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An alternative method for transient and random responses of structures subject to support motions

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In this paper, we propose the Stokes' transformation technique for extracting the finite part of divergent series resulting from modal dynamic analysis for support motion problems. From the computational point of view, Stokes' transformation is the best method as not only does it avoid calculating the quasistatic solution, it also has the same convergence rate as the mode acceleration method. It is found that the present method should only integrate a known series instead of solving a partial differential equation (PDE) for the quasistatic solution. The general formulation for a finite elastic body subjected to support motions is derived. Finally, two examples, a shear and a flexural beam subjected to multisupport motions, are analysed. Both derivations for transient and statistical responses are considered. The numerical results for random responses are compared with the quasistatic decomposition method proposed by Mindlin and Goodman and the exact solution by Tsaur. Good agreement is obtained. Copyright © 1996 Elsevier Science Ltd.

Keywords: Stokes' transformation, support motions, random response, series solution

1. Introduction

Support motion problems are often encountered in earthquake engineering and structural dynamics. It is well known that the quasistatic decomposition method is a feasible technique for obtaining a solution using the modal dynamic concept[1]. The mode acceleration method is so-named because of its accelerating convergence rate. Nevertheless, as mentioned by Eringen and Suhubi[2], the quasistatic solution may still be very difficult to find for a continuous system, especially for two-dimensional (2D) and three-dimensional (3D) problems, since it is necessary to directly solve a partial differential equation. In a discrete system, the quasistatic part is obtained at the cost of a large matrix inversion[3]. Therefore, the quasistatic solution was not present and merged into the total solution[2] after employing Betti's law. In avoiding this calculation by merging the quasistatic solution into the total solution, the Gibbs phenomenon for displacement and divergent (oscillating) series for slope occurs as mentioned by Strenkowski[4] and Chen *et al.*[5]. Therefore, Yeh *et al.*[6] and Hong and Chen[7] proposed the Cesàro sum technique to deal with the Gibbs phenomenon and the divergent series for the displacement and shear force responses, respectively. Although the quasistatic solution does not have to be determined, low convergence makes the Cesàro sum technique inefficient in direct problems. Also, the Cesàro sum technique is not suitable for a stiffer structure or low frequency input since the quasistatic part is not determined separately. However, the Cesàro sum technique is still a useful tool in the regularization method for inverse problems. Using the idea of the Cesàro sum technique, deconvolution in site response analysis has been successfully applied[8]. Any structure that tends to make the system more 'static' favours the use of the quasistatic decomposition method in order to accelerate convergence as with the mode acceleration method[9]. Finding a method with accelerating convergence as in the quasistatic decomposition method but which does not require the calculation of the quasistatic solution for general structures was the goal of the present study.

Here, we will propose the Stokes' transformation that not only omits the calculation of the quasistatic part but also accelerates the convergence rate. Although the Stokes' transformation was found in 1880[10], there have been only a few applications in the literature[11–15]. Recently, the technique of Stokes' transformation has been referred to in books on the Fourier series and partial differential equation[16–21], although the technique was termed differently. Mathematically speaking, Stokes' transformation is a legal method for series differentiation when term-by-term differentiation is not permissible. For clarity, a systematic formulation for a finite elastic body is derived. For simplicity, we choose a shear and a flexural beam subjected to support motions as examples to demonstrate the validity. The results are also compared with the exact solutions derived by Tsaur[22–24] using dynamic shape functions.

2. Transient response of a finite elastic body to boundary excitations

2.1. *Problem statement*

Consider a homogeneous, isotropic, linear, elastic continuum with finite domain D bounded by boundary $B = B_t \cup B_u$; the governing equation for the displacement $\boldsymbol{u}(\boldsymbol{x},t)$ at a domain point $\boldsymbol{x}$ at time t can be written as

$$\rho\, \ddot{u} + (2\alpha\rho + \beta\mathcal{L})\{\dot{u}\} + \mathcal{L}\{u\} = 0\,, \quad \boldsymbol{x} \in D,\ t \in (0, \infty) \tag{1}$$

where α and β are the Rayleigh-damping coefficients, ρ is the mass density and the operator $\mathcal{L}$ is

$$\mathcal{L}\{\boldsymbol{u}\} = \begin{cases} -(\lambda + G)\nabla\nabla \cdot \boldsymbol{u} - G\nabla^2\boldsymbol{u}, & \text{elastic body,} \\ -G\dfrac{\partial^2 \boldsymbol{u}}{\partial x^2}, & \text{elastic shear beam,} \\ EI\dfrac{\partial^4 \boldsymbol{u}}{\partial x^4}, & \text{elastic flexural beam} \end{cases} \tag{2}$$

where λ and G are Lame's constants, E is Young's modulus and I is the moment of inertia of the cross-section of the flexural beam. The shear beam and flexural beam are shown in *Figure 1*. The time-dependent boundary conditions are

$$\mathcal{T}\{\boldsymbol{u}(\boldsymbol{x},t)\} \equiv \boldsymbol{t}(\boldsymbol{x},t) = \hat{\boldsymbol{t}}(\boldsymbol{x},t),\ \boldsymbol{x} \in B_t\,, \tag{3}$$

$$\boldsymbol{u}(\boldsymbol{x},t) = \hat{\boldsymbol{u}}(\boldsymbol{x},t),\ \boldsymbol{x} \in B_u\,, \tag{4}$$

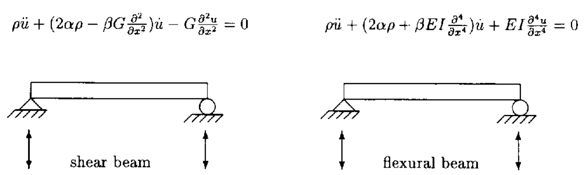

Figure 1 Shear beam and flexural beam

where $\hat{\boldsymbol{u}}$ is the prescribed displacement on B_u, $\boldsymbol{t}$ is the traction on B, $\hat{\boldsymbol{t}}$ is the prescribed traction on B_t, and $\mathcal{T}$ is the traction operator defined as

$$\mathcal{T}\{\boldsymbol{u}\} = \begin{cases} [\lambda \boldsymbol{I}(\nabla \cdot \boldsymbol{u}) + G\nabla\boldsymbol{u} + G\boldsymbol{I} \cdot (\boldsymbol{u}\nabla)] \cdot \boldsymbol{n}, & \text{elastic body} \\ G\dfrac{\partial \boldsymbol{u}}{\partial x}, & \text{elastic shear beam} \\ -\mathrm{EI}\dfrac{\partial^3 \boldsymbol{u}}{\partial x^3}, & \text{elastic flexural beam} \end{cases} \tag{5}$$

The initial conditions are

$$\boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x}) \tag{6}$$

$$\dot{\boldsymbol{u}}(\boldsymbol{x},0) = \boldsymbol{v}_0(\boldsymbol{x}) \tag{7}$$

For comparison purposes, both of the integral formulations for direct and modal elastodynamics are derived in the next section.

