} \tag{37}$$

By multiplying the auxiliary system in Eq. (36) with respect to the governing equation and integrating by parts, we have the boundary integral equation as

$$u(s) = \int_0^L \left[u(x)\frac{\partial^4 U(x,s)}{\partial x^4} - U(x,s)\frac{\partial^4 u(x)}{\partial x^4} \right] \mathrm{d}x. \tag{38}$$

The boundary integral equation is derived as

$$u(s) = \left[u(x)\frac{\partial^3 U(x,s)}{\partial x^3} - u'(x)\frac{\partial^2 U(x,s)}{\partial x^2} + u''(x)\frac{\partial U(x,s)}{\partial x} - u'''(x)U(x,s) \right]\Bigg|_{x=0}^{x=L}. \tag{39}$$

3.1. Direct method

By rewriting the displacement field, we have

$$u(s) = [-U(x,s)u'''(x) + \Theta(x,s)u''(x) - M(x,s)u'(x) + V(x,s)u(x)]\big|_{x=0}^{x=L}. \tag{40}$$

By differentiating Eq. (40) with respect to x, the displacement, the slope, moment and shear force fields can be obtained

$$u'(s) = [-U_\theta(x,s)u'''(x) + \Theta_\theta(x,s)u''(x) - M_\theta(x,s)u'(x) + V_\theta(x,s)u(x)]\big|_{x=0}^{x=L}, \tag{41}$$

$$u''(s) = [-U_m(x,s)u'''(x) + \Theta_m(x,s)u''(x) - M_m(x,s)u'(x) + V_m(x,s)u(x)]\big|_{x=0}^{x=L}, \tag{42}$$

$$u'''(s) = [-U_v(x,s)u'''(x) + \Theta_v(x,s)u''(x) - M_v(x,s)u'(x) + V_v(x,s)u(x)]\big|_{x=0}^{x=L}, \tag{43}$$

where $u(s)$ is the deflection, $\theta(s)$ is the slope, $m(s)$ is the moment and $v(s)$ is the shear force, respectively, and the relations of the 16 kernels are shown in Fig. 4. Degenerate kernels of the 16 kernels in a one-dimensional biharmonic problem are shown in Tables 4–7.

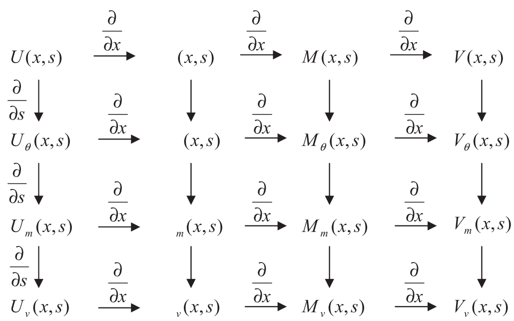

Fig. 4. Differential operators for the 16 kernels of the Euler beam.

Table 4
Degenerate kernels for beam problems (U)

Domain	Kernels			
	$U(x,s)$	$U_\theta(x,s)$	$U_m(x,s)$	$U_v(x,s)$
$x>s$	$\frac{(x-s)^3}{12}$	$-\frac{(x-s)^2}{4}$	$\frac{x-s}{2}$	$-\frac{1}{2}$
$x<s$	$-\frac{(x-s)^3}{12}$	$\frac{(x-s)^2}{4}$	$\frac{s-x}{2}$	$\frac{1}{2}$

Table 5
Degenerate kernels for beam problems (Θ)

Domain	Kernels			
	$\Theta(x,s)$	$\Theta_\theta(x,s)$	$\Theta_m(x,s)$	$\Theta_v(x,s)$
$x>s$	$\frac{(x-s)^2}{4}$	$\frac{s-x}{2}$	$\frac{1}{2}$	0
$x<s$	$-\frac{(x-s)^2}{4}$	$\frac{x-s}{2}$	$-\frac{1}{2}$	0

Table 6
Degenerate kernels for beam problems (M)

Domain	Kernels			
	$M(x,s)$	$M_\theta(x,s)$	$M_m(x,s)$	$M_v(x,s)$
$x>s$	$\frac{x-s}{2}$	$-\frac{1}{2}$	0	0
$x<s$	$\frac{s-x}{2}$	$\frac{1}{2}$	0	0

Table 7
Degenerate kernels for beam problems (V)

Domain	Kernels			
	$V(x,s)$	$V_\theta(x,s)$	$V_m(x,s)$	$V_v(x,s)$
$x>s$	$\frac{1}{2}$	0	0	0
$x<s$	$-\frac{1}{2}$	0	0	0

By approaching s to 0^+ and L^- into Eq. (40), we have

$$\frac{1}{2}u(0)-\frac{1}{2}u(L)+\frac{L}{2}u'(L)=\frac{L^2}{4}u''(L)-\frac{L^3}{12}u'''(L), \tag{44}$$

$$-\frac{1}{2}u(0)+\frac{1}{2}u(L)-\frac{L}{2}u'(0)=\frac{L^2}{4}u''(0)+\frac{L^3}{12}u'''(0). \tag{45}$$

By approaching s to 0^+ and L^- into Eq. (41), we have

$$\frac{1}{2}u'(0)-\frac{1}{2}u'(L)=-\frac{L}{2}u''(L)+\frac{L^2}{4}u'''(L), \tag{46}$$

$$-\frac{1}{2}u'(0)+\frac{1}{2}u'(L)=\frac{L}{2}u''(0)+\frac{L^2}{4}u'''(0). \tag{47}$$

