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Normalized quasi-static mass – A new definition for multi-support motion problems

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Abstract

An efficient algorithm called the modal reaction method for calculating the modal participation factors in support motion problems has been proposed by Chen et al. [1]. In this paper, we extend this method to determine the number of modes needed to satisfy 90% of the sum of the base-shear modal mass as UBC(uniform building code) suggests. The sum of all the modes for each support in multi-support motions is found to be equal to the normalized quasi-static mass which is defined in this paper. The normalized quasi-static mass is equivalent to the total structure mass in the case of single supported structure. By extracting the reaction from the SPC force in data recovery using SOL 3 (linear modal analysis) or SOL 106 (nonlinear modal analysis) in MSC/NASTRAN, the modal participation factor and the base-shear modal mass ratio can be directly determined free from calculation of the influence vector, or the so-called quasi-static solution. To demonstrate this new concept of the normalized quasi-static mass, several examples including rod, beam, tower structures are given to check the validity of the proposed method using MSC/NASTRAN program. Finally, the minimum number of modes needed to reach 90% of the normalized quasi-static mass for each support is proposed as a reference for analysis and design engineers.

Keywords: Modal reaction method; Support motions; Base shear; MSC/NASTRAN; Mode superposition and normalized quasi-static mass

1. Introduction

In solving dynamic problems, either direct transient analysis or modal analysis can be utilized. For support motion problems, many approaches are available, e.g., the Laplace transform, the generalized eigenfunction expansion method, Mindlin and Goodman method, Eringen and Suhubi method and Stokes' transformation [2]. In the above-mentioned methods, the mode superposition method is employed except for the Laplace transform. For the case of modal analysis, the total response is obtained by superimposing the contributions of natural modes, and each of the generalized coordinates represents the weight of the contribution made by the corresponding mode. While the

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modal coordinate and the base-shear modal mass ratio are modal outputs for displacement and base shear, respectively, the modal participation factor plays the role of modal input by distributing external excitations to the corresponding mode. The derivations of those factors caused by body force, boundary force and support excitations are well known in the literature [3]. When the excitation is body force or boundary force, the modal displacement at the application point of the body force or the boundary force governs the magnitude of the modal participation factor. Therefore, if a concentrated load is applied at the node of a certain mode shape, then the modal participation factor of the mode is zero since no work is done. For a long time, the modal participation factor for support motion has been calculated by using the influence vector; in other words, the quasi-static solution as Mindlin and Goodman proposed, even though the resulting calculation is very time-consuming and has little physical meaning [3]. Recently, Chen et al. [1, 2, 16, 17] found that the modal participation factor is proportional to modal reaction, which is the constraint force on the support. Therefore, if support excitation is imposed at the constraint point where the modal reaction is zero, the modal participation factor is zero. The modal reaction method not only has a clear physical meaning, but also saves a large amount of computational time. However, this technique has not been used in the available programs, e.g., ABAQUS, NASTRAN [4] and ANSYS. Geyer [5] applied NASTRAN to solve the dynamic problem of a piping system subjected to multiple-support motions. Schiavello and Sinkiewicz [6] have provided a DMAP Rigid Format Alter(RFA) to calculate the modal participation factor and modal effective mass. Also, William [7] used a DMAP Alter to determine the significant modes in support motion problems according to the corresponding modal participation factor. Palmieri [8] has also shown some example problems which illustrate the effect of multiple cross-correlated excitations on the response of linear systems to Gaussian random excitations. However, all the above-mentioned papers utilized the Mindlin–Goodman method. Motivated by the significance of support motion problems in earthquake engineering, we synthesize and extend the idea to multi-supported structures and find that the sum of the base-shear modal mass is equal to the normalized quasi-static mass defined in this paper, which is equal to the total structure mass in the single support case. A base-shear modal mass ratio is defined by dividing the base-shear modal mass over the normalized quasi-static mass. After employing Parseval's equality in the discrete system, the sum of all the base-shear modal mass ratios can be proved to be one. Furthermore, the minimum number of modes needed to reach 90% of the modal mass ratio of the normalized quasi-static mass is solved by using the finite element method for several typical structures, including rod, beam and tower structures subjected to axial, transverse motions and rocking excitations.

2. Modal formulation for support motion in discrete systems

Consider the discrete system with the governing equation

$$[M]\{\ddot{U}\} + [C]\{\dot{U}\} + [K]\{U\} = \{P(t)\}, \tag{1}$$

where $[M]$ is the mass matrix, $[C]$ the damping matrix, $[K]$ stiffness matrix, and all three are symmetric square matrices of order N (N is the total number of degrees of freedom). $\{U\}$ represents displacement, and $\{P\}$ represents force, both being column matrices of order N. We then decompose the degrees of freedom into two sets, one supported and one unsupported, denoted by the subscripts r(redundant/reaction) and ℓ(left/load), respectively. Then, Eq. (1) can be rewritten as

$$\begin{bmatrix} M_{\ell\ell} & M_{\ell r} \\ M_{r\ell} & M_{rr} \end{bmatrix} \begin{Bmatrix} \ddot{U}_\ell \\ \ddot{U}_r \end{Bmatrix} + \begin{bmatrix} C_{\ell\ell} & C_{\ell r} \\ C_{r\ell} & C_{rr} \end{bmatrix} \begin{Bmatrix} \dot{U}_\ell \\ \dot{U}_r \end{Bmatrix} + \begin{bmatrix} K_{\ell\ell} & K_{\ell r} \\ K_{r\ell} & K_{rr} \end{bmatrix} \begin{Bmatrix} U_\ell \\ U_r \end{Bmatrix} = \begin{Bmatrix} P_\ell \\ P_r \end{Bmatrix}, \tag{2}$$