2.2. *Direct dynamic elasticity*

Extending the dual integral representation[25,26] to transient elastodynamics, the displacement $\boldsymbol{u}(\boldsymbol{x},t)$ and traction $\boldsymbol{t}(\boldsymbol{x},t)$ for a domain point $\boldsymbol{x}$ at time t can be written as

$$\begin{aligned} \boldsymbol{u}(\boldsymbol{x},t) = & \int_0^t \int_B \boldsymbol{U}(\boldsymbol{s},\boldsymbol{x};\ \tau,t) \cdot \boldsymbol{t}(\boldsymbol{s},\tau)\mathrm{d}B(\boldsymbol{s})\mathrm{d}\tau \\ & - \int_0^t \int_B \boldsymbol{T}(\boldsymbol{s},\boldsymbol{x};\ \tau,t) \cdot \boldsymbol{u}(\boldsymbol{s},\tau)\mathrm{d}B(\boldsymbol{s})\mathrm{d}\tau \\ & + \int_D \boldsymbol{U}(\boldsymbol{s},\boldsymbol{x};\ 0,t) \cdot \rho\boldsymbol{v}_0(\boldsymbol{s})\mathrm{d}D(\boldsymbol{s}) \\ & + \int_D \dot{\boldsymbol{U}}(\boldsymbol{s},\boldsymbol{x};\ 0,t) \cdot \rho\boldsymbol{u}_0(\boldsymbol{s})\mathrm{d}D(\boldsymbol{s}) \end{aligned} \tag{8}$$

$$\begin{aligned} \boldsymbol{t}(\boldsymbol{x},t) = & \int_0^t \int_B \boldsymbol{L}(\boldsymbol{s},\boldsymbol{x};\ \tau,t) \cdot \boldsymbol{t}(\boldsymbol{s},\tau)\mathrm{d}B(\boldsymbol{s})\mathrm{d}\tau \\ & - \int_0^t \int_B \boldsymbol{M}(\boldsymbol{s},\boldsymbol{x};\ \tau,t) \cdot \boldsymbol{u}(\boldsymbol{s},\tau)\mathrm{d}B(\boldsymbol{s})\mathrm{d}\tau \\ & + \int_D \boldsymbol{L}(\boldsymbol{s},\boldsymbol{x};\ 0,t) \cdot \rho\boldsymbol{v}_0(\boldsymbol{s})\mathrm{d}D(\boldsymbol{s}) \\ & + \int_D \dot{\boldsymbol{L}}(\boldsymbol{s},\boldsymbol{x};\ 0,t) \cdot \rho\boldsymbol{u}_0(\boldsymbol{s})\mathrm{d}D(\boldsymbol{s}) \end{aligned} \tag{9}$$

where $\boldsymbol{U}(\boldsymbol{s},\boldsymbol{x};\ \tau,t)$, $\boldsymbol{T}(\boldsymbol{s},\boldsymbol{x};\ \tau,t)$, $\boldsymbol{L}(\boldsymbol{s},\boldsymbol{x};\ \tau,t)$ and $\boldsymbol{M}(\boldsymbol{s},\boldsymbol{x};\ \tau,t)$ are four kernel functions. The closed-form solutions can be found in Reference 26. The dual integral formulations for the displacement and traction on a smooth boundary point $\boldsymbol{x}$ at time t are

$$\begin{aligned} c\,\boldsymbol{u}(\boldsymbol{x},t) = & RPV \int_0^t \int_B \boldsymbol{U}(\boldsymbol{s},\boldsymbol{x};\ \tau,t) \cdot \boldsymbol{t}(\boldsymbol{s},\tau)\mathrm{d}B(\boldsymbol{s})\mathrm{d}\tau \\ & - CPV \int_0^t \int_B \boldsymbol{T}(\boldsymbol{s},\boldsymbol{x};\ \tau,t) \cdot \boldsymbol{u}(\boldsymbol{s},\tau)\mathrm{d}B(\boldsymbol{s})\mathrm{d}\tau \\ & + \int_D \boldsymbol{U}(\boldsymbol{s},\boldsymbol{x};\ 0,t) \cdot \rho\boldsymbol{v}_0(\boldsymbol{s})\mathrm{d}D(\boldsymbol{s}) + \int_D \dot{\boldsymbol{U}}(\boldsymbol{s},\boldsymbol{x};\ 0,t) \\ & \cdot \rho\boldsymbol{u}_0(\boldsymbol{s})\mathrm{d}D(\boldsymbol{s}) \end{aligned} \tag{10}$$

$$c\, \boldsymbol{t}(\boldsymbol{x},t) = CPV \int_0^t \int_B \boldsymbol{L}(s,\boldsymbol{x};\, \tau,t) \cdot \boldsymbol{t}(s,\tau)\mathrm{d}B(s)\mathrm{d}\tau$$
$$- HPV \int_0^t \int_B \boldsymbol{M}(s,\boldsymbol{x};\, \tau,t) \cdot \boldsymbol{u}(s,\tau)\mathrm{d}B(s)\mathrm{d}\tau$$
$$+ \int_D \boldsymbol{L}(s,\boldsymbol{x};\, 0,t) \cdot \rho v_0(s)\mathrm{d}D(s) + \int_D \dot{\boldsymbol{L}}(s,\boldsymbol{x};\, 0,t)$$
$$\cdot\, \rho \boldsymbol{u}_0(s)\mathrm{d}D(s) \qquad (11)$$

where c is 1/2 for 1D, π for 2D, or 2π for 3D, and *RPV*, *CPV* and *HPV* denote the Riemann integral, the Cauchy principal value and the Hadamard (or Mangler) principal value, respectively. In discretized numerical calculations, it is found that matrix inversion is necessary at each time-marching stage for the direct dynamic method. To avoid these repetitive time-consuming inversions, the modal dynamic method is employed, as in the next section.

2.3. *Modal dynamic elasticity*

In the derivation procedure, let the solution be decomposed into two parts[1]

$$\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{U}(\boldsymbol{x},t) + \sum_{k=1}^{\infty} q_k(t)\, \boldsymbol{u}_k(\boldsymbol{x}) \qquad (12)$$

where $\boldsymbol{U}(\boldsymbol{x},t)$ denotes the quasistatic solution, and the second term, which is composed of various eigenfunctions $\boldsymbol{u}_k(\boldsymbol{x})$, $k \in N$, N that are the set of natural numbers, and is weighted by generalized co-ordinates $q_k(t)$, is the dynamic contribution due to the inertia effect. Following the same procedures of Tsaur[22], the displacement $\boldsymbol{u}(\boldsymbol{x},t)$ can be represented by introducing a more generalized co-ordinate, $\bar{q}_k(t)$, as in the following series

$$\boldsymbol{u}(\boldsymbol{x},t) = \sum_{k=1}^{\infty} \bar{q}_k(t)\, \boldsymbol{u}_k(\boldsymbol{x})$$
$$= \sum_{m=1}^{\infty} \left\{ \frac{\lambda_m}{N_m} e^{-\xi_m\omega_m t} \left[\cos(\omega_m^d t) + \frac{\xi_m}{\sqrt{1-\xi_m^2}} \sin(\omega_m^d t) \right] + \frac{\kappa_m}{\omega_m^d N_m} \left[e^{-\xi_m\omega_m t} \sin(\omega_m^d) \right] + \frac{1}{\omega_m^d N_m} \int_0^t [U_m^B(\tau) - T_m^B(\tau)]\, e^{-\xi_m\omega_m(t-\tau)} \sin(\omega_m^d(t-\tau))\, \mathrm{d}\tau \right\} \boldsymbol{u}_m(\boldsymbol{x}) \qquad (13)$$

The above equation can also be derived by expanding $\boldsymbol{U}(\boldsymbol{x},t)$ in equation (12) into series using Betti's law[5]. We can easily apply a traction operator to the summation sign of equation (13) to obtain

$$\boldsymbol{t}(\boldsymbol{x},t) = \sum_{m=1}^{\infty} \left\{ \frac{\lambda_m}{N_m} \left[e^{-\xi_m\omega_m t} \cos(\omega_m^d t) + \frac{\xi_m}{\sqrt{1-\xi_m^2}} \sin(\omega_m^d t) \right] + \frac{\kappa_m}{\omega_m^d N_m} [e^{-\xi_m\omega_m t} \sin(\omega_m^d)] + \frac{1}{\omega_m^d N_m} \int_0^t [U_m^B(\tau) - T_m^B(\tau)]\, e^{-\xi_m\omega_m(t-\tau)} \sin(\omega_m^d(t-\tau))\, \mathrm{d}\tau \right\} \boldsymbol{t}_m(\boldsymbol{x}) \qquad (14)$$

where

$$\delta_{ml} N_m = \int_D \rho \boldsymbol{u}_m(x) \cdot \boldsymbol{u}_l(x)\mathrm{d}D(x) \qquad (15)$$