By approaching s to 0^+ and L^- into Eq. (42), we have

$$-\frac{1}{2}u''(0)+\frac{1}{2}u''(L)-\frac{L}{2}u'''(L)=0, \tag{48}$$

$$\frac{1}{2}u''(0)-\frac{1}{2}u''(L)+\frac{L}{2}u'''(0)=0. \tag{49}$$

By approaching s to 0^+ and L^- into Eq. (43), we have

$$-\tfrac{1}{2}u'''(0)+\tfrac{1}{2}u'''(L)=0, \tag{50}$$

$$\tfrac{1}{2}u'''(0)-\tfrac{1}{2}u'''(L)=0. \tag{51}$$

Any two boundary integral equations can be chosen, six options can be considered. We utilize the degenerate kernel expansion in Tables 4–7 and substitute them into the two boundary integral equations which are chosen. By approaching s to 0^+ and L^-, we have the matrix form as follows:

$$[A]\begin{bmatrix}u(0)\\u'(0)\\u(L)\\u'(L)\end{bmatrix}=[B]\begin{bmatrix}u'''(0)\\u''(0)\\u'''(L)\\u''(L)\end{bmatrix}, \tag{52}$$

where $[A]$ and $[B]$ are obtained through the six formulations ($u-\theta$, $u-m$, $u-v$, $\theta-m$, $\theta-v$, $m-v$) as shown in Table 8. Also, the ranks of influence matrices are calculated.

3.1.1. The stiffness matrix of the Euler beam

We utilize a simple structure in Fig. 5 to define the notations of generalized displacement and generalized force to connect the FEM notations.

For the DOF of generalized displacements, we have

$$\begin{bmatrix}u(0)\\u'(0)\\u(L)\\u'(L)\end{bmatrix}=\begin{bmatrix}-1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix}\begin{bmatrix}u_0\\\theta_0\\u_L\\\theta_L\end{bmatrix}=T_{bu}\begin{bmatrix}u_0\\\theta_0\\u_L\\\theta_L\end{bmatrix}, \tag{53}$$

since $u(x)$ is defined downward.

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Table 8
Stiffness matrix for the Euler beam by using the direct method

Eqs.	[A]	[B]	[K_B]	[K_F]
u–θ (Eqs. (40) and (41))	$\begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} & \frac{L}{2} \\ -\frac{1}{2} & -\frac{L}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} Rank(A)=2$	$L^3 \begin{bmatrix} 0 & 0 & -\frac{1}{12} & \frac{1}{4L} \\ \frac{1}{12} & \frac{1}{4L} & 0 & 0 \\ 0 & 0 & \frac{1}{4L} & -\frac{1}{2L^2} \\ \frac{1}{4L} & \frac{1}{2L^2} & 0 & 0 \end{bmatrix} Rank(B)=4$	$\frac{1}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ -6L & -4L^2 & 6L & -2L^2 \\ 12 & 6L & -12 & 6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} Rank(K_B)=2$	$\frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} Rank(K_F)=2$
u–m (Eqs. (40) and (42))	$\begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} & \frac{L}{2} \\ -\frac{1}{2} & -\frac{L}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} Rank(A)=2$	$L^3 \begin{bmatrix} 0 & 0 & -\frac{1}{12} & \frac{1}{4L} \\ \frac{1}{12} & \frac{1}{4L} & 0 & 0 \\ 0 & \frac{-1}{2L^3} & \frac{-1}{2L^2} & \frac{1}{2L^3} \\ \frac{1}{2L^2} & \frac{1}{2L^3} & 0 & \frac{-1}{2L^3} \end{bmatrix} Rank(B)=4$	$\frac{1}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ -6L & -4L^2 & 6L & -2L^2 \\ 12 & 6L & -12 & 6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} Rank(K_B)=2$	$\frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} Rank(K_F)=2$
u–v (Eqs. (40) and (43))	$\begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} Rank(A)=2$	$L^3 \begin{bmatrix} 0 & 0 & -\frac{1}{12} & \frac{1}{4L} \\ \frac{1}{12} & \frac{1}{4L} & 0 & 0 \\ \frac{-1}{2L^3} & 0 & \frac{1}{2L^3} & 0 \\ \frac{1}{2L^3} & 0 & \frac{-1}{2L^3} & 0 \end{bmatrix} Rank(B)=3$	NA	NA
θ–m (Eqs. (41) and (42))	$\begin{bmatrix} 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} Rank(A)=1$	$L^3 \begin{bmatrix} 0 & 0 & \frac{1}{4L} & \frac{-1}{2L^2} \\ \frac{1}{4L} & \frac{1}{2L^2} & 0 & 0 \\ 0 & \frac{-1}{2L^3} & \frac{-1}{2L^2} & \frac{1}{2L^3} \\ \frac{1}{2L^2} & \frac{1}{2L^3} & 0 & \frac{-1}{2L^3} \end{bmatrix} Rank(B)=3$	NA	NA
θ–v (Eqs. (41) and (43))	$\begin{bmatrix} 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} Rank(A)=1$	$L^3 \begin{bmatrix} 0 & 0 & \frac{1}{4L} & -\frac{1}{2L^2} \\ \frac{1}{4L} & \frac{1}{2L^2} & 0 & 0 \\ -\frac{1}{2L^3} & 0 & \frac{1}{2L^3} & 0 \\ \frac{1}{2L^3} & 0 & -\frac{1}{2L^3} & 0 \end{bmatrix} Rank(B)=3$	NA	NA
m–v (Eqs. (42) and (43))	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} Rank(A)=0$	$L^3 \begin{bmatrix} 0 & -\frac{1}{2L^3} & -\frac{1}{2L^2} & \frac{1}{2L^3} \\ \frac{1}{2L^2} & \frac{1}{2L^3} & 0 & -\frac{1}{2L^3} \\ -\frac{1}{2L^3} & 0 & \frac{1}{2L^3} & 0 \\ \frac{1}{2L^3} & 0 & -\frac{1}{2L^3} & 0 \end{bmatrix} Rank(B)=2$	NA	NA

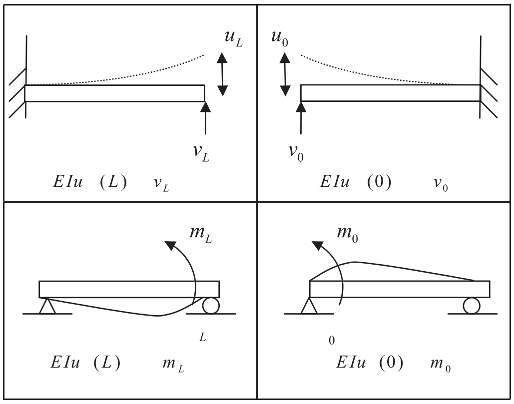

Fig. 5. Notations of generalized displacements and generalized forces in simple structure.

