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A NEW POINT OF VIEW FOR THE POLAR DECOMPOSITION USING SINGULAR VALUE DECOMPOSITION

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Abstract: In this paper, the singular value decomposition and polar decomposition in continuum mechanics are compared with and the relation is constructed. The matrix analysis is studied and the geometric interpretation is explained. The dual bases can be extracted from the right and left vectors of singular value decomposition. An illustrative example of the simple shear case is shown to see the validity of the proposed formulation.

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1. Introduction

The polar decomposition theorem in the continuum mechanics can be found in the textbooks [1, 2, 3]. It is well known that the deformation gradient (F) can be decomposed into (VR) or (RU), where R is a rotation matrix, U and V are stretching matrices. The former one

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(VR) can be explained that the total deformation process can be decomposed into rotation first and then stretching, while the latter one (RU) is stretching first and then rotation. In the matrix computation, singular value decomposition (SVD) [4] is a very powerful technique for the matrix decomposition and has been applied to engineering problems successfully [5, 6]. However, the relation between the SVD and the polar decomposition was not discussed before and their geometric interpretations in continuum mechanics were not fully understood to the authors' best knowledge.

In this paper, singular value decomposition technique is employed to understand the deformation mechanism in continuum mechanics. The role of the right and left unitary matrices in the singular value decomposition and their relation to the orthogonal matrix (R) in polar decomposition will be examined. One illustrative example with plane deformation, will be demonstrated to show the deformation mechanism by using the SVD technique. It is shown that the two unitary matrices (Φ and Ψ) in SVD provide dual bases for the deformed and undeformed systems. If the deformed and undeformed infinitesimal elements are expanded according to the dual bases, the transformed coordinates between the deformed and undeformed states can be mapped by a diagonal matrix only.

2. Polar Decomposition and SVD Technique

From the textbooks on continuum mechanics [1, 2, 3], we have

$$F = RU = VR, \tag{1}$$

where F is the deformation gradient matrix, which maps undeformed element $\mathbf{dX}$ to deformed element $\mathbf{dx}$ ($\mathbf{dx} = F\mathbf{dX}$), R is an orthogonal matrix, U and V are positive definite symmetric matrices. The U, V and R matrices can be obtained by

$$U = \sqrt{F^T F}, \tag{2}$$

$$V = \sqrt{F F^T}, \tag{3}$$

$$R = F U^{-1}, \tag{4}$$

where the superscript "T" denotes the transpose of a matrix. By employing the SVD technique [4], the F matrix can be decomposed into

$$F = \Phi\Sigma\Psi^T, \tag{5}$$

where Σ is a diagonal matrix with elements of singular values of F, Φ and Ψ are the right and left unitary matrices, respectively. By substituting Eq. (5) into Eqs.(2) and (3), we obtain

$$U = \Psi\Sigma\Psi^T, \tag{6}$$

$$V = \Phi\Sigma\Phi^T. \tag{7}$$

By substituting Eq.(6) into Eq.(4), we have

$$R = \Phi\Psi^T. \tag{8}$$

According to the property of SVD, we have

$$F\psi_i = \sigma_i\phi_i, \tag{9}$$

$$F^T\phi_i = \sigma_i\psi_i, \tag{10}$$

where σ_i is the ith singular value of F, ϕ_i and ψ_i are the ith column vectors for Φ and Ψ, respectively. According to Eqs.(6) and (7), it is easily found that U and V matrices have the same singular values (eigenvalues) (σ_i) and their eigenvectors are ψ_i and ϕ_i, respectively. If the undeformed element, $\mathbf{dX}$, is expanded in terms of the ψ_i ($i = 1, 2, 3$) bases, we have the new coordinate, $\mathbf{dY}$,

$$\mathbf{dY} = \Psi^T\mathbf{dX}. \tag{11}$$

Similarly, the deformed element, $\mathbf{dx}$, can be expanded in terms of ϕ_i ($i = 1, 2, 3$) bases and the new coordinate for $\mathbf{dy}$ is

$$\mathbf{dy} = \Phi^T\mathbf{dx}. \tag{12}$$

According to $\mathbf{dx} = F\mathbf{dX}$, the formula between the transformed coordinates, $\mathbf{dy}$ and $\mathbf{dY}$, can be derived as

$$\mathbf{dy} = \Sigma\mathbf{dY}. \tag{13}$$

It is found that the two transformed vectors ($\mathbf{dy}$ and $\mathbf{dY}$) can be mapped by the diagonal matrix (Σ) only.