where $\{U_r\}$ prescribes N_r support motions, $\{P_\ell\}$ prescribes N_ℓ loads, $\{P_r\}$ contains the resulting N_r reaction forces, and $\{U_\ell\}$ contains the resulting N_ℓ displacements. It is obvious that N is equal to the sum of N_r and N_ℓ. For multi-support motion problems in earthquake engineering, the N_r entries of $\{U_r\}$ are prescribed time histories at N_r supports, and it is assumed that $\{P_\ell\} = 0$, i.e., the system is free of external loadings. For problems with external loadings, we need to superimpose the effects due to driving forces $\{P_\ell\}$, a relatively easier job, onto the results obtained in this section which is due to support motions $\{U_r\}$ only.

Similar to the quasi-static decomposition method for a continuous system considered by Mindlin and Goodman [9], the solution can be decomposed into two parts, the quasi-static part with superscript s and the inertia-dynamic part with superscript d:

$$\{U\} = \begin{Bmatrix} U_\ell \\ U_r \end{Bmatrix} = \{U^s\} + \{U^d\} = \begin{Bmatrix} U_\ell^s \\ U_r^s \end{Bmatrix} + \begin{Bmatrix} U_\ell^d \\ U_r^d \end{Bmatrix}. \tag{3}$$

By definition, the quasi-static solution satisfies

$$\begin{bmatrix} K_{\ell\ell} & K_{\ell r} \\ K_{r\ell} & K_{rr} \end{bmatrix} \begin{Bmatrix} U_\ell^s \\ U_r^s \end{Bmatrix} = \begin{Bmatrix} P_\ell^s \\ P_r^s \end{Bmatrix} \tag{4}$$

without considering acceleration and velocity terms. Also, for the inertia-dynamic part,

$$\{P_\ell^d\} = \{0\}, \tag{5}$$

$$\{U_r^d\} = \{0\} \tag{6}$$

by definition.

The eigenequation corresponding to Eq. (2) is

$$-\omega_i^2 \begin{bmatrix} M_{\ell\ell} & M_{\ell r} \\ M_{r\ell} & M_{rr} \end{bmatrix} \begin{Bmatrix} \phi_i \\ 0 \end{Bmatrix} + \begin{bmatrix} K_{\ell\ell} & K_{\ell r} \\ K_{r\ell} & K_{rr} \end{bmatrix} \begin{Bmatrix} \phi_i \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ R_i \end{Bmatrix}, \quad i = 1, 2, \ldots, N_\ell, \tag{7}$$

where $\{\phi_i\}$ denotes the ith mode shape with N_ℓ entries, and $\{R_i\}$ is the ith modal reaction forces with N_r entries. The modal matrix $[\Phi_{\ell\ell}]$ is the collection of all the N_ℓ mode shape

$$[\Phi_{\ell\ell}] = [\{\phi_1\}, \{\phi_2\}, \ldots, \{\phi_{N_\ell}\}], \tag{8}$$

with the following properties:

$$[\Phi_{\ell\ell}]^T [M_{\ell\ell}] [\Phi_{\ell\ell}] = diag(1, 1, \ldots, 1), \tag{9}$$

$$[\Phi_{\ell\ell}]^T [C_{\ell\ell}] [\Phi_{\ell\ell}] = diag(2\xi_1\omega_1, 2\xi_2\omega_2, \ldots, 2\xi_{N_\ell}\omega_{N_\ell}), \tag{10}$$

$$[\Phi_{\ell\ell}]^T [K_{\ell\ell}] [\Phi_{\ell\ell}] = diag(\omega_1^2, \omega_2^2, \ldots, \omega_{N_\ell}^2), \tag{11}$$

in which the ith modal damping ratio ξ_i and the ith modal (or natural) frequency ω_i in the case of the Rayleigh damping model with coefficients α and β satisfy

$$2\xi_i\omega_i = 2\alpha + \beta\omega_i^2, \quad i = 1,2,\ldots,N_\ell, \tag{12}$$

that is,

$$[C] = 2\alpha[M] + \beta[K]. \tag{13}$$

Denote as $\{Q^{\mathrm{d}}\}$ the column matrix of the modal (or generalized) coordinates of the inertia-dynamic part. Then, the inertia-dynamic displacements have mode superposition

$$\{U_\ell^{\mathrm{d}}\} = [\Phi_{\ell\ell}]\{Q^{\mathrm{d}}\}, \qquad \{U^{\mathrm{d}}\} = [\Phi_{N\ell}]\{Q^{\mathrm{d}}\}, \tag{14}$$

where we have defined the augmented matrix $[\Phi_{N\ell}]$ of order $N \times N_\ell$ as

$$[\Phi_{N\ell}] = \begin{bmatrix} \Phi_{\ell\ell} \\ 0 \end{bmatrix}. \tag{15}$$