Also, we can obtain

$$\begin{bmatrix} u'''(0) \\ u''(0) \\ u'''(L) \\ u''(L) \end{bmatrix} = \frac{1}{EI}\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} v_0 \\ m_0 \\ v_L \\ m_L \end{bmatrix} = T_{bt}\begin{bmatrix} v_0 \\ m_0 \\ v_L \\ m_L \end{bmatrix} \tag{54}$$

for the DOF of generalized forces.

By substituting Eqs. (53) and (54) into Eq. (52), we have

$$[A]T_{bu}\begin{bmatrix} u_0 \\ \theta_0 \\ u_L \\ \theta_L \end{bmatrix} = [B]T_{bt}\begin{bmatrix} v_0 \\ m_0 \\ v_L \\ m_L \end{bmatrix} \tag{55}$$

in which

$$[A]T_{bu} = [A^{\mathrm{T}}], \tag{56}$$

$$[B]T_{bt} = [B^{\mathrm{T}}]. \tag{57}$$

The relation between generalized displacement and generalized force is shown below as

$$[A^{\mathrm{T}}]\begin{bmatrix} u_0 \\ \theta_0 \\ u_L \\ \theta_L \end{bmatrix} = [B^{\mathrm{T}}]\begin{bmatrix} v_0 \\ m_0 \\ v_L \\ m_L \end{bmatrix}. \tag{58}$$

where the stiffness matrices are defined as

$$\begin{bmatrix} u'''(0) \\ u''(0) \\ u'''(L) \\ u''(L) \end{bmatrix} = [K_B]\begin{bmatrix} u(0) \\ u'(0) \\ u(L) \\ u'(L) \end{bmatrix}, \tag{59}$$

$$\begin{bmatrix} v_0 \\ m_0 \\ v_L \\ m_L \end{bmatrix} = [K_F]\begin{bmatrix} u_0 \\ \theta_0 \\ u_L \\ \theta_L \end{bmatrix}. \tag{60}$$

It is found that only two combinations of Eqs. (40) and (41) and Eqs. (40) and (42) can construct the stiffness matrix, and the stiffness matrix can be expressed as the same form of that derived by FEM as shown in Table 8. Other approaches fail to obtain the stiffness matrix due to the rank deficiency of matrix [B].

3.1.2. The flexibility matrix of the Euler beam

The flexibility matrix cannot be obtained, because the [A] matrix is singular in Table 8. We utilize the SVD method to calculate $[A]^{-1}$ and get the flexibility matrix of the beam. By employing SVD technique, we have the Eq. (26). The [A] matrix can be expressed as Eq. (30).

The inverse of the [A] matrix is expressed as Eq. (31). For the $u-\theta$ formulation, we obtain

$$[A]^{-1} = \begin{bmatrix} \frac{2(-4+L^2)}{(4+L^2)^2} & \frac{2(-4+L^2)}{(4+L^2)^2} & \frac{2L(-6+L^2)}{3(4+L^2)^2} & \frac{-2L(-6+L^2)}{3(4+L^2)^2} \\ \frac{-(12+4L^2)}{4L+L^3} & \frac{4(12+5L^2+L^4)}{L(4+L^2)^2} & \frac{-2(24+15L^2+2L^4)}{3(4+L^2)^2} & \frac{-2(24+9L^2+2L^4)}{3(4+L^2)^2} \\ \frac{-2(-4+L^2)}{(4+L^2)^2} & \frac{-2(-4+L^2)}{(4+L^2)^2} & \frac{-2L(-6+L^2)}{3(4+L^2)^2} & \frac{2L(-6+L^2)}{3(4+L^2)^2} \\ \frac{4(12+5L^2+L^4)}{L(4+L^2)^2} & \frac{-(12+4L^2)}{4L+L^3} & \frac{2(24+9L^2+2L^4)}{3(4+L^2)^2} & \frac{2(24+15L^2+2L^4)}{3(4+L^2)^2} \end{bmatrix}. \tag{61}$$

The flexibility matrices are defined as

$$\begin{bmatrix} u(0) \\ u'(0) \\ u(L) \\ u'(L) \end{bmatrix} = [F_B]\begin{bmatrix} u'''(0) \\ u''(0) \\ u'''(L) \\ u''(L) \end{bmatrix}, \tag{62}$$

$$\begin{bmatrix} u_0 \\ \theta_0 \\ u_L \\ \theta_L \end{bmatrix} = [F_F]\begin{bmatrix} v_0 \\ m_0 \\ v_L \\ m_L \end{bmatrix}. \tag{63}$$

It is found that only two combinations Eqs. (40) and (41) and Eqs. (40) and (42) can construct the flexibility matrix, and the flexibility matrix can be expressed as the same form of that derived by FEM as shown in Table 8.

3.2. Indirect method

Instead of choosing two equations from the dual formulation in the direct BEM, we can also adopt two potentials from single, double, triple and quadruple potentials as denoted by $U-\Theta$, $U-M$, $U-V$, $\Theta-M$, $\Theta-V$, and $M-V$ formulations.