3. An Illustrative Example

Considering a simple shear problem [7] defined by

$$x_1 = X_1 + \frac{2}{\sqrt{3}}X_2, \tag{14}$$

$$x_2 = X_2, \tag{15}$$

$$x_3 = X_3, \tag{16}$$

we have

$$F = \begin{bmatrix} 1 & \frac{2}{\sqrt{3}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{17}$$

$$U = \sqrt{F^T F} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{5}{2\sqrt{3}} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{18}$$

$$R = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(-30^o) & -\sin(-30^o) & 0 \\ \sin(-30^o) & \cos(-30^o) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{19}$$

$$V = FR^{-1} = \begin{bmatrix} \frac{5\sqrt{3}}{6} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{20}$$

Based on the SVD technique, F can be decomposed into

$$F = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad (21)$$

where

$$[\Phi] = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \cos(30^o) & -\sin(30^o) & 0 \\ \sin(30^o) & \cos(30^o) & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad (22)$$

$$[\Sigma] = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (23)$$

$$[\Psi]^T = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \cos(-60^o) & -\sin(-60^o) & 0 \\ \sin(-60^o) & \cos(-60^o) & 0 \\ 0 & 0 & -1 \end{bmatrix}. \quad (24)$$

By substituting Eqs.(22) $\sim$ (24) into Eqs.(6) $\sim$ (8) and comparing with the results of Eqs.(18) $\sim$ (20), the relations between (U, V, R) and (Φ, Σ, Ψ) in Eqs.(6) $\sim$ (8) are all verified. The dual bases for the undeformed (ψ_1, ψ_2, ψ_3) and deformed states (ϕ_1, ϕ_2, ϕ_3) are shown in Figure 1(a). For the undeformed vector ψ_1, the deformation process $(F = RU)$ can be decomposed into stretching with ratio $\sqrt{3}$ and then rotation -30 degrees as shown in Figure 1(b). A reverse process $(F = VR)$ can be understood that rotation -30 degrees first and then stretching with ratio $\sqrt{3}$ as shown in Figure 1(c). By considering the undeformed vector at the corner of the square as shown in Figure 1(b),

$$\mathbf{dX} = (1, 1, 0)^T \tag{25}$$

we have the transformed vector by using Eq.(11),

$$\mathbf{dY} = (\frac{1+\sqrt{3}}{2}, \frac{1-\sqrt{3}}{2}, 0)^T. \tag{26}$$

By substituting $\mathbf{dY}$ into Eq.(13), we have

$$\mathbf{dy} = (\frac{3+\sqrt{3}}{2}, \frac{-3+\sqrt{3}}{6}, 0)^T. \tag{27}$$

By substituting $\mathbf{dy}$ in Eq.(27) into Eq.(12), we have

$$\mathbf{dx} = (1 + \frac{2}{\sqrt{3}}, 1, 0)^T, \tag{28}$$

which is exactly the same as $F\mathbf{dX}$. Although the derivation is lengthy, the geometric interpretation in the rotation and stretching stages is clear. Also, the relation of polar decomposition in continuum mechanics and SVD in linear algebra is constructed.

4. Concluding Remarks

The mechanism of deformation can be understood by using the SVD technique instead of polar decomposition in this paper. The relation between the matrices in the SVD and those in the polar decomposition was constructed. Also, the deformation stages of stretching and rotation were clearly interpreted in the shown example for plane deformation. Dual bases for the deformed (ϕ_i) and undeformed (ψ_i) states are imbedded in the two unitary matrices of Φ and Ψ. The transformed coordinates for the deformed state can be mapped into that of the deformed state by a diagonal matrix if the dual bases are adopted.