$$2\xi_m\omega_m \equiv 2\alpha + \beta\omega_m^2 \qquad (16)$$

$$\lambda_m \equiv \int_D \rho \boldsymbol{u}_0(\boldsymbol{x}) \cdot \boldsymbol{u}_m(\boldsymbol{x})\mathrm{d}D(\boldsymbol{x}) \qquad (17)$$

$$\kappa_m \equiv \int_D \rho\, \boldsymbol{v}_0(\boldsymbol{x}) \cdot \boldsymbol{u}_m(\boldsymbol{x})\mathrm{d}D(\boldsymbol{x}) \qquad (18)$$

$$U_m^B(t) \equiv \int_{B_t} \boldsymbol{u}_m(s) \cdot \hat{\boldsymbol{t}}(s,t)\mathrm{d}B(s) + \beta \frac{\mathrm{d}}{\mathrm{d}t} \left\{ \int_{B_t} \boldsymbol{u}_m(s) \cdot \hat{\boldsymbol{t}}(s,t)\mathrm{d}B(s) \right\} \qquad (19)$$

$$T_m^B(t) \equiv \int_{B_u} \boldsymbol{t}_m(s) \cdot \hat{\boldsymbol{u}}(s,t)\mathrm{d}B(s) + \beta \frac{\mathrm{d}}{\mathrm{d}t} \left\{ \int_{B_u} \boldsymbol{t}_m(s) \cdot \hat{\boldsymbol{u}}(s,t)\mathrm{d}B(s) \right\} \qquad (20)$$

$$\omega_m^d \equiv \omega_m \sqrt{1-\xi_m^2} \qquad (21)$$

in which ξ_m and ω_m are the mth modal damping ratio and modal frequency. Comparing equations (13) and (14) with equations (8) and (9), we have

$$\boldsymbol{U}(s,\boldsymbol{x};\, \tau,t) = \sum_{m=1}^{\infty} \frac{1}{N_m\omega_m^d} e^{-\xi_m\omega_m(t-\tau)} \sin(\omega_m^d(t-\tau))\boldsymbol{u}_m(\boldsymbol{x}) \otimes \boldsymbol{u}_m(s) \qquad (22)$$

$$\boldsymbol{T}(s,\boldsymbol{x};\, \tau,t) = \sum_{m=1}^{\infty} \frac{1}{N_m\omega_m^d} e^{-\xi_m\omega_m(t-\tau)} \sin(\omega_m^d(t-\tau))\boldsymbol{u}_m(\boldsymbol{x}) \otimes \boldsymbol{t}_m(s) \qquad (23)$$

$$\boldsymbol{L}(s,\boldsymbol{x};\, \tau,t) = \sum_{m=1}^{\infty} \frac{1}{N_m\omega_m^d} e^{-\xi_m\omega_m(t-\tau)} \sin(\omega_m^d(t-\tau))\boldsymbol{t}_m(\boldsymbol{x}) \otimes \boldsymbol{u}_m(s) \qquad (24)$$

$$\boldsymbol{M}(s,\boldsymbol{x};\, \tau,t) = \sum_{m=1}^{\infty} \frac{1}{N_m\omega_m^d} e^{-\xi_m\omega_m(t-\tau)} \sin(\omega_m^d(t-\tau))\boldsymbol{t}_m(\boldsymbol{x}) \otimes \boldsymbol{t}_m(s) \qquad (25)$$

where $\otimes$ is the dyadic product, and $\boldsymbol{t}_m(\boldsymbol{x})$ is the mth modal reaction. Equations (22)–(25) can be seen as the spectral decomposition for closed-form kernels. In order to accelerate the convergence to deal with the Gibbs phenomenon, the Cesàro sum of order $(C,1)$ is applied to equation (13) while the $(C,2)$ operator is utilized to extract the finite part of the divergent series of equation (14) as follows

$$\boldsymbol{u}(\boldsymbol{x},t) = (C,1)\left\{\sum_{m=1}^{\infty}\left\{\frac{\lambda_m}{N_m} e^{-\xi_m\omega_m t}\left[\cos(\omega_m^d t) + \frac{\xi_m}{\sqrt{1-\xi_m^2}}\sin(\omega_m^d t)\right] + \frac{\kappa_m}{\omega_m^d N_m}[e^{-\xi_m\omega_m t}\sin(\omega_m^d)] + \frac{1}{\omega_m^d N_m}\int_0^t [U_m^B(\tau) - T_m^B(\tau)] e^{-\xi_m\omega_m(t-\tau)}\sin(\omega_m^d(t-\tau))\,\mathrm{d}\tau\right\}\boldsymbol{u}_m(\boldsymbol{x})\right\} \tag{26}$$

$$\boldsymbol{t}(\boldsymbol{x},t) = (C,2)\left\{\sum_{m=1}^{\infty}\left\{\frac{\lambda_m}{N_m}[e^{-\xi_m\omega_m t}\cos(\omega_m^d t) + \frac{\xi_m}{\sqrt{1-\xi_m^2}}\sin(\omega_m^d t)] + \frac{\kappa_m}{\omega_m^d N_m}[e^{-\xi_m\omega_m t}\sin(\omega_m^d)] + \frac{1}{\omega_m^d N_m}\int_0^t [U_m^B(\tau) - T_m^B(\tau)] e^{-\xi_m\omega_m(t-\tau)}\sin(\omega_m^d(t-\tau))\,\mathrm{d}\tau\right\}\boldsymbol{t}_m(\boldsymbol{x})\right\} \tag{27}$$

If regularized, equations (26) and (27) are the dual integral representation in series forms or, briefly, the dual series representation[5]. Each term of the series is seen to be a general Duhamel integral. Its kernel functions $\boldsymbol{U}(\boldsymbol{s},\boldsymbol{x};\tau,t)$, $\boldsymbol{T}(\boldsymbol{s},\boldsymbol{x};\tau,t)$, $\boldsymbol{L}(\boldsymbol{s},\boldsymbol{x};\tau,t)$ and $\boldsymbol{M}(\boldsymbol{s},\boldsymbol{x};\tau,t)$ all have the same decaying oscillating factor $e^{-\xi_m\omega_m(t-\tau)}\sin(\omega_m^d(t-\tau))$ and represent system characteristics whereas its density functions represent input excitations, but the initial disturbances appear in the free terms, λ_m and κ_m, outside the integral signs.

3. Regularization by the Stokes' transformation

From the standpoint of ordinary convergence, the legitimacy of the term-by-term differentiation of series can only be guaranteed by rather strong requirements. We shall relax this constraint by using the regularization techniques of Stokes' transformation instead of a posterior treatment of the Cesàro sum, which has been discussed in the previous section and in detail elsewhere[5,6].

4. One-dimensional case of second-order operator $\partial^2/\partial x^2$ for a shear beam

A shear beam subjected to support motions is considered in this subsection. The series representation for displacement can be written as

$$u(x,t) = \sum_{l=1}^{\infty} \bar{q}_l(t)\, u_l(x),\ 0 < x < L \tag{28}$$

where $\bar{q}_l(t)$ is the generalized co-ordinate, and $u_l(x)$ is the modal shape with the following properties

$$\delta_{lp} N_l = \int_0^L u_l(x)\, u_p(x)\,\mathrm{d}x \tag{29}$$

$$\bar{q}_l(t) = \frac{1}{N_l}\int_0^L u(x,t)\, u_l(x)\,\mathrm{d}x \tag{30}$$

Since termwise differentiation for series is not always permissible, the legal way of series differentiation for equation (28) by Stokes' transformation shows

$$u'(x,t) = \sum_{l=1}^{\infty} \bar{q}_l'(t)\, u_l'(x),\ 0 \le x \le L \tag{31}$$

where

$$\bar{q}_l'(t) = \frac{1}{N_l\,\lambda_l}\{u(y,t)\, u_l'(y)\}\,\Big|_{y=0}^{y=L} + \bar{q}_l(t) \tag{32}$$

$$\delta_{lp}\, N_l\, \lambda_l = \int_0^L u_l'(x)\, u_p'(x)\,\mathrm{d}x \tag{33}$$

$$u_l''(x) = -\lambda_l\, u_l(x) \tag{34}$$

Therefore,

$$u'(x,t) = \sum_{l=1}^{\infty}\frac{1}{N_l\,\lambda_l}\{u(y,t)\, u_l'(y)\}\,\Big|_{y=0}^{y=L} u_l'(x) + \sum_{l=1}^{\infty}\bar{q}_l(t)\, u_l'(x),\ 0 \le x \le L \tag{35}$$

In equation (35), it is easily found that the term-by-term differentiation for equation (28) drives away the boundary terms $(1/N_l\,\lambda_l)\{u(y,t)\, u_l'(y)\}\,|_{y=0}^{y=L}\, u_l'(x)$ and makes the series diverge. This finding reveals that Stokes' transformation is a legal way for series differentiation.