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(1) Single and double-layer approach ($U-\Theta$)

$$u(s) = U(0,s)\phi_0 + U(L,s)\phi_L + \Theta(0,s)\psi_0 + \Theta(L,s)\psi_L, \tag{64}$$

where ϕ_0, ϕ_L, ψ_0 and ψ_L are fictitious densities.

By approaching s to L^- and to 0^+, we have

$$\begin{bmatrix} 0 & \frac{L^3}{12} & 0 & \frac{L^2}{4} \\ 0 & \frac{-L^2}{4} & 0 & \frac{-L}{2} \\ \frac{-L^3}{12} & 0 & \frac{L^2}{4} & 0 \\ \frac{-L^2}{4} & 0 & \frac{L}{2} & 0 \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_L \\ \psi_0 \\ \psi_L \end{bmatrix} = \begin{bmatrix} u(0) \\ \theta(0) \\ u(L) \\ \theta(L) \end{bmatrix}, \tag{65}$$

$$\begin{bmatrix} \frac{-1}{2} & \frac{-1}{2} & 0 & 0 \\ 0 & \frac{L}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{-1}{2} & 0 & 0 \\ \frac{-L}{2} & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_L \\ \psi_0 \\ \psi_L \end{bmatrix} = \begin{bmatrix} v(0) \\ m(0) \\ v(L) \\ m(L) \end{bmatrix}. \tag{66}$$

(2) Single and triple-layer approach ($U-M$)

$$u(s) = U(0,s)\phi_0 + U(L,s)\phi_L + M(0,s)\psi_0 + M(L,s)\psi_L. \tag{67}$$

By approaching s to L^- and to 0^+, we have the matrix form as shown below

$$\begin{bmatrix} 0 & \frac{L^3}{12} & 0 & \frac{L}{2} \\ 0 & \frac{-L^2}{4} & \frac{-1}{2} & \frac{-1}{2} \\ \frac{-L^3}{12} & 0 & \frac{-L}{2} & 0 \\ \frac{-L^2}{4} & 0 & \frac{-1}{2} & \frac{-1}{2} \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_L \\ \psi_0 \\ \psi_L \end{bmatrix} = \begin{bmatrix} u(0) \\ \theta(0) \\ u(L) \\ \theta(L) \end{bmatrix}, \tag{68}$$

$$\begin{bmatrix} \frac{-1}{2} & \frac{-1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ \frac{-1}{2} & \frac{-1}{2} & 0 & 0 \\ \frac{-1}{2} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_L \\ \psi_0 \\ \psi_L \end{bmatrix} = \begin{bmatrix} v(0) \\ m(0) \\ v(L) \\ m(L) \end{bmatrix}. \tag{69}$$

(3) Single and quadruple layer approach ($U-V$)

$$u(s) = U(0,s)\phi_0 + U(L,s)\phi_L + V(0,s)\psi_0 + V(L,s)\psi_L. \tag{70}$$

By approaching s to L^- and to 0^+, we have

$$\begin{bmatrix} 0 & \frac{L^3}{12} & \frac{1}{2} & \frac{1}{2} \\ \frac{-L^2}{4} & 0 & 0 & 0 \\ \frac{-L^3}{12} & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{-L^2}{4} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_L \\ \psi_0 \\ \psi_L \end{bmatrix} = \begin{bmatrix} u(0) \\ \theta(0) \\ u(L) \\ \theta(L) \end{bmatrix}, \tag{71}$$

$$\begin{bmatrix} \frac{-1}{2} & \frac{-1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ \frac{-1}{2} & \frac{-1}{2} & 0 & 0 \\ \frac{-1}{2} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_L \\ \psi_0 \\ \psi_L \end{bmatrix} = \begin{bmatrix} v(0) \\ m(0) \\ v(L) \\ m(L) \end{bmatrix}. \tag{72}$$

(4) Double and triple-layer approach ($\Theta-M$)

$$u(s) = \Theta(0,s)\phi_0 + \Theta(L,s)\phi_L + M(0,s)\psi_0 + M(L,s)\psi_L. \tag{73}$$

By approaching s to L^- and to 0^+, we have

$$\begin{bmatrix} 0 & \frac{L^2}{4} & 0 & \frac{L}{2} \\ 0 & \frac{-L}{2} & \frac{-1}{2} & \frac{-1}{2} \\ \frac{L^2}{4} & 0 & \frac{-L}{2} & 0 \\ \frac{L}{2} & 0 & \frac{-1}{2} & \frac{-1}{2} \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_L \\ \psi_0 \\ \psi_L \end{bmatrix} = \begin{bmatrix} u(0) \\ \theta(0) \\ u(L) \\ \theta(L) \end{bmatrix}, \tag{74}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_L \\ \psi_0 \\ \psi_L \end{bmatrix} = \begin{bmatrix} v(0) \\ m(0) \\ v(L) \\ m(L) \end{bmatrix}. \tag{75}$$

(5) Double and quadruple layer approach ($\Theta-V$)

$$u(s) = \Theta(0,s)\phi_0 + \Theta(L,s)\phi_L + V(0,s)\psi_0 + V(L,s)\psi_L. \tag{76}$$

By approaching s to L^- and to 0^+, we have

$$\begin{bmatrix} 0 & \frac{L^2}{4} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{-L}{2} & 0 & 0 \\ \frac{L^2}{4} & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{L}{2} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_L \\ \psi_0 \\ \psi_L \end{bmatrix} = \begin{bmatrix} u(0) \\ \theta(0) \\ u(L) \\ \theta(L) \end{bmatrix}, \tag{77}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_L \\ \psi_0 \\ \psi_L \end{bmatrix} = \begin{bmatrix} v(0) \\ m(0) \\ v(L) \\ m(L) \end{bmatrix}. \tag{78}$$