Substituting first Eq. (3) and then (14) into (1), and premultiplying by $[\Phi_{N\ell}]^{\mathrm{T}}$ yields

$$\begin{aligned} &[\Phi_{N\ell}]^{\mathrm{T}}[M][\Phi_{N\ell}]\{\ddot{Q}^{\mathrm{d}}(t)\} + [\Phi_{N\ell}]^{\mathrm{T}}[C][\Phi_{N\ell}]\{\dot{Q}^{\mathrm{d}}(t)\} + [\Phi_{N\ell}]^{\mathrm{T}}[K][\Phi_{N\ell}]\{Q^{\mathrm{d}}(t)\} \\ &\quad = -[\Phi_{N\ell}]^{\mathrm{T}}[M]\,\{\ddot{U}^{\mathrm{s}}\} - [\Phi_{N\ell}]^{\mathrm{T}}[C]\,\{\dot{U}^{\mathrm{s}}\}, \end{aligned} \tag{16}$$

where the superscript T denotes the transpose of a matrix. Note that in the present formulation, we encounter no difficulty in retaining the $[\Phi_{N\ell}]^{\mathrm{T}}[C]\{\dot{U}^{\mathrm{s}}\}$ term in Eq. (16), in contrast to some papers in which this term was always neglected [10]. Substitutions of Eqs. (9)–(11) into (16) lead to

$$\ddot{Q}_i^{\mathrm{d}} + 2\xi_i\omega_i\dot{Q}_i^{\mathrm{d}} + \omega_i^2 Q_i^{\mathrm{d}} = -\begin{Bmatrix} \phi_i \\ 0 \end{Bmatrix}^{\mathrm{T}}[M]\{\ddot{U}^{\mathrm{s}}\} - \begin{Bmatrix} \phi_i \\ 0 \end{Bmatrix}^{\mathrm{T}}[C]\{\dot{U}^{\mathrm{s}}\} \tag{17}$$

for the modal coordinate of the inertia-dynamic part Q_i^{d}.

3. Modal reaction method for determining the modal participation factor

Since $\{P_\ell\} = \{P_\ell^{\mathrm{s}}\} = \{0\}$, we obtain from Eq. (4)

$$\{U_\ell^{\mathrm{s}}\} = -[K_{\ell\ell}]^{-1}[K_{\ell\mathrm{r}}]\,\{U_{\mathrm{r}}^{\mathrm{s}}\}. \tag{18}$$

Given the following two systems: one is the quasi-static system of Eq. (18) and the other is the eigensystem for the ith mode of Eq. (7). We apply Betti's law for discrete systems [11] to prove that the two discrete systems obey

$$-\omega_i^2\{\phi_i\}^{\mathrm{T}}[M_{\ell\ell}\ M_{\ell\mathrm{r}}]\,\{U^{\mathrm{s}}\} = -\omega_i^2\begin{Bmatrix} \phi_i \\ 0 \end{Bmatrix}^{\mathrm{T}}[M]\{U^{\mathrm{s}}\} = \{R_i\}^{\mathrm{T}}\{U_{\mathrm{r}}^{\mathrm{s}}\}. \tag{19}$$

Based on Eq. (7), we obtain the ith modal reaction as follows:

$$\{R_i\} = -\omega_i^2[M_{r\ell}]\{\phi_i\} + [K_{r\ell}]\{\phi_i\}. \tag{20}$$

Then, inserting Eq. (20) into the right-hand side of Eq. (19), we obtain

$$\begin{aligned}\{R_i\}^{\mathrm{T}}\,\{U_{\mathrm{r}}^{\mathrm{s}}\} &= \{-\omega_i^2[M_{r\ell}]\{\phi_i\} + [K_{r\ell}]\{\phi_i\}\}^{\mathrm{T}}\,\{U_{\mathrm{r}}^{\mathrm{s}}\}\\ &= \{-\omega_i^2\{\phi_i\}^{\mathrm{T}}[M_{\ell r}] + \{\phi_i\}^{\mathrm{T}}[K_{\ell r}]\}\,\{U_{\mathrm{r}}^{\mathrm{s}}\}.\end{aligned} \tag{21}$$

Substitution of Eq. (18) into the left-hand side of Eq. (19) yields

$$-\,\omega_i^2\{\phi_i\}^{\mathrm{T}}[M_{\ell\ell}\;M_{\ell r}]\,\{U^{\mathrm{s}}\} = \omega_i^2\,\{\{\phi_i\}^{\mathrm{T}}[M_{\ell\ell}][K_{\ell\ell}]^{-1}[K_{\ell r}] - \{\phi_i\}^{\mathrm{T}}[M_{\ell r}]\}\,\{U_{\mathrm{r}}^{\mathrm{s}}\}. \tag{22}$$

After comparing Eq. (21) with (22), to derive Eq. (19) is equivalent to finding

$$\omega_i^2\{\phi_i\}^{\mathrm{T}}[M_{\ell\ell}][K_{\ell\ell}]^{-1}[K_{\ell r}]\,\{U_{\mathrm{r}}^{\mathrm{s}}\} = \{\phi_i\}^{\mathrm{T}}[K_{\ell r}]\,\{U_{\mathrm{r}}^{\mathrm{s}}\}. \tag{23}$$

From the eigensystem of Eq. (7), we have

$$-\,\omega_i^2[M_{\ell\ell}]\{\phi_i\} + [K_{\ell\ell}]\{\phi_i\} = \{0\}. \tag{24}$$

Taking the transpose of Eq. (24) and postmultiplying with $[K_{\ell\ell}]^{-1}[K_{\ell r}]\{U_{\mathrm{r}}^{\mathrm{s}}\}$, we have

$$-\,\omega_i^2\{\phi_i\}^{\mathrm{T}}[M_{\ell\ell}][K_{\ell\ell}]^{-1}[K_{\ell r}]\,\{U_{\mathrm{r}}^{\mathrm{s}}\} + \{\phi_i\}^{\mathrm{T}}[K_{\ell\ell}][K_{\ell\ell}]^{-1}[K_{\ell r}]\,\{U_{\mathrm{r}}^{\mathrm{s}}\} = 0. \tag{25}$$

Therefore, we have derived Eq. (23) and, hence, Eq. (19).