5. One-dimensional case of fourth-order operator $\partial^4/\partial x^4$ for a flexural beam

For the flexural beam, the displacement can be superimposed by the following mode superposition

$$u(x,t) = \sum_{l=1}^{\infty}\bar{q}_l(t)\, u_l(x),\ 0 < x < L \tag{36}$$

where

$$\delta_{lp} = \int_0^L u_l(x)\, u_p(x)\,\mathrm{d}x \tag{37}$$

$$\bar{q}_p(t) = \int_0^L u(x,t)\, u_p(x)\,\mathrm{d}x \tag{38}$$

The series differentiation by Stokes' transformation shows

$$u'(x,t) = \sum_{l=1}^{\infty} \bar{q}_l'(t)\, u_l'(x),\ 0 < x < L \tag{39}$$

where

$$\bar{q}_l'(t) = \frac{1}{\lambda_l}\left\{\int_0^L u'(x,t)\, u_l'(x)\, dx\right\}$$
$$= \frac{1}{\lambda_l}\{u(x,t)\, u_l'(x)\}\,|_0^L + \bar{q}_l(t) \tag{40}$$

in which

$$\delta_{lp}\,\lambda_l = \int_0^L u_l'(x)\, u_p'(x)\, dx \tag{41}$$

$$u_l''(x) = -\lambda_l\, u_l(x) \tag{42}$$

Similarly, we have

$$u''(x,t) = \sum_{l=1}^{\infty} \bar{q}_l''(t)\, u_l''(x),\ 0 < x < L \tag{43}$$

where

$$\bar{q}_l''(t) = \frac{1}{\lambda_l^2}\left\{\int_0^L u''(x,t)\, u_l''(x)\, dx\right\}$$
$$= \frac{1}{\lambda_l^2}\{u'(x,t)\, u_l''(x)\}\,|_0^L + \bar{q}_l'(t) \tag{44}$$

in which

$$\delta_{lp}\,\lambda_l^2 = \int_0^L u_l''(x)\, u_p''(x)\, dx \tag{45}$$

$$u_l'''(x) = -\lambda_l\, u_l'(x) \tag{46}$$

For the three-fold differentiation

$$u'''(x,t) = \sum_{l=1}^{\infty} \bar{q}_l'''(t)\, u_l'''(x),\ 0 < x < L \tag{47}$$

where

$$\bar{q}_l'''(t) = \frac{1}{\lambda_l^3}\left\{\int_0^L u'''(x,t)\, u_l'''(x)\, dx\right\}$$
$$= \frac{1}{\lambda_l^3}\{u''(x,t)\, u_l'''(x)\}\,|_0^L + \bar{q}_l''(t) \tag{48}$$

in which

$$\delta_{lp}\,\lambda_l^3 = \int_0^L u_l'''(x)\, u_p'''(x)\, dx \tag{49}$$

$$u_l''''(x) = -\lambda_l\, u_l''(x) \tag{50}$$

In equations (40), (43) and (47), the termwise differentiation loses the boundary term, which is present if the essential boundary condition is time dependent.

6. Navier operator for dynamic elasticity

The representation of the displacement field for a finite elastic body can be expressed in component form

$$u_i(\boldsymbol{x},t) = \sum_{l=1}^{\infty} \bar{q}_l(t)\, u_i^l(\boldsymbol{x}) \tag{51}$$

where

$$\delta_{lp} = \int_D u_i^p(\boldsymbol{x})\, u_i^l(\boldsymbol{x})\, dD \tag{52}$$

$$\bar{q}_l(t) = \int_D u_i(\boldsymbol{x},t)\, u_i^l(\boldsymbol{x})\, dD \tag{53}$$

and the subscript of $u_i^l(\boldsymbol{x})$ indicates the ith component of displacement, and the superscript denotes the lth mode. First, we define the Navier operator

$$\mathcal{D}_{ij} \equiv (\lambda + G)\, \partial_i\partial_j + G\, \delta_{ij}\, \partial_k\partial_k \tag{54}$$

Applying the Navier operator to u_j, we have

$$\mathcal{D}_{ij}u_j = (\lambda + G)u_{j,ij} + G\, u_{i,kk} = \sigma_{ij,j} \tag{55}$$

The strain field is

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{56}$$

and the stress field is

$$\sigma_{ij} = \lambda\epsilon_{kk}\delta_{ij} + 2G\epsilon_{ij} = \lambda u_{k,k}\delta_{ij}$$
$$+ G(u_{i,j} + u_{j,i}) \tag{57}$$

The above equation can be written in terms of the $\mathcal{D}_{ijk}^{\sigma}$ operator

$$\sigma_{ij} = \lambda\delta_{ij}\partial_k\, u_k + G(\partial_j\delta_{ik} + \partial_i\delta_{jk})$$
$$= \mathcal{D}_{ijk}^{\sigma}\, u_k \tag{58}$$

where the $\mathcal{D}_{ijk}^{\sigma}$ operator is

$$\mathcal{D}_{ijk}^{\sigma} \equiv \lambda\delta_{ij}\partial_k + G(\partial_j\delta_{ik} + \partial_i\delta_{jk}) \tag{59}$$

Comparing equations (54) and (59), the relation between the $\mathcal{D}_{ijk}^{\sigma}$ and $\mathcal{D}_{ij}$ operators is

$$\partial_j\mathcal{D}_{ijk}^{\sigma} = \mathcal{D}_{ik} \tag{60}$$

Changing k and j in equation (59), we have

$$\mathcal{D}_{ikj}^{\sigma} \equiv \lambda\delta_{ik}\partial_j + G(\partial_k\delta_{ij} + \partial_i\delta_{kj}) \tag{61}$$

Define the traction operator $\mathcal{B}_{ij}$ as

$$\mathcal{B}_{ij} \equiv \mathcal{D}^{\sigma}_{ikj} n_k \equiv \lambda n_i \partial_j + G(\delta_{ij} n_k \partial_k + n_j \partial_i) \tag{62}$$

The field representation of stress can be written as

$$\sigma_{ik} = \mathcal{D}^{\sigma}_{ikj} u_j = \sum_{l=1}^{\infty} \bar{q}^{\sigma}_l(t)\, \mathcal{D}^{\sigma}_{ikj} u^l_j$$

$$= \sum_{l=1}^{\infty} \bar{q}^{\sigma}_l(t)\, \sigma^l_{ik} \tag{63}$$

where

$$\bar{q}^{\sigma}_p(t) = \frac{1}{\lambda_p} \left\{ \int_D u^p_{i,k}\, \mathcal{D}^{\sigma}_{ikj} u_j \, \mathrm{d}D \right\} \tag{64}$$

in which

$$\delta_{lp}\, \lambda_l = \int_D \mathcal{D}^{\sigma}_{ikj}\, u^l_j\, u^p_{i,k}\, \mathrm{d}D \tag{65}$$

$$\mathcal{D}_{ij}\, u^l_j(\boldsymbol{x}) = -\,\lambda_l\, u^l_i(\boldsymbol{x}) \tag{66}$$

where λ_l is an eigenvalue. After using

$$\int_B u^l_i\, \mathcal{B}_{ij} u^p_j \, \mathrm{d}B = \int_B u^l_i\, \mathcal{D}^{\sigma}_{ikj} u^p_j n_k \, \mathrm{d}B$$

$$= \int_D \partial_k \{ u^l_i \mathcal{D}^{\sigma}_{ikj} u^p_j \} \, \mathrm{d}D = \int_D u^l_{i,k}\, \mathcal{D}^{\sigma}_{ikj} u^p_j \, \mathrm{d}D$$

$$+ \int_D u^l_i\, \partial_k \{ \mathcal{D}^{\sigma}_{ikj} u^p_j \} \, \mathrm{d}D \tag{67}$$

changing l and p and using Betti's law, we have

$$\int_D u^l_{i,k}\, \mathcal{D}^{\sigma}_{ikj} u^p_j \, \mathrm{d}D = \int_D u^p_{i,k}\, \mathcal{D}^{\sigma}_{ikj} u^l_j \, \mathrm{d}D \tag{68}$$