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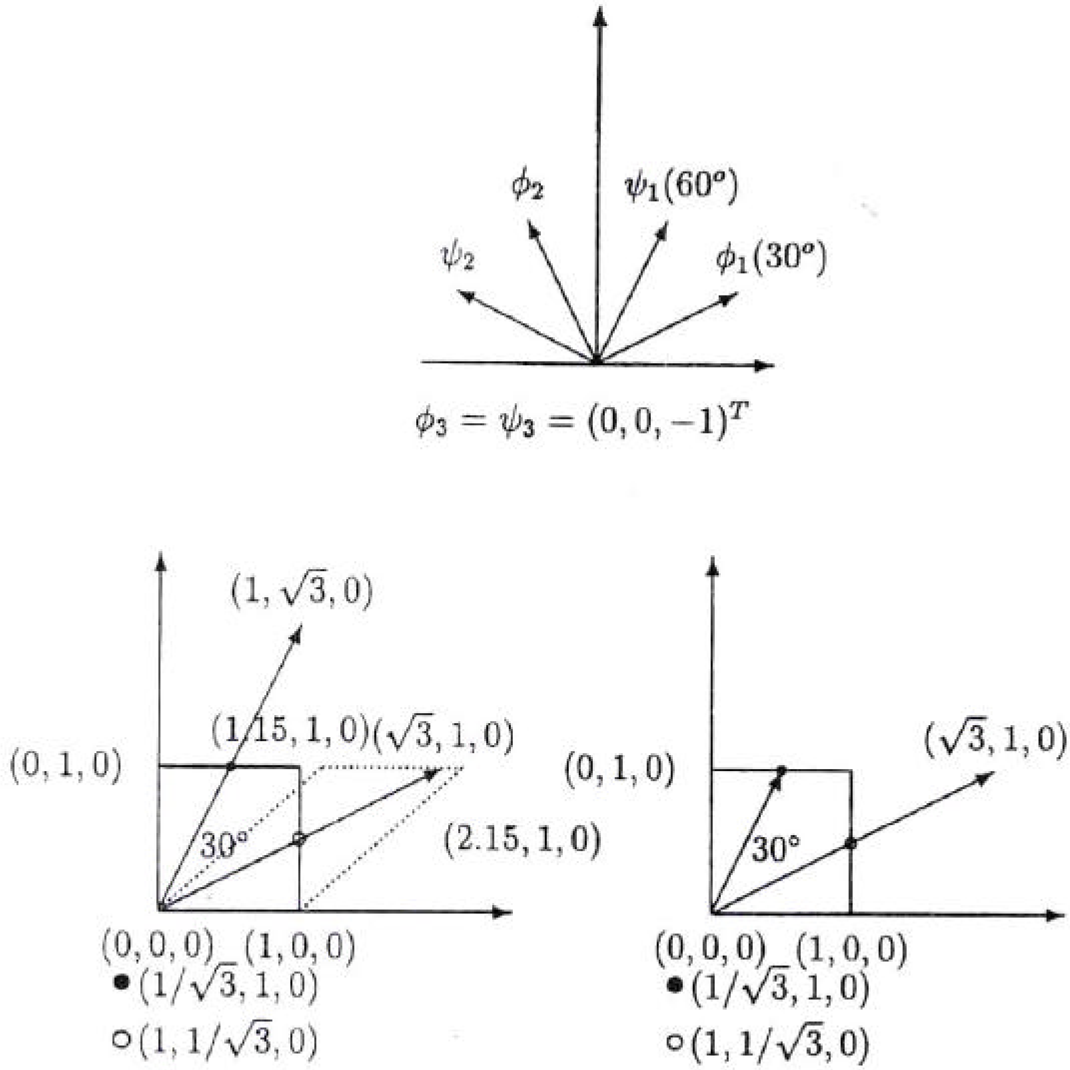

Figure 1(a): The dual bases for the deformed shape and undeformed shape.

Figure 1(b): The undeformed shape (solid line) and deformed shape (dotted line).

Figure 1(c): The undeformed element $(\frac{1}{\sqrt{3}}, 1, 0)$ and deformed element $(1, \frac{1}{\sqrt{3}}, 0)$