Now, considering Eqs. (3), (6) and (17), we can express the quasi-static part as

$$\{U^{\mathrm{s}}\} = \begin{bmatrix} -K_{\ell\ell}^{-1}K_{\ell r} \\ I \end{bmatrix}\{U_{\mathrm{r}}\} \equiv [\{G_1\}\{G_2\}\cdots\{G_j\}\cdots\{G_{N_{\mathrm{r}}}\}]\begin{Bmatrix} U_{\mathrm{r}1}(t) \\ U_{\mathrm{r}2}(t) \\ \vdots \\ U_{\mathrm{r}j}(t) \\ \vdots \\ U_{\mathrm{r}N_{\mathrm{r}}}(t) \end{Bmatrix}, \tag{26}$$

where $U_{\mathrm{r}j}(t)$ is the jth support history of the r set, and $\{G_j\}$ denotes the quasi-static influence vector for $\{U^{\mathrm{s}}\}$ when only the jth entry of $\{U_{\mathrm{r}}\}$ is 1; otherwise it is zero. Eq. (26) can be rewritten as

$$\{U^{\mathrm{s}}\} = \sum_{j=1}^{N_{\mathrm{r}}}\{G_j\}\,U_{\mathrm{r}j}(t). \tag{27}$$

Substituting Eq. (27) into (17), we have

$$\begin{aligned}\ddot{Q}_i^{\mathrm{d}} + 2\xi_i\omega_i\dot{Q}_i^{\mathrm{d}} + \omega_i^2 Q_i^{\mathrm{d}} &= \sum_{j=1}^{N_{\mathrm{r}}}\left(-\begin{Bmatrix}\phi_i\\0\end{Bmatrix}^{\mathrm{T}}[M]\{G_j\}\ddot{U}_{\mathrm{r}j}(t) - \begin{Bmatrix}\phi_i\\0\end{Bmatrix}^{\mathrm{T}}[C]\{G_j\}\dot{U}_{\mathrm{r}j}(t)\right)\\ &= -\sum_{j=1}^{N_{\mathrm{r}}}(\Gamma_{ij}^{\mathrm{d}}\ddot{U}_{\mathrm{r}j}(t)) - 2\alpha\sum_{j=1}^{N_{\mathrm{r}}}(\Gamma_{ij}^{\mathrm{d}}\dot{U}_{\mathrm{r}j}(t))\end{aligned} \tag{28}$$

by using Eqs. (4) and (13), where Γ_{ij}^{d} is the ith modal participation factor for the inertia-dynamic part Q_i^{d} subjected to the jth support motion $\ddot{U}_{rj}(t)$, and is defined as

$$\Gamma_{ij}^{d} \equiv \begin{Bmatrix} \phi_i \\ 0 \end{Bmatrix}^{T} [M]\{G_j\}, \quad i = 1,2,\ldots,N_\ell, \; j = 1,2,\ldots,N_r. \tag{29}$$

By substituting the particular selection,

$$\{U^s\} = \{G_j\}, \tag{30}$$

$$\{U_r\} = \{U_r^s\} = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ U_{rj}(t) \\ \vdots \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{Bmatrix} \tag{31}$$

into Eq. (19), we obtain the general equality

$$-\omega_i^2 \begin{Bmatrix} \phi_i \\ 0 \end{Bmatrix}^{T} [M]\{G_j\} = R_{ij}, \quad i = 1,2,\ldots,N_\ell, \; j = 1,2,\ldots,N_r, \tag{32}$$

where R_{ij} is the jth entry of the ith modal reaction, i.e., the jth support reaction of the ith mode. From Eqs. (29) and (32), we have

$$\Gamma_{ij}^{d} = \frac{R_{ij}}{(-\omega_i^2)}, \quad i = 1,2,\ldots,N_\ell, \; j = 1,2,\ldots,N_r, \tag{33}$$

which expresses that the modal participation factor for the inertia-dynamic part is the modal reaction divided by the negative of the modal frequency squared. Note that the minus sign of $-\omega_i^2$ comes from the square of the imaginary unit.

As a consequence, the techniques for calculating the modal participation factors can be remarkably improved; those in accordance with Eq. (33) may be called the "modal reaction method."