Using equation (68), equation (64) can be rewritten as

$$\bar{q}^{\sigma}_p(t) = \frac{1}{\lambda_p} \left\{ \int_D u^p_{i,k}\, \mathcal{D}^{\sigma}_{ikj} u_j \, \mathrm{d}D \right\}$$

$$= \frac{1}{\lambda_p} \left\{ \int_{B_u} u_i\, t^p_i \, \mathrm{d}B \right\} + \bar{q}_p(t) \tag{69}$$

Therefore, the traction field can be represented by

$$t_i(\boldsymbol{x}) = \sum_{l=1}^{\infty} \bar{q}^{\sigma}_l(t)\, t^l_i(\boldsymbol{x}) \tag{70}$$

Substituting equation (69) into equation (70), we have

$$t_i(\boldsymbol{x}) = \sum_{l=1}^{\infty} \frac{1}{\lambda_l} \left\{ \int_{B_u} u_i\, t^l_i \, \mathrm{d}B \right\} t^l_i(\boldsymbol{x})$$

$$+ \sum_{l=1}^{\infty} \bar{q}_l(t)\, t^l_i(\boldsymbol{x}) \tag{71}$$

Equation (71) also reveals that the term-by-term traction derivative of equation (51) loses the boundary terms $\Sigma_{l=1}^{\infty}(1/\lambda_l)\{\int_{B_u} u\, t^l_i\, \mathrm{d}B\}\, t^l_i(\boldsymbol{x})$ on the right-hand side of the equals sign in equation (71) and makes the series diverge.

After the secondary fields (stress/traction) are determined, the primary field (displacement) can be easily integrated, and the essential time-dependent boundary condition, i.e., the support motions, can be automatically included.

7. Transient and random responses for general structure subjected to multiple support motions

For the transient response, three representations for support excitation problems are shown in *Table 1*. In *Table 1*, U is the quasistatic solution, $\underline{U'}$ is the boundary solution by Stokes' transformation and $\bar{U}$ is derived by integrating the secondary field.

After some lengthy, but otherwise straightforward manipulations, the power spectrum density of the displacement at location x is found to be

$$S_{pp}(x,\omega) = \sum_{i=1}^{N_s} \sum_{j=1}^{N_s} r_i(x) r_j(x)\, S_{ij}(\omega)$$

$$+ \sum_{i=1}^{N_s} \sum_{j=1}^{N_s} \sum_{m=1}^{N} [-u_m(x)(r_i(x) P_{mj} H_m(\omega)$$

$$+ r_j(x) P_{mi} H^*_m(\omega))]\omega^2 S_{ij}(\omega)$$

$$+ \sum_{i=1}^{N_s} \sum_{j=1}^{N_s} \sum_{m=1}^{N} \sum_{n=1}^{N} [u_m(x) u_n(x) P_{mi} P_{nj} H^*_n(\omega)$$

$$H_m(\omega)]\omega^4 S_{ij}(\omega) \tag{72}$$

if

$$u(x,t) = \sum_{i=1}^{N_s} U^i(t) r_i(x) + \sum_{i=1}^{N} q_i(t) u_i(x) \tag{73}$$

where $r_i(x)$, P_{mi} and P_{mj} are shown in *Table 2*, $H_m(\omega)$ denotes the transfer function and p can denote displacement u or slope u' as shown in *Table 2*. For the random response, the three representations for support excitation problems are shown in *Table 2*, where N_s is the support number, N is the mode number, Γ_{mi} is the mth modal participation factor for the i support to q_m, and $\bar{\Gamma}_{mi}$ is the mth modal participation factor for the i support to $\bar{q}_m$.

The mean square response can be obtained by

$$\sigma^2_{uu}(\boldsymbol{x}) = \int_{-\infty}^{\infty} S_{uu}(\boldsymbol{x},\omega)\, \mathrm{d}\omega \tag{74}$$

$$\sigma^2_{u'u'}(\boldsymbol{x}) = \int_{-\infty}^{\infty} S_{u'u'}(\boldsymbol{x},\omega)\, \mathrm{d}\omega \tag{75}$$

8. Numerical example for a shear beam

Consider a simply supported uniform shear beam excited by random boundary support motion at both ends. The data prepared are shown below[6,27]

Table 1 Transient response of three different methods

Transient response	Quasistatic decomposition	Cesàro sum	Stokes' transformation
$u(x,t)$	$U+\Sigma\, q_n(t)u_n(x)$	$(C,1)\{\Sigma\, \bar{q}_n(t)u_n(x)\}$	$\bar{U}+\Sigma\, \bar{q}_n(t)u_n(x)$
$u'(x,t)$	$U'+\Sigma\, q_n(t)u'_n(x)$	$(C,2)\{\Sigma\, \bar{q}_n(t)u'_n(x)\}$	$\bar{U}'+\Sigma\, \bar{q}_n(t)u'_n(x)$

Table 2 Random response of three different methods

Random response	Quasistatic decomposition	Cesàro sum	Stokes' transformation
$S_{pp}\rightarrow S_{uu}$	$r_i(x)\rightarrow U_i$ $P_{mj}\rightarrow \Gamma_{mi}$	$r_i(x)\rightarrow 0$ $P_{mj}\rightarrow \bar{\Gamma}_{mi}$ $(C,1)\{S_{uu}\}$	$r_i(x)\rightarrow \bar{U}_i$ $P_{mj}\rightarrow \bar{\Gamma}_{mi}$
$S_{pp}\rightarrow S_{u'u'}$	$r'_i(x)\rightarrow U'_i$ $P_{mj}\rightarrow \Gamma_{mi}$	$r_i(x)\rightarrow 0$ $P_{mj}\rightarrow \bar{\Gamma}_{mi}$ $(C,2)\{S_{u'u'}\}$	$r_i(x)\rightarrow \bar{U}'_i$ $P_{mj}\rightarrow \bar{\Gamma}_{mi}$

governing equation

$$-G\frac{\partial^2 u(x,t)}{\partial x^2}+\left(2\alpha\rho-\beta G\frac{\partial^2}{\partial x^2}\right)\dot{u}+\rho\ddot{u}=0,$$

$$0<x<l$$

natural frequency

$$\omega_n=n\pi c/l,\ \text{where}\ c=\sqrt{G/\rho}$$

natural mode

$$u_n(x)=\sin(n\pi x/l)$$

modal reaction

$$R_n(0)=\frac{-Gn\pi}{l} \tag{76}$$

$$R_n(l)=\frac{Gn\pi(-1)^n}{l} \tag{77}$$

Modal generalized mass

$$N_m=\rho l/2$$

The input power spectrum

$$S=\begin{bmatrix}S_{aa} & S_{ab}\\ S_{ba} & S_{bb}\end{bmatrix}$$

$$=\begin{bmatrix}\dfrac{6A\delta^5}{(\delta^6+\omega^6)} & \dfrac{6A\delta^5}{(\delta^6+\omega^6)}e^{-i\omega\tau}\\ \dfrac{6A\delta^5}{(\delta^6+\omega^6)}e^{i\omega\tau} & \dfrac{6A\delta^5}{(\delta^6+\omega^6)}\end{bmatrix} \tag{78}$$