(6) Triple and quadruple layer approach ($M-V$)

$$u(s) = M(0,s)\phi_0 + M(L,s)\phi_L + V(0,s)\psi_0 + V(L,s)\psi_L. \tag{79}$$

By approaching s to L^- and to 0^+, we have

$$\begin{bmatrix} 0 & \frac{L}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{-1}{2} & 0 & 0 \\ \frac{-L}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{-1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_L \\ \psi_0 \\ \psi_L \end{bmatrix} = \begin{bmatrix} u(0) \\ \theta(0) \\ u(L) \\ \theta(L) \end{bmatrix}, \tag{80}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_L \\ \psi_0 \\ \psi_L \end{bmatrix} = \begin{bmatrix} v(0) \\ m(0) \\ v(L) \\ m(L) \end{bmatrix}. \tag{81}$$

The unknown fictitious densities (ϕ, ψ) can be obtained by

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Table 9
Stiffness matrix for the Euler beam by using the indirect method

Portential	[A]	[B]	$[K]=[B][A]^{-1}$
$U-\Theta$, single and double layer	$\begin{bmatrix} 0 & \frac{L^3}{12} & 0 & \frac{L^2}{4} \\ 0 & \frac{-L^2}{4} & 0 & \frac{-L}{2} \\ \frac{-L^3}{12} & 0 & \frac{L^2}{4} & 0 \\ \frac{-L^2}{4} & 0 & \frac{L}{2} & 0 \end{bmatrix} Rank(A)=4$	$\begin{bmatrix} \frac{-1}{2} & \frac{-1}{2} & 0 & 0 \\ 0 & \frac{L}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{-1}{2} & 0 & 0 \\ \frac{-L}{2} & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} Rank(B)=2$	$\frac{1}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ -6L & -4L^2 & 6L & -2L^2 \\ 12 & 6L & -12 & 6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} Rank(K)=3$
$U-M$, single and triple layer	$\begin{bmatrix} 0 & \frac{L^3}{12} & 0 & \frac{L}{2} \\ 0 & \frac{-L^2}{4} & \frac{-1}{2} & \frac{-1}{2} \\ \frac{-L^3}{12} & 0 & \frac{-L}{2} & 0 \\ \frac{-L^2}{4} & 0 & \frac{-1}{2} & \frac{-1}{2} \end{bmatrix} Rank(A)=4$	$\begin{bmatrix} \frac{-1}{2} & \frac{-1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ \frac{-1}{2} & \frac{-1}{2} & 0 & 0 \\ \frac{-1}{2} & 0 & 0 & 0 \end{bmatrix} Rank(B)=2$	$\frac{1}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ -6L & -4L^2 & 6L & -2L^2 \\ 12 & 6L & -12 & 6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} Rank(K)=3$
$U-V$, single and guadrupole layer	$\begin{bmatrix} 0 & \frac{L^3}{12} & \frac{1}{2} & \frac{1}{2} \\ \frac{-L^2}{4} & 0 & 0 & 0 \\ \frac{-L^3}{12} & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{-L^2}{4} & 0 & 0 & 0 \end{bmatrix} Rank(A)=3$	$\begin{bmatrix} \frac{-1}{2} & \frac{-1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ \frac{-1}{2} & \frac{-1}{2} & 0 & 0 \\ \frac{-1}{2} & 0 & 0 & 0 \end{bmatrix} Rank(B)=2$	NA
$\Theta-M$, double and triple layer	$\begin{bmatrix} 0 & \frac{L^2}{4} & 0 & \frac{L}{2} \\ 0 & \frac{-L}{2} & \frac{-1}{2} & \frac{-1}{2} \\ \frac{L^2}{4} & 0 & \frac{-L}{2} & 0 \\ \frac{L}{2} & 0 & \frac{-1}{2} & \frac{-1}{2} \end{bmatrix} Rank(A)=3$	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix} Rank(B)=1$	NA
$\Theta-V$, double and guadrupole layer	$\begin{bmatrix} 0 & \frac{L^2}{4} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{-L}{2} & 0 & 0 \\ \frac{L^2}{4} & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{L}{2} & 0 & 0 & 0 \end{bmatrix} Rank(A)=3$	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix} Rank(B)=1$	NA
$M-V$, triple and guadrupole layer	$\begin{bmatrix} 0 & \frac{L}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{-1}{2} & 0 & 0 \\ \frac{-L}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{-1}{2} & 0 & 0 \end{bmatrix} Rank(A)=2$	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} Rank(B)=0$	NA

$$[A]\begin{bmatrix}\phi_0\\ \phi_L\\ \psi_0\\ \psi_L\end{bmatrix}=\begin{bmatrix}u(0)\\ \theta(0)\\ u(L)\\ \theta(L)\end{bmatrix}\rightarrow\begin{bmatrix}\phi_0\\ \phi_L\\ \psi_0\\ \psi_L\end{bmatrix}=[A]^{-1}\begin{bmatrix}u(0)\\ \theta(0)\\ u(L)\\ \theta(L)\end{bmatrix}, \quad (82)$$

$$[B]\begin{bmatrix}\phi_0\\ \phi_L\\ \psi_0\\ \psi_L\end{bmatrix}=\begin{bmatrix}v(0)\\ m(0)\\ v(L)\\ m(L)\end{bmatrix}\rightarrow\begin{bmatrix}\phi_0\\ \phi_L\\ \psi_0\\ \psi_L\end{bmatrix}=[B]^{-1}\begin{bmatrix}v(0)\\ m(0)\\ v(L)\\ m(L)\end{bmatrix}. \quad (83)$$

It is found that the stiffness matrix can be obtained by selecting $U-\Theta$ and $U-M$ formulations and the other combinations fail, due to the rank deficiency of $[A]$. All the above results are collected in Table 9.