4. Normalized quasi-static mass – a new definition

In earthquake-resistant design [3], the partial sum of the ith base-shear modal mass for the first support (single support), m_{i1}, plays the role of determining the modes needed since

$$\sum_{i=1}^{N_\ell} m_{i1} = \text{total structural mass}, \tag{34}$$

where the ith base-shear modal mass subjected to only one support motion can be calculated by the modal reaction as follows:

$$m_{i1} = (\Gamma^{\mathrm{d}}_{i1})^2 = \frac{(R_{i1})^2}{\omega_i^4}, \quad i = 1,2,\ldots,N_l. \tag{35}$$

In UBC code [15], the number of modes should be chosen to reach 90% of the total structural mass in the design and analysis procedures. By extending the single support to multi-support motions, the influence vector can be expressed as

$$\{G_j\} = [\Phi]\{\Gamma^{\mathrm{d}}_{ij}\} \tag{36}$$

according to Eqs. (29) and (9). If one defines the normalized quasi-static mass for the jth support as

$$M_j = \{G_j\}^{\mathrm{T}}[M]\{G_j\} \quad \text{(no summation on } j), \tag{37}$$

then by substituting Eq. (36) into Eq. (37), we obtain the norm for the vector of the modal participation factor Γ_{ij} as follows:

$$M_j = \{\Gamma^{\mathrm{d}}_{ij}\}^{\mathrm{T}}\{\Gamma^{\mathrm{d}}_{ij}\} \quad \text{(no summation on } j, \text{ summation on } i). \tag{38}$$

To construct a nondimensional number, the definition of the base-shear modal mass ratio is generalized as

$$\text{ratio of } m_{ij} = \frac{(\Gamma^{\mathrm{d}}_{ij})^2}{M_j}, \tag{39}$$

where the subscripts i and j denote the ith mode and the jth support, respectively. According to Eq. (38), the sum of the modal ratio in Eq. (39) for all the modes is equal to one for each support. For the special case of a single support structure with an influence vector of rigid-body motion as

$$\{G_j\} = \begin{Bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{Bmatrix}, \tag{40}$$

the normalized quasi-static mass in Eq. (37) is reduced to

$$M_j = \sum_i M_{ii} = \text{total structure mass}. \tag{41}$$

Comparing Eq. (41) with Eq. (34), we obtain consistent results. Eq. (38) shows that the normalized quasi-static mass for a multi-support structure is equal to the norm of the vector of the modal participation factor, Γ^{d}_{ij}. It is interesting to see that Eq. (38) is a discrete version of Parseval's equality, which states that

$$\int_D \rho(x) f^2(x)\,\mathrm{d}x = \sum_{n=1}^{\infty} a_n^2, \tag{42}$$

where $f(x)$ is a function, $\rho(x)$ a weighting function(density here), D the considered domain, and the generalized Fourier coefficient a_n is determined by

$$a_n = \frac{\int_D \rho(x) f(x) u_n(x)\,\mathrm{d}x}{\int_D \rho(x) u_n^2(x)\,\mathrm{d}x}, \tag{43}$$

in which $u_n(x)$ is the modal shape. Eqs. (38) and (42) are the Parseval's equation in discrete and continuous systems, respectively. Based on the equality in Eq. (38), the criterion for determining the minimum number of modes N_{90} needed to reach more than 90% of the normalized quasi-static mass for the modal sum of the base-shear mass can be constructed as follows:

$$\sum_{i=1}^{N_{90}} m_{ij} = \frac{\sum_{i=1}^{N_{90}} (\Gamma_{ij}^{\mathrm{d}})^2}{M_j} \geqslant 90\%. \tag{44}$$

N_{90} is suggested in modal analysis and design procedures for practical engineers.

5. MSC/NASTRAN implementations

Since the modal participation factor depends on the properties of the structure and the support point, the eigendata including the modal frequency, modal shape and modal reaction at the support point should be determined first. Any program can be utilized to demonstrate the validity of this new concept if the modal reaction can be extracted out. In this study, MSC/NASTRAN was adopted [12, 14]. By using the rigid format of SOL 3 or the structural solution sequence SOL 106 in MSC/NASTRAN, we can obtain data in the output file with the extension name F06. In the literature, the modal reaction has been often overlooked; however, it is just as important as modal frequency and modal shape for support motion problems. The modal reaction can be directly determined at the same time once the modal frequency and modal shape are obtained without matrix inversion. Therefore, the influence vector with matrix inversion as Eq. (18) shows can be avoided. It must be noted that the modal reaction at the support is extracted out by the SPC force on the SPC constraint point in MSC/NASTRAN even though it can be generated by the RBMG3 module with output data block DM if the support degree of freedom is in the r set by matrix inversion. In [1], a numerical example with 640 degrees of freedoms shows 99% of the CPU time can be saved by using the modal reaction method in comparison with the Mindlin–Goodman method. We will extend the modal reaction method to determine the minimum number of modes to satisfy the 90% requirement of the equivalent base-shear mass.

In MSC/NASTRAN implementation, when the parameter of the WTMASS card is defined as a value of α, then the modal base-shear mass ratio must be multiplied by $1/\alpha$ in order to make the sum equal to one for the consistency of units.