where τ is time lag, S_{aa} and S_{bb} denote the autopower spectral densities of support displacement excitations at $x=0,l$, respectively, and S_{ab} and S_{ba} are cross-spectral densities which are assumed to be

$$S_{ab}=S^*_{ba}=S_{aa}e^{i\omega\tau}=S_{bb}e^{i\omega\tau} \tag{79}$$

in which the superscript * denotes a complex conjugate. Model parameters

$$\delta=1, A=0.5, l=1, T_1=2, \xi_1=\xi_2=0.1,$$

$$\rho=1, G=1$$

All the parameters are in consistent units[6]. According to the representation in *Tables 1* and *2*, the data can be obtained as follows

$$U_1(x)=\left(1-\frac{x}{l}\right) \tag{80}$$

$$U_2(x)=\frac{x}{l} \tag{81}$$

$$\bar{U}_1(x)=\left(1-\frac{x}{l}\right)+\sum_{n=1}^{\infty}\frac{-2}{n\pi}\sin(n\pi x/l) \tag{82}$$

$$\bar{U}_2(x)=\frac{x}{l}+\sum_{n=1}^{\infty}\frac{2}{n\pi}(-1)^n\sin(n\pi x/l) \tag{83}$$

$$\Gamma_{m1}=\frac{Gm\pi}{\omega_m^2N_ml} \tag{84}$$

$$\Gamma_{m2}=\frac{-Gm\pi(-1)^m}{\omega_m^2N_ml} \tag{85}$$

$$\bar{\Gamma}_{m1}=\frac{-Gm\pi}{N_ml} \tag{86}$$

$$\bar{\Gamma}_{m2}=\frac{Gm\pi(-1)^m}{N_ml} \tag{87}$$

It is interesting to find that the modal participation factors in equations (84)–(87) are proportional to the modal reactions in equations (76) and (77). More detail can be found in Reference 3. Equations (84)–(87) also show that the modal participation factor is proportional to the modal reaction as mentioned in Reference 3.

For such a simple case, an exact solution has been derived by Tsaur[23] using dynamic shape functions as follows

$$\begin{aligned}S_{uu}(x,\omega)&=|H_a(x,\omega)|^2S_{aa}(\omega)\\&+|H_b(x,\omega)|^2S_{bb}(\omega)\\&+2\mathrm{Re}\{H_a^*(x,\omega)H_b(x,\omega)S_{ab}(\omega)\}\end{aligned} \tag{88}$$

$$\begin{aligned}S_{u'u'}(x,\omega)&=|H'_a(x,\omega)|^2S_{aa}(\omega)\\&+|H'_b(x,\omega)|^2S_{bb}(\omega)\\&+2\mathrm{Re}\{H'^*_a(x,\omega)H'_b(x,\omega)S_{ab}(\omega)\}\end{aligned} \tag{89}$$

where

$$H_a(x,\omega)=\frac{1}{1-e^{-2\lambda l}}\{e^{-\lambda x}-e^{\lambda(x-2l)}\} \tag{90}$$

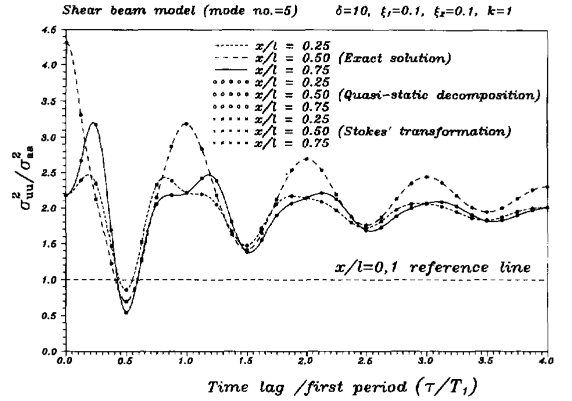

Figure 2 Mean square displacement spectra for shear beam

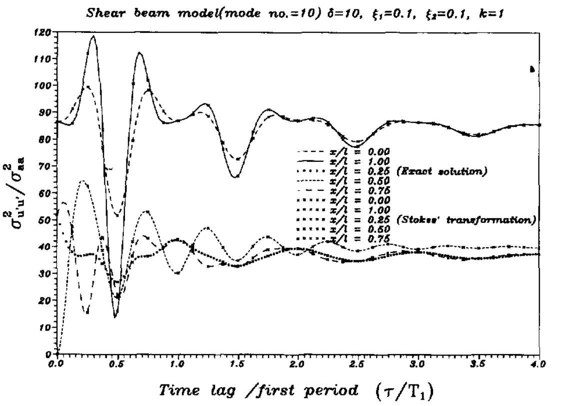

Figure 3 Mean square slope spectra for shear beam

$$H_b(x,\omega) = \frac{1}{1 - e^{-2\lambda l}} \{e^{\lambda(x-l)} - e^{-\lambda(x+l)}\} \tag{91}$$

$$\lambda = \sqrt{\frac{\rho\omega(-\omega + i2\alpha)}{G(1 + i\beta\omega)}} \tag{92}$$

The mean square displacements and slopes at $x/l = 0$, 0.25, 0.5, 0.75 and 1.0 versus time delay are shown in *Figures 2* and *3*. The mean square displacement and slope along the x-axis for fully coherent excitation without time delay ($\tau = 0$) are shown in *Figures 4* and *5*, respectively. The Gibbs phenomenon and series divergence can be avoided by using the method of Cesàro sum and Stokes' transformation, respectively. The Cesàro sum requires a larger number of modes to obtain the same accuracy than does the Stokes' transformation. The results of the present formulation, Mindlin and Goodmans' method and the exact solution are in good agreement. These figures exhibit the same features observed by Masri and Udwadia[27] and Yeh *et al.*[6] in terms of magnitude and spectral peaks using both regularization techniques.

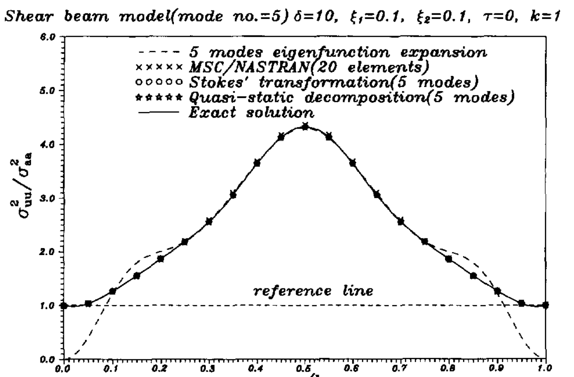

Figure 4 Mean square displacement along x-axis for shear beam

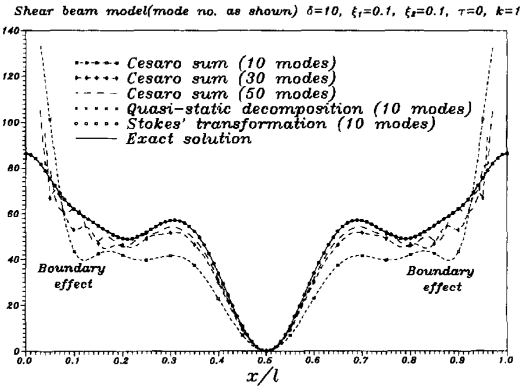

Figure 5 Mean square slope along x-axis for shear beam

9. Random responses for a flexural beam

Consider a simply supported uniform flexural beam excited by random boundary support motion at both ends. The data prepared are shown below

governing equation

$$EI\,\frac{\partial^4 u(x,t)}{\partial x^4} + c\dot{u} + \rho\,\ddot{u} = 0,\ \ 0 < x < l$$