4. Discussion of the rigid body mode and spurious mode in case of degenerate scale using the SVD updating technique for rods and beams

If the rigid body term, c, and the linear term, ax, are superimposed in the fundamental solution, we have $U_r(x,s)=U(x,s)+ax+c$. By substituting the auxiliary system $U(x,s)$ into Eq. (5), and setting $EA=1$, $L=1$, we have

$$\begin{bmatrix}\frac{1}{2}+a & -\frac{1}{2}-a\\ -\frac{1}{2}+a & \frac{1}{2}-a\end{bmatrix}\begin{Bmatrix}u(0)\\ u(1)\end{Bmatrix}=\begin{bmatrix}c & -\frac{1}{2}-a-c\\ \frac{1}{2}+c & -a-c\end{bmatrix}\begin{Bmatrix}u'(0)\\ u'(1)\end{Bmatrix}. \quad (84)$$

The matrix $[B_1]$ for a rod is singular when $(1+2a)=-4c$. This results in the degenerate scale problem. According to the Fredholm alternative theorem, the degenerate scale depends on the rigid body term. When $a=0$ and $c=-1/4$, $[B_1]$ matrix is not invertible and results in a degenerate scale. By employing the SVD technique, we can decompose $[A_1]$ and $[B_1]$ into

$$[A_1]=\begin{bmatrix}\frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}^{\mathrm{T}}, \quad (85)$$

$$[B_1]=\begin{bmatrix}\frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\frac{1}{2} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}^{\mathrm{T}}. \quad (86)$$

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The spurious mode $[\phi]$ satisfies

$$\begin{bmatrix} [A]^{\mathrm{T}} \\ [B]^{\mathrm{T}} \end{bmatrix} [\phi] = 0, \tag{87}$$

$$[A_1] = \begin{bmatrix} \frac{1}{2} & 0 & \frac{-1}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{-1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{-1}{2} \\ 0 & \frac{-1}{2} & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0.556 & -0.539 & \mathbf{-0.632} & 0 \\ 0.031 & 0.774 & \mathbf{-0.632} & 0 \\ -0.587 & -0.234 & \mathbf{-0.316} & 0.707 \\ 0.587 & 0.234 & \mathbf{0.316} & 0.707 \end{bmatrix} \begin{bmatrix} 1.118 & 0 & 0 & 0 \\ 0 & 1.118 & 0 & 0 \\ 0 & 0 & \mathbf{0} & 0 \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix} \begin{bmatrix} 0.234 & -0.587 & \mathbf{-0.774} & \mathbf{0} \\ -0.539 & -0.556 & \mathbf{0.258} & \mathbf{0.577} \\ -0.234 & 0.587 & \mathbf{-0.516} & \mathbf{0.577} \\ 0.774 & -0.031 & \mathbf{0.258} & \mathbf{0.577} \end{bmatrix}^{\mathrm{T}} \tag{91}$$

where the spurious mode $[\phi]$ is

$$[\phi] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{88}$$

and the rigid body mode $[\psi]$ is

$$[\psi] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{89}$$

The spurious mode $[\phi]$ and the rigid body mode $[\psi]$ are shown in Table 10.

If the rigid body term, c, and the linear, quadratic and cubic terms, ax, bx^2 and dx^3 are superimposed in the fundamental solution, we have $U_b(x,s) = U(x,s) + ax + bx^2 + dx^3 + c$. By substituting the auxiliary system $U_b(x,s)$ into Eqs. (39) for the u–θ formulation, and setting $EI = 1$, $L = 1$, we have

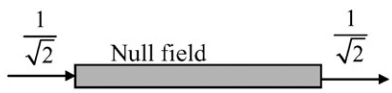
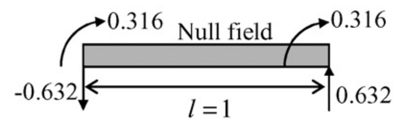
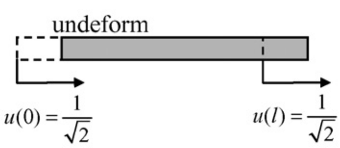
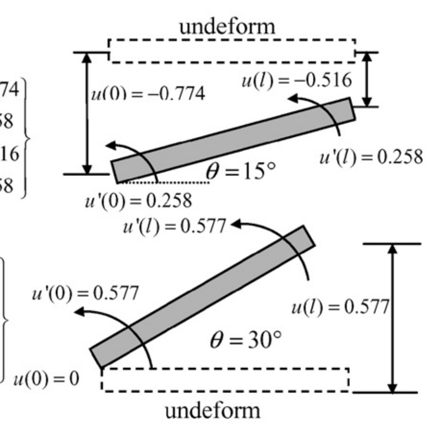
$$\begin{bmatrix} \frac{1}{2}+6d & -2b & -\frac{1}{2}-6d & \frac{1}{2}+2b+6d \\ -\frac{1}{2}+6d & -\frac{1}{2}-2b & \frac{1}{2}-6d & 2b+6d \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} u(0) \\ u'(0) \\ u(1) \\ u'(1) \end{Bmatrix} = \begin{bmatrix} c & -a & -\frac{1}{12}-c-a-b-d & \frac{1}{4}+a+2b+3d \\ \frac{1}{12}+c & \frac{1}{4}-a & -c-a-b-d & a+2b+3d \\ 0 & 0 & \frac{1}{4} & -\frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & 0 & 0 \end{bmatrix} \times \begin{Bmatrix} u'''(0) \\ u''(0) \\ u'''(1) \\ u''(1) \end{Bmatrix}. \tag{90}$$