6. Numerical examples

Five examples are demonstrated to show the validity of the modal reaction method. The structure types and sources of excitation (translation or rocking) are shown in Figs. 1(a) – (e). The structure properties in Fig. 1(d) are based on data of the Tower of Golden Gate Bridge given in [13]. The normalized quasi-static mass is shown in Table 1. The modal frequencies, modal shapes and modal reactions were determined by using two methods, continuous system and discrete system by FEM, respectively. In the FEM implementation, the MSC/NASTRAN program was utilized to obtain the eigen data of the structure. Based on the modal reaction method, the modal participation factor and modal base-shear-equivalent mass could be determined using Eqs. (33) and (35), respectively. The partial sum of the modal base-shear mass ratios is summarized in Tables 2 and 3 by continuous system and discrete system, respectively. The partial sums of the modal base-shear mass ratio for Figs. 1(a), (b), (c), (d) and (e) by continuous system and discrete system are shown in Figs. 2(a), (b), (c), (d) and (e), respectively. The results using the two methods agree very well. To

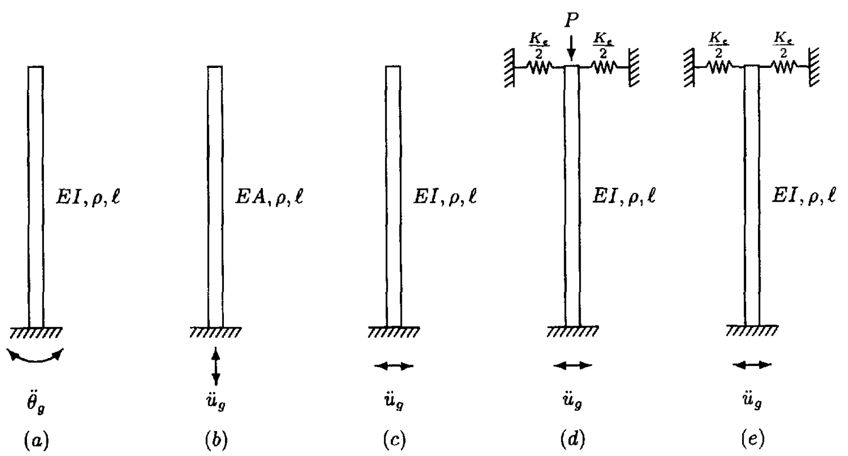

Fig. 1. (a) A cantilever beam subjected to rocking excitation; (b) A rod subjected to axial excitation; (c) A cantilever beam subjected to support excitation; (d) A restrained cantilever with axial load subjected to support excitation. (e) A restrained cantilever subjected to support excitation.

Table 1
Normalized quasi-static mass for the structures shown in Figs. 1(a)–(e)

Case	(a)	(b)	(c)	(d)	(e)
Normalized quasi-static mass $\int \rho U^2(x)\,dx$	$\frac{1}{3}\rho l$	ρl	ρl	$0.486\,\rho l$	$0.487\,\rho l$

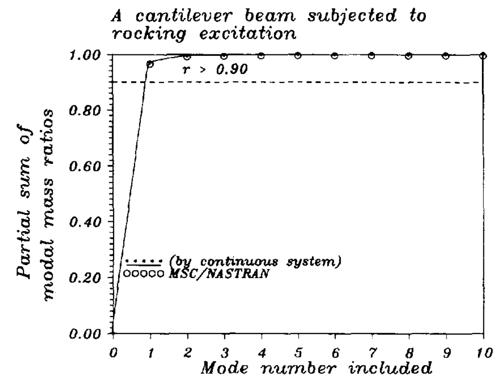

Fig. 2(a). Partial sum of the modal mass ratio for Fig. 1 (a) by continuous system and discrete system.

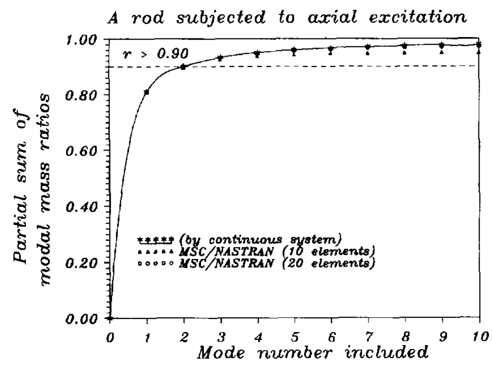

Fig. 2(b) Partial sum of the modal mass ratio for Fig. 1(b) by continuous system and discrete system.

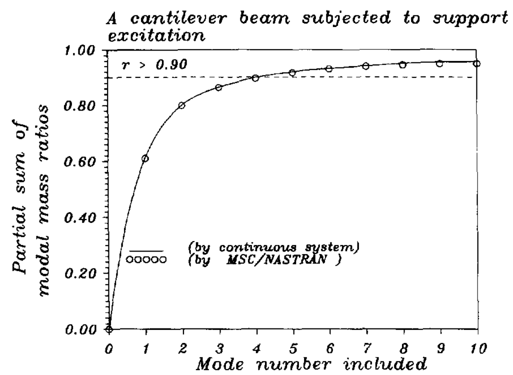

Fig. 2(c). Partial sum of the modal mass ratio for Fig. 1 (c) by continuous system and discrete system.

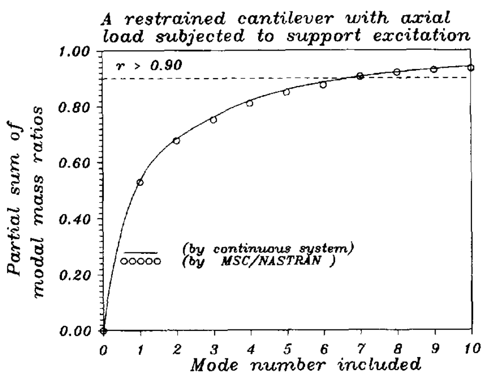

Fig. 2(d). Partial sum of the modal mass ratio for Fig. 1 (d) by continuous system and discrete system.