natural frequency

$$\omega_n = (n\pi/l)^2\,\sqrt{EI/\rho}$$

normal mode

$$u_n(x) = \sin(n\pi x/l)$$

modal generalized mass

$$N_m = \rho l/2$$

modal reaction vector

$$R_m(0) = \frac{EI(m\pi)^3}{l^3} \tag{93}$$

$$R_m(l) = \frac{-EI(m\pi)^3}{l^3}(-1)^m \tag{94}$$

The input power spectrum: modified Kanai–Tajimi spectrum model

$$S_{aa} = S_{bb} = \frac{(\omega_g^4 + 4\omega^2\omega_g^2\xi_g^2)}{[(\omega_g^2 - \omega^2)^2 + 4\omega^2\omega_g^2\xi_g^2][(\omega_f^2 - \omega^2)^2 + 4\omega^2\omega_f^2\xi_f^2]}\,S_0 \tag{95}$$

$$S_{ab} = S^*_{ba} = S_{aa}e^{i\omega\tau} = S_{bb}e^{i\omega\tau} \tag{96}$$

Model parameters

$\omega_1 = 6.06$ rad/s, $\xi_1 = 0.05$, $\omega_g = 8.82$ rad/s

$\omega_f = 5.45$ rad/s, $\xi_g = 0.60$, $\xi_f = 0.27$, $S_0 = 1$

$E = 2.0 \times 10^{11}$ N/m², $I = 0.06$ m⁴,

$\rho = 2450$ kg/m³, $l = 60$ m

According to the representation in *Tables 1* and *2*, the data can be obtained as

$$U_1(x) = \left(1 - \frac{x}{l}\right) \tag{97}$$

$$U_2(x) = \frac{x}{l} \tag{98}$$

$$\overline{U}_1(x) = \left(1 - \frac{x}{l}\right) + \sum_{n=1}^{\infty} \frac{-2}{n\pi} \sin(n\pi x/l) \tag{99}$$

$$\overline{U}_2(x) = \frac{x}{l} + \sum_{n=1}^{\infty} \frac{2}{n\pi} (-1)^n \sin(n\pi x/l) \tag{100}$$

$$\Gamma_{m1} = \frac{-EI(m\pi)^3}{\omega_m^2 l^2} \tag{101}$$

$$\Gamma_{m2} = \frac{EI(m\pi)^3(-1)^m}{\omega_m^2 l^2} \tag{102}$$

$$\overline{\Gamma}_{m1} = \frac{-EI(m\pi)^3}{l^2} \tag{103}$$

$$\overline{\Gamma}_{m2} = \frac{EI(m\pi)^3(-1)^m}{l^2} \tag{104}$$

Similarly, the modal participation factors in equations (101)–(104) are proportional to the modal reactions in equations (93) and (94). More detail can be found elsewhere[3]. Similarly for a shear beam, the Stokes' transformation technique is considered here and the autopower spectrum can be written as

$$S_{uu}(x,\omega) = \sum_{i=1}^{N_s}\sum_{j=1}^{N_s} r_i(x)r_j(x)\, S_{ij}(\omega)$$
$$+ \sum_{i=1}^{N_s}\sum_{j=1}^{N_s}\sum_{m=1}^{N} [-u_m(x)(r_i(x)P_{mj}H_m(\omega)$$
$$+ r_j(x)P_{mi}H_m^*(\omega))]\omega^2 S_{ij}(\omega)$$
$$+ \sum_{i=1}^{N_s}\sum_{j=1}^{N_s}\sum_{m=1}^{N}\sum_{n=1}^{N} [u_m(x)u_n(x)P_{mi}P_{nj}H_n^*(\omega)$$
$$H_m(\omega))]\omega^4 S_{ij}(\omega)$$

$$S_{u'u'}(x,\omega) = \sum_{i=1}^{N_s}\sum_{j=1}^{N_s} r_i'(x)r_j'(x)\, S_{ij}(\omega)$$
$$+ \sum_{i=1}^{N_s}\sum_{j=1}^{N_s}\sum_{m=1}^{N} [-u_m'(x)(r_i'(x)P_{mj}H_m(\omega)$$
$$+ r_j'(x)P_{mi}H_m^*(\omega))]\omega^2 S_{ij}(\omega)$$
$$+ \sum_{i=1}^{N_s}\sum_{j=1}^{N_s}\sum_{m=1}^{N}\sum_{n=1}^{N} [u_m'(x)u_n'(x)P_{mi}P_{nj}H_n^*(\omega)$$
$$H_m(\omega))]\omega^4 S_{ij}(\omega)$$

$$S_{MM}(x,\omega) = \sum_{i=1}^{N_s}\sum_{j=1}^{N_s} EIr_i''(x)EIr_j''(x)\, S_{ij}(\omega)$$
$$+ \sum_{i=1}^{N_s}\sum_{j=1}^{N_s}\sum_{m=1}^{N} [-EIu_m''(x)(EIr_i''(x)P_{mj}H_m(\omega)$$
$$+ EIr_j''(x)P_{mi}H_m^*(\omega))]\omega^2 S_{ij}(\omega)$$
$$+ \sum_{i=1}^{N_s}\sum_{j=1}^{N_s}\sum_{m=1}^{N}\sum_{n=1}^{N} [EIu_m''(x)EIu_n''(x)P_{mi}P_{nj}H_n^*(\omega)$$
$$H_m(\omega))]\omega^4 S_{ij}(\omega)$$

$$S_{VV}(x,\omega) = \sum_{i=1}^{N_s}\sum_{j=1}^{N_s} EIr_i'''(x)EIr_j'''(x)\, S_{ij}(\omega)$$
$$+ \sum_{i=1}^{N_s}\sum_{j=1}^{N_s}\sum_{m=1}^{N} [-EIu_m'''(x)(EIr_i'''(x)P_{mj}H_m(\omega)$$
$$+ EIr_j'''(x)P_{mi}H_m^*(\omega))]\omega^2 S_{ij}(\omega)$$
$$+ \sum_{i=1}^{N_s}\sum_{j=1}^{N_s}\sum_{m=1}^{N}\sum_{n=1}^{N} [EIu_m'''(x)EIu_n'''(x)P_{mi}P_{nj}H_n^*(\omega)$$
$$H_m(\omega))]\omega^4 S_{ij}(\omega)$$

For this case, an exact solution has also been derived by Tsaur[24] using dynamic shape functions as follows

$$S_{uu}(x,\omega) = |H_a(x,\omega)|^2\, S_{aa}(\omega)$$
$$+ |H_b(x,\omega)|^2\, S_{bb}(\omega)$$
$$+ 2\mathrm{Re}\{H_a^*(x,\omega)H_b(x,\omega)S_{ab}(\omega)\} \tag{105}$$

$$S_{u'u'}(x,\omega) = |H_a'(x,\omega)|^2\, S_{aa}(\omega)$$
$$+ |H_b'(x,\omega)|^2\, S_{bb}(\omega)$$
$$+ 2\mathrm{Re}\{H_a'^*(x,\omega)H_b'(x,\omega)S_{ab}(\omega)\} \tag{106}$$

$$S_{MM}(x,\omega) = EI\{|H_a''(x,\omega)|^2\, S_{aa}(\omega)$$
$$+ |H_b''(x,\omega)|^2\, S_{bb}(\omega)$$
$$+ 2\mathrm{Re}\{H_a''^*(x,\omega)H_b''(x,\omega)S_{ab}(\omega)\}\} \tag{107}$$

$$S_{VV}(x,\omega) = EI\{|H_a'''(x,\omega)|^2\, S_{aa}(\omega)$$
$$+ |H_b'''(x,\omega)|^2\, S_{bb}(\omega)$$
$$+ 2\mathrm{Re}\{H_a'''^*(x,\omega)H_b'''(x,\omega)S_{ab}(\omega)\}\} \tag{108}$$

where

$$H_a(x,\omega) = \frac{1-\mu}{1-e^{2\lambda_1 l}}\{e^{\lambda_1 x} - e^{\lambda_1(2l-x)}\}$$
$$+ \frac{\mu}{1-e^{2\lambda_2 l}}\{e^{\lambda_2 x} - e^{\lambda_2(2l-x)}\} \tag{109}$$