The matrix $[B_1]$ for a beam is singular when $(1+12d) - 24a = 48c$. This results in the degenerate scale problem. When $a = 0$, $b = 0$, $c = 1/48$ and $d = 0$, $[B_1]$ matrix is not invertible and results in a degenerate scale. By employing the SVD technique with respect to the influence matrix in the u–θ formulation, for $[A_1]$, $[A_2]$ and $[B_1]$ matrices, we have

$$[B_1] = \begin{bmatrix} \frac{1}{48} & 0 & \frac{-5}{48} & \frac{1}{4} \\ \frac{5}{48} & \frac{1}{48} & \frac{-1}{48} & 0 \\ 0 & 0 & \frac{1}{4} & \frac{-1}{2} \\ \frac{1}{4} & \frac{1}{2} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0.316 & 0.299 & -0.640 & \mathbf{-0.632} \\ 0.316 & -0.299 & 0.640 & \mathbf{-0.632} \\ -0.632 & -0.640 & -0.299 & \mathbf{-0.316} \\ 0.632 & -0.640 & -0.299 & \mathbf{0.316} \end{bmatrix} \begin{bmatrix} 0.625 & 0 & 0 & 0 \\ 0 & 0.617 & 0 & 0 \\ 0 & 0 & 0.033 & 0 \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix} \times \begin{bmatrix} 0.316 & -0.299 & -0.640 & -0.632 \\ 0.632 & -0.640 & 0.299 & 0.316 \\ -0.316 & -0.299 & -0.640 & 0.632 \\ 0.632 & 0.640 & -0.299 & 0.316 \end{bmatrix}^{\mathrm{T}} \tag{92}$$

$$[A_2] = \begin{bmatrix} \frac{1}{2} & 0 & \frac{-1}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{-1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -0.707 & 0.707 & 0 & 0 \\ 0.707 & 0.707 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1.118 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & \mathbf{0} & 0 \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix} \times \begin{bmatrix} -0.632 & 0 & \mathbf{-0.774} & \mathbf{0} \\ -0.316 & -0.707 & \mathbf{0.258} & \mathbf{0.577} \\ 0.632 & 0 & \mathbf{-0.516} & \mathbf{0.577} \\ -0.316 & 0.707 & \mathbf{0.258} & \mathbf{0.577} \end{bmatrix}^{\mathrm{T}} \tag{93}$$

Table 10
Spurious modes and the rigid body modes for a rod and a beam in BEM

	Rod	Beam
	$U^T\phi=0$	$U^T\phi=0$
Spurious mode	 $[\phi]=\begin{Bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{Bmatrix}$	 $[\phi]=\begin{Bmatrix}-0.632\\ -0.632\\ -0.316\\ 0.316\end{Bmatrix}$
	$T^T\psi=0$	$T^T\psi=0$
Rigid body mode	 $\psi_1=\begin{Bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{Bmatrix}$	$\psi_1=\begin{Bmatrix}-0.774\\ 0.258\\ -0.516\\ 0.258\end{Bmatrix}$ $\psi_2=\begin{Bmatrix}0\\ 0.577\\ 0.577\\ 0.577\end{Bmatrix}$ 

According to the Fredholm alternative theorem [11], the spurious mode $[\phi]$ satisfies

$$\begin{bmatrix}[A]^{\mathrm{T}}\\ [B]^{\mathrm{T}}\end{bmatrix}[\phi]=0, \tag{94}$$

where the spurious mode is found to be imbedded in the Eqs. (91) and (92) by using the bold face, as shown below

$$[\phi]=\begin{bmatrix}-0.632\\ -0.632\\ -0.316\\ 0.316\end{bmatrix} \tag{95}$$

and the rigid body mode $[\psi]$ satisfies

$$\begin{bmatrix}[A_1]\\ [A_2]\end{bmatrix}[\psi]=0. \tag{96}$$

It is found that the two rigid body modes are shown below

$$[\psi]=\begin{bmatrix}-0.774\\ 0.258\\ -0.516\\ 0.258\end{bmatrix},\quad \begin{bmatrix}0\\ 0.577\\ 0.577\\ 0.577\end{bmatrix}. \tag{97}$$

The spurious mode $[\phi]$ and the rigid body mode $[\psi]$ are shown in Table 10. By substituting the auxiliary system $U_b(x,s)$ into Eq. (39) for the $u{-}m$ formulation, and setting $EI=1$, $L=1$, we have

$$\begin{bmatrix}\frac{1}{2}+6d & -2b & -\frac{1}{2}-6d & \frac{1}{2}+2b+6d\\ -\frac{1}{2}+6d & -\frac{1}{2}-2b & \frac{1}{2}-6d & 2b+6d\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{Bmatrix}u(0)\\ u'(0)\\ u(1)\\ u'(1)\end{Bmatrix}$$

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EABE : 1978

$$= \begin{bmatrix} c & -a & -\frac{1}{12}-c-a-b-d & \frac{1}{4}+a+2b+3d \\ \frac{1}{12}+c & \frac{1}{4}-a & -c-a-b-d & a+2b+3d \\ 0 & \frac{-1}{2} & \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{-1}{2} \end{bmatrix} \times \begin{Bmatrix} u'''(0) \\ u''(0) \\ u'''(1) \\ u''(1) \end{Bmatrix}. \qquad (98)$$