provide a guide for the number of modes needed in the modal analysis, the requirement of more than 90% of the normalized quasi-static mass for the modal sum of the base-shear mass ratios in UBC is also shown. It is easily found that the minimum number of modes needed to meet the requirement of UBC code for each structure is 1, 2, 4, 7 and 8, respectively, as shown in Table 4. Based on the continuous system, the effect of a restrained spring on a cantilever beam is shown in Fig. 3 with larger N_{90} while the influence of axial compression on a cantilever beam is shown in Fig. 4 with

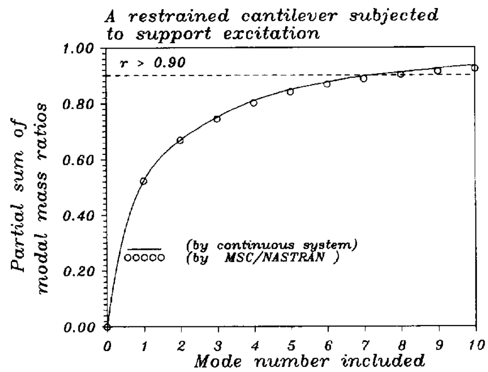

Fig. 2(e). Partial sum of the modal mass ratio for Fig. 1(e) by continuous system and discrete system.

Table 2
Partial sum of the modal mass ratios needed to meet 90% of the normalized quasi-static mass for the five structures in Figs. 1(a)–(e) by continuous system

Case	Normalized quasi-static mass	1	2	3	4	5	6	7	8
a	$\frac{1}{3}\rho l^3$	97.0%							
b	ρl	81.1%	90.1%						
c	ρl	61.3%	80.0%	86.6%	90.0%				
d	0.488 ρl	53.5%	68.4%	76.2%	82.0%	85.9%	88.6%	90.6%	
e	0.487 ρl	52.3%	67.0%	75.0%	81.1%	84.8%	87.6%	89.7%	91.2%

Table 3
Partial sum of the modal mass ratios needed to meet 90% of the normalized quasi-static mass for the five structures in Figs. 1(a)–(e) by MSC/NASTRAN

Case	Normalized quasi-static mass	1	2	3	4	5	6	7	8
a	$\frac{1}{3}\rho l^3$	96.6%							
b	ρl	80.9%	90.0%						
c	ρl	61.0%	79.9%	86.4%	91.7%				
d	0.488 ρl	52.9%	67.7%	75.2%	81.0%	84.9%	87.6%	90.5%	
e	0.487 ρl	52.3%	66.9%	74.5%	80.2%	84.1%	86.8%	88.7%	90.2%

smaller N_{90}. Also, results can be obtained using MSC/NASTRAN which agree with results based on the continuous system, as shown in Figs. 5 and 6. All of the results can be understood from the fact that a spring restraint makes the structure stiffer; however, axial compression reduces the stiffness. The greater the structure stiffness is, the larger is the N_{90} needed.

Table 4
The minimum number of modes needed to meet 90% of the normalized quasi-static mass for the five structures in Figs. 1(a)–(e)

Case	(a)	(b)	(c)	(d)	(e)
N_{90}	1	2	4	7	8

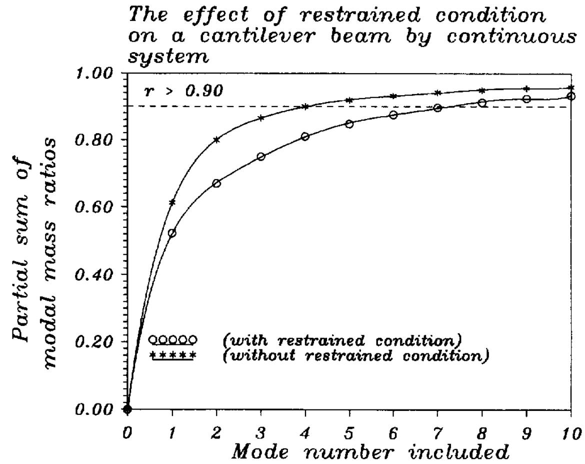

Fig. 3. The effect of a restrained condition on a cantilever beam by continuous system.

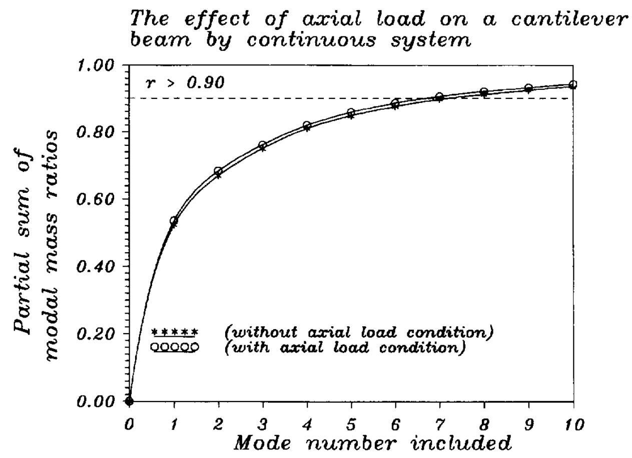

Fig. 4. The effect of an axial load on a cantilever beam by continuous system.