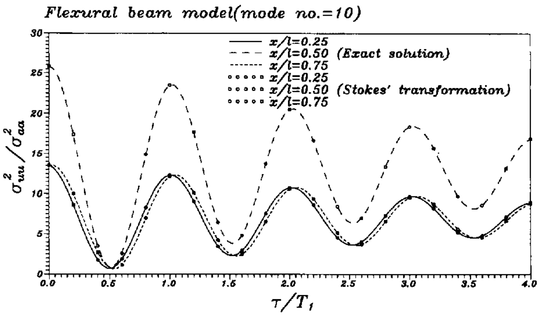

Figure 6 Mean square displacement spectra for flexural beam

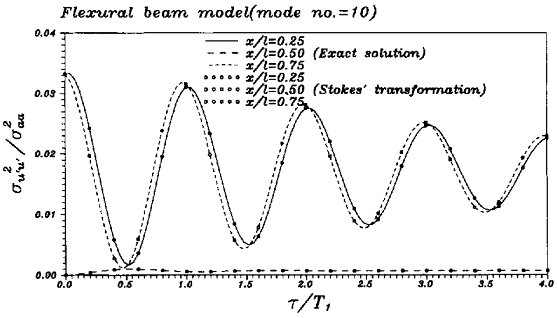

Figure 7 Mean square slope spectra for flexural beam

$$H_b(x,\omega) = \frac{1-\mu}{1-e^{2\lambda_1 l}}\{e^{\lambda_1(l-x)} - e^{\lambda_1(x+l)}\}$$

$$+ \frac{\mu}{1-e^{2\lambda_2 l}}\{e^{\lambda_2(l-x)} - e^{\lambda_2(x+l)}\} \quad (110)$$

$$\mu = \frac{\lambda_1^2}{\lambda_1^2 - \lambda_2^2} \quad (111)$$

$$\lambda_1 = \left\{\frac{m\omega^2 - i2\omega c}{EI}\right\}^{1/4} \quad (112)$$

$$\lambda_2 = i\left\{\frac{m\omega^2 - i2\omega c}{EI}\right\}^{1/4} \quad (113)$$

The mean square displacements, slopes, moments and shear forces at $x/l = 0$, 0.25, 0.5, 0.75 and 1.0 versus time delay are shown in *Figures 6–9*. The mean square displacement, slope, moment and shear force of x for three different excitations, including fully coherent, fully incoherent and phase shift with one natural period of the first mode are shown in *Figures 10–13*, respectively. The results of the present

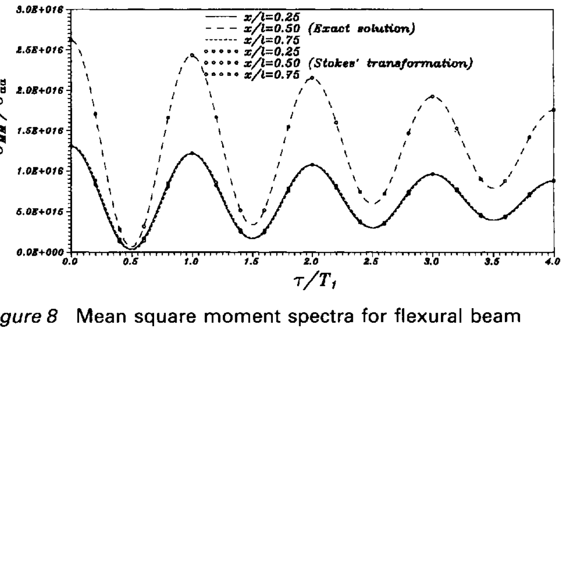

Figure 8 Mean square moment spectra for flexural beam

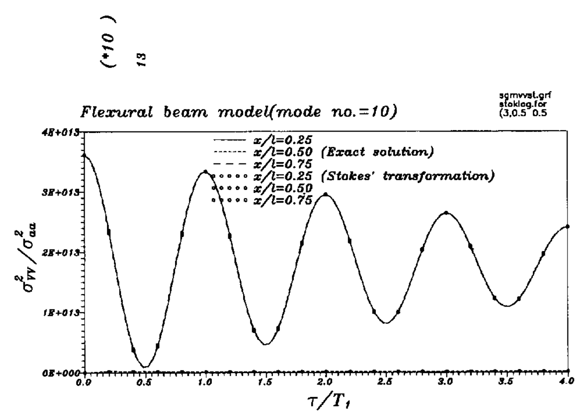

Figure 9 Mean square shear spectra for flexural beam

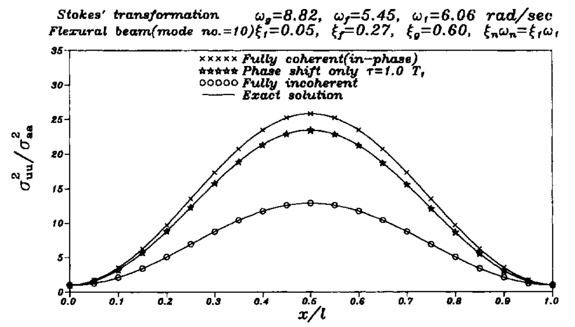

Figure 10 Mean square displacement along x-axis for three cases

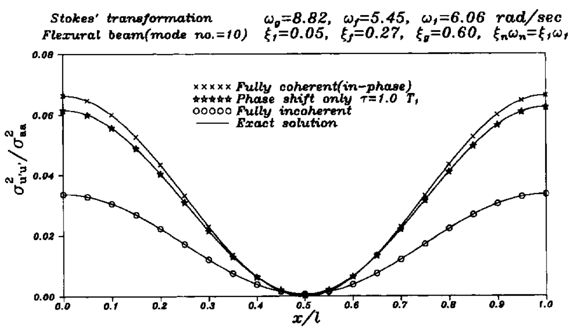

Figure 11 Mean square slope along x-axis for three cases

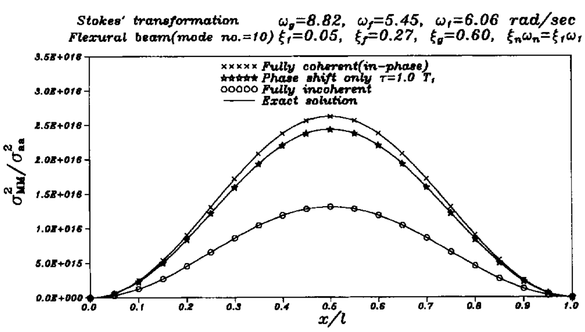

Figure 12 Mean square moment along x-axis for three cases

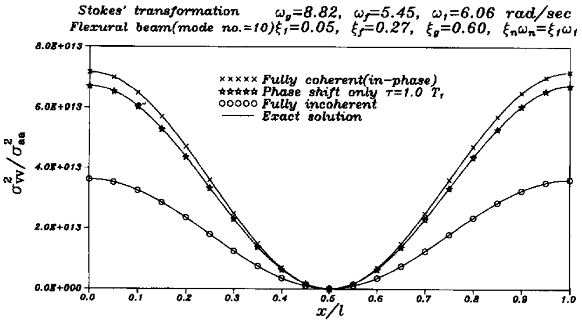

Figure 13 Mean square shear along *x*-axis for three cases

formulation, Mindlin and Goodmans' method and the exact solution are in good agreement. The oscillation behaviours are qualitatively similar to the shear beam results as investigated above; i.e., marked amplification is present for certain dimensionless time delays. *Figures 10–13* show that the maximum variance occurs in the fully coherent case, and the minimum variance occurs in the fully incoherent case. This result was also found in Reference 28.

10. Conclusions

The Stokes' transformation has been applied to a shear beam, a flexural beam and a finite elastic body subjected to multiple support motions. Only integrating a known series is needed by the Stokes' transformation instead of directly solving a PDE by the quasistatic decomposition method. The results of three available methods, quasistatic decomposition, the Cesàro sum and the exact solution have been compared with those of the proposed Stokes' transformation method. The numerical results for random responses demonstrate the validity of applying the Stokes' transformation. The FEM results by NASTRAN are also satisfactory. This research paper has proposed a regularization technique for divergent series, which is also a useful tool for analytical formulation in series representation. The proposed technique has also been successfully applied to the string problem[29].

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