The $[B_2]$ matrix for a beam is singular when $(1+12d)-24a=48c$. This results in the nonuniqueness problem. When $a=0$, $b=0$, $c=0$ and $d=-1/12$, $[B_2]$ matrix is not invertible and results in a degenerate scale. By employing the SVD technique with respect to the influence matrix in the $u-\theta$ formulation, for $[A_2]$ and $[B_2]$ matrix, we have

$$[A_2] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & \frac{-1}{2} & 1 & \frac{-1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & \mathbf{1} \\ 1 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{0} \\ 0 & 1 & 0 & \mathbf{0} \end{bmatrix} \begin{bmatrix} 1.58114 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{0} & 0 \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix} \times \begin{bmatrix} 0.632 & -0.447 & \mathbf{0} & \mathbf{-0.774} \\ 0.316 & 0 & \mathbf{0.577} & \mathbf{0.258} \\ -0.632 & 0 & \mathbf{0.577} & \mathbf{-0.516} \\ 0.316 & 0.894 & \mathbf{0.577} & \mathbf{0.258} \end{bmatrix}^{T}, \qquad (99)$$

$$[B_2] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{12} & \frac{1}{4} & \frac{1}{12} & -\frac{1}{4} \\ 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & \mathbf{1} \\ 0.313 & 0 & -0.949 & \mathbf{0} \\ -0.671 & 0.707 & -0.221 & \mathbf{0} \\ 0.671 & 0.707 & 0.221 & \mathbf{0} \end{bmatrix} \times \begin{bmatrix} 1.17748 & 0 & 0 & 0 \\ 0 & 0.500 & 0 & 0 \\ 0 & 0 & 0.050 & 0 \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix}$$

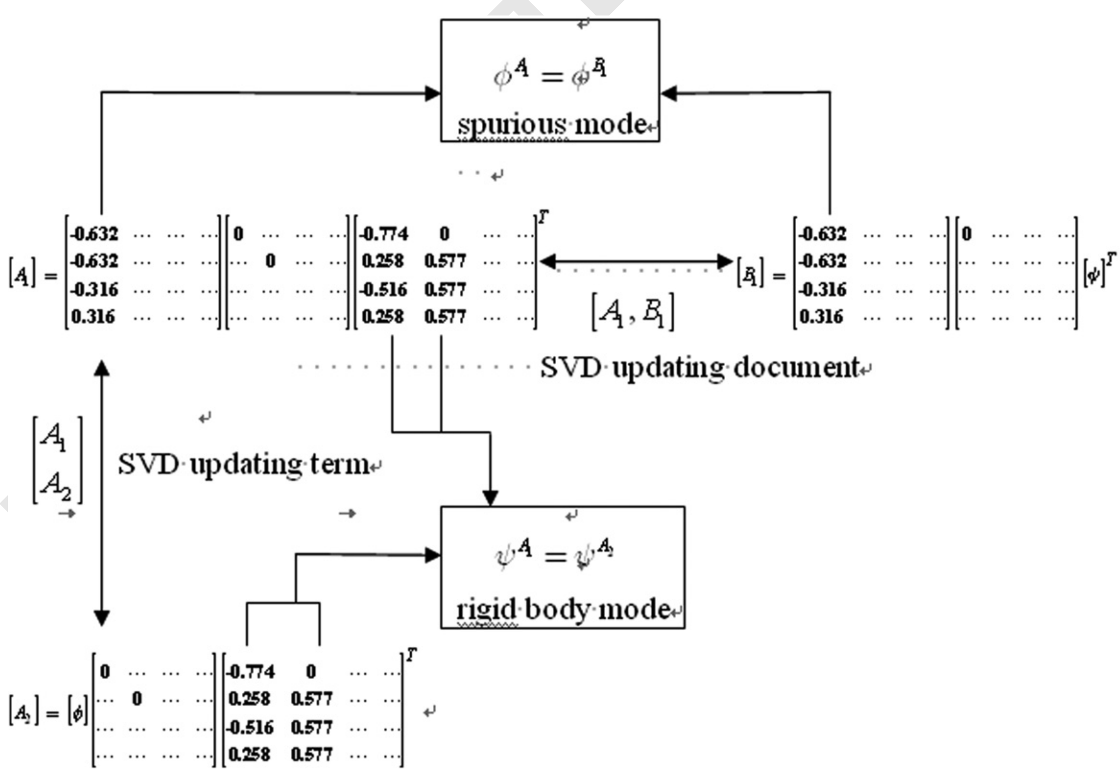

Fig. 6. Mathematical SVD structures of the influence matrices using updating techniques.

$$\times\begin{bmatrix} -0.307 & -0.707 & 0.636 & 0 \\ -0.636 & 0 & -0.307 & 0.707 \\ -0.307 & 0.707 & 0.636 & 0 \\ 0.636 & 0 & 0.307 & 0.707 \end{bmatrix}^{\mathrm{T}}. \tag{100}$$

The spurious mode $[\phi]$ satisfies Eq. (94), and the spurious mode is found to be imbedded in Eqs. (99) and (100) by using the bold face as shown below

$$[\phi] = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{101}$$

It is found that the rigid body modes is shown below

$$[\psi] = \begin{bmatrix} 0 \\ 0.577 \\ 0.577 \\ 0.577 \end{bmatrix}, \quad \begin{bmatrix} -0.774 \\ 0.258 \\ -0.516 \\ 0.258 \end{bmatrix}. \tag{102}$$

The mathematical framework of $[A]$ and $[B]$ are shown in Fig. 6.

5. Conclusions

Dual boundary integral equations were employed to derive the stiffness and flexibility of the rod and beam which match well with those of FEM. Not only the direct method but also the indirect method were used. It is found that displacement-slope (u–θ) and displacement-moment (u–m) formulations in the direct method can construct the stiffness matrix. Similarly, the single–double-layer approach (U–Θ) and single–triple-layer approach (U–M) work for the constructing of stiffness matrix in the indirect method. For choosing a special fundamental solution, the stiffness matrix cannot be obtained for the degenerate scale. Rigid body mode and spurious mode were studied by using the SVD updating term and document technique. It is found that rigid body mode and spurious mode are imbedded in the right and left unitary vectors of the influence matrices through SVD. Flexibility is also derived from the inversion of singular stiffness matrix by using SVD.

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