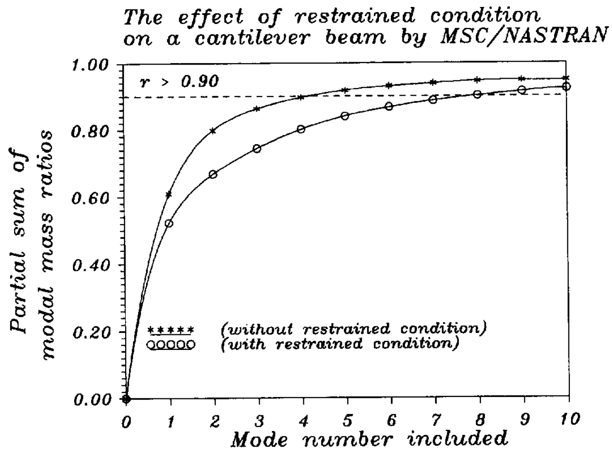

Fig. 5. The effect of a restrained condition on a cantilever beam by MSC/NASTRAN.

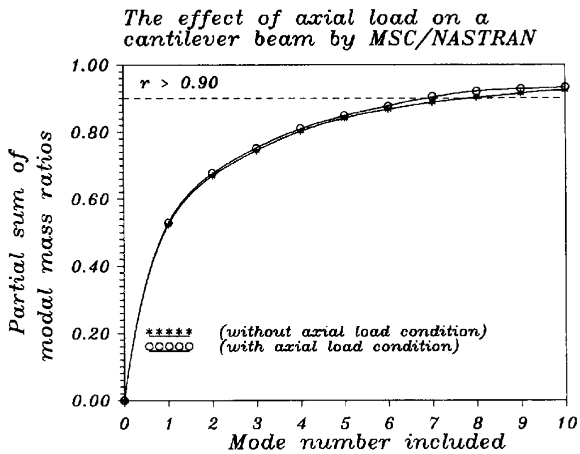

Fig. 6. The effect of an axial load on a cantilever beam by MSC/NASTRAN.

7. Conclusions

The sum of all the base-shear modal mass ratios has been found to be equal to the normalized quasi-static mass, which has been defined in this paper for multi-support structures. Several examples including rod, beam and tower structures have been given to demonstrate the validity of the proposed method. The minimum number of modes needed to reach 90% of the normalized quasi-static mass ratio for each support has been proposed as a reference for design engineers.

References

[1] J.T. Chen, H.-K. Hong and C.S. Yeh, "Modal reaction method for modal participation factors in support motion problems", *Commun. Numer. Methods Eng.* **11**, pp. 479–490, 1995.

[2] J.T. Chen, H.-K. Hong, C.S. Yeh and S.W. Chyuan, "Integral representations and regularizations for divergent series solution of a beam subjected to multi-support motion, *Earthquake Eng. Struct. Dyn.* **25**, pp. 909–925, 1996.

[3] G.V. Berg, *Elements of Structural Dynamics*, Prentice-Hall, Englewood Cliffs, NJ, 1988.

[4] M.A. Gockel, *Handbook of Dynamic Analysis*, pp. 3.5-6–3.5-11, 4.5-3–4.5-9, 1983.

[5] A. Geyer, "Multiple support excitation for NASTRAN piping analysis", *Proc. MSC/NASTRAN Eur. User's Conf.*, 1982.

[6] D.V. Schiavello and J.E. Sinkiewicz, "DMAP for determining modal participation factor", *Proc. MSC/NASTRAN World Users' Conf.*, 1983.

[7] C.R. William, "A NASTRAN DMAP procedure for calculation of base excitation modal participation factors", 11th *NASTRAN Users' Colloq.*, pp. 113–140, 1983.

[8] F.M. Palmieri, "Example problems illustrating the effect of multiple cross correlated excitations on the response of linear systems to Gaussian random excitations", *MSC 1988 World User's Conf.*, Vol. 1, 1988.

[9] R.D. Mindlin and L.E. Goodman, "Beam vibrations with time-dependent boundary conditions", *J. Appl. Mech. ASME* **17**, pp. 377–380, 1950.

[10] A. Der Kiureghian and A. Neuenhofer, "Response spectrum method for multi-support seismic excitations", *Earthquake Eng. Struct. Dyn.* **21**, pp. 713–740, 1992.

[11] J.T. Chen and H.-K. Hong, *Boundary Element Method*, 2nd edn., New World Press, Taipei, Taiwan, 1992 (in Chinese).

[12] H.G. Schaeffer, *MSC/NASTRAN Primer – Static and Normal Modes Analysis*, 3rd edn., Schaeffer Analysis Inc., 1979.

[13] A.M. Abdel-Ghaffar and J.D. Rood, Simplified earthquake analysis of suspension bridge towers, *ASCE* **108**(EM2), pp. 291–308, 1982.

[14] J.T. Chen, S.L. Lin, C.Y. Chiou, S.W. Chyuan, J.Y. Hwang, W.R. Harn and W.T. Chin, *Finite Element Analysis and Engineering Applications Using MSC/NASTRAN*, Northern Gate Publ., Taipei, Taiwan, 1996 (in Chinese).

[15] Uniform Building Code, p. 2312, International Conference of Building Officials, Whitter, CA, 1991.

[16] J.T. Chen and Y.S. Jeng, "Dual series representation and its applications to a string subjected to support motions", Adv. Engng. Software, 27, pp. 227–238, 1996.

[17] J.T. Chen and D.H. Tsaur, "An alternative method for transient and random responses subjected to support motions", Engng. Stru., 19, pp. 162–172, 1996.