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ON FICTITIOUS FREQUENCIES USING DUAL SERIES REPRESENTATION

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Introduction

Acoustic problems are generally modeled by the wave equation, which is transient, or by the Helmholtz equation, which is time harmonic. Terai [1] used the dual integral formulation to solve the acoustic problem with a screen. Wu and Wan [2] also applied this formulation to thin-body acoustic problems. Burton and Miller [3] first combined the dual integral equations to avoid fictitious eigenvalues and to ensure a unique solution for all wave numbers. For exterior problems, it is well known that fictitious eigenvalues stem from the numerical resonance instead of the physical resonance. In the literature, Martin [5] pointed out that the Neumann boundary condition has fictitious frequencies corresponding to the interior Dirichlet eigenvalues while the Dirichlet boundary condition has fictitious frequencies corresponding to interior Neumann eigenvalues if the integral equation of second kind is used. However, the relations between various boundary integral equation formulations of the Dirichlet and Neumann problems has been clarified using indirect BEM by Kleinman and Roach [6]. As quoted from the Martin's paper [5]:

"It is well known that both of these methods(potential method and Green's theorem) yield integral equations which have unique solutions, except at the same discrete set of wave numbers(the irregular values), corresponding to the eigenfrequencies of the interior Dirichlet problem. The same methods can be modified to solve the exterior Dirichlet problem, and both yield integral equations of the second kind which have unique solutions except at the eigenfrequencies of the interior Neumann problem."

A similar statement can be found in Shaw's [7] paper as quoted below:

"Exterior Dirichlet → Interior Neumann eigenvalues
Exterior Neumann → Interior Dirichlet eigenvalues"

The two quoted sentences are easily misleading. Therefore, many researchers, *e.g.*, Rizzo *et al.* [8, 9] and Huang and Fan [10], have taken it for granted that boundary conditions will change fictitious eigenvalues. For example, Rizzo [9] pointed out that

"Fictitious eigenvalues are equal to eigenvalues of interior domain with " reverse " boundary conditions."

This paper will confirm the conclusion in [6] that the positions of fictitious eigenvalues are independent of the boundary conditions once the method is chosen. From the numerical point of view, this nonunique problem can be seen as the indefinite form of zero divided by zero. If L'hospital's rule is employed analytically, no fictitious eigenvalues will occur. However, L'hospital's rule can not be applied in the numerical computation.

In this paper, degenerate kernels in [11] will be employed to demonstrate the mechanism of fictitious eigenvalues using dual integral equations. As we shall see in the following, the theory of dual integral equations with degenerate kernels involves nothing more than linear algebra[12]. Therefore, some analytical results can be derived. Based on the degenerate kernels, the relations between the natural frequencies for the interior domain and the fictitious eigenvalues for the exterior domain are examined. It will also be shown that boundary conditions can not change the fictitious eigenvalues once the integral representation for the solution is chosen. Finally, three examples, one-dimensional, two dimensional and three dimensional cases, are illustrated to show the mechanism of fictitious eigenvalues in radiation problems solved using the direct method. Some misleading statements will become clear and will be corrected.

Dual integral formulation for an acoustic problem with a degenerate boundary

The linearized acoustic equation can be written as

$$-\nabla^2 u + \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = Q(x,t), \tag{1}$$

where u is the velocity potential, c is the wave velocity and $Q(x,t)$ is a source term. In the frequency domain, the governing equation is changed to

$$\nabla^2 \bar{u} + k^2 \bar{u} = \bar{Q}(x), \tag{2}$$

where the bar over the symbol means the complex amplitude, and k is the wave number defined by $k = \omega/c$. In choosing an auxiliary system, the fundamental solution $U(x,s)$ satisfies

$$\nabla^2 U(x,s) + k^2 U(x,s) = \delta(x-s) \tag{3}$$

where

$$U(x,s) = \frac{-e^{-ikr}}{4\pi r}, \quad r = \mid x - s \mid \tag{4}$$

for three-dimensional case. Using Green's third identity, we have the first equation of dual integral equations as follows:

$$u(x) = \int_B \{T(s,x)u(s) - U(s,x)t(s)\} dB(s), \quad x \in D, \tag{5}$$

where D is the domain, $t = \frac{\partial u}{\partial n}$ and the bar is omitted for simplicity. Applying the normal derivative operator to the above equation, we have

$$t(x) = \int_B \{M(s,x)u(s) - L(s,x)t(s)\} dB(s), \quad x \in D, \tag{6}$$

where U, T, L and M are the four kernels in the dual integral equations.

Kernel decomposition using degenerate series

Let $R_m(kx)$ and $I_m(kx)$, $m \in \{0,1,2,3\cdots\}$, be two linearly independent bases which satisfy

$$(\nabla^2 + k^2) R_m(kx) = 0, \quad x \in R^n, \; n = 1,2,3 \tag{7}$$

$$(\nabla^2 + k^2) I_m(kx) = 0, \quad x \in R^n, \; n = 1,2,3. \tag{8}$$

It follows that we can write

$$I_m(kx)\,(\nabla^2+k^2)R_m(kx)=0, \tag{9}$$

$$R_m(kx)\,(\nabla^2+k^2)I_m(kx)=0. \tag{10}$$

Subtracting Eq.(10) from Eq.(9), we obtain

$$\nabla\cdot\{R_m(kx)\nabla I_m(kx)-I_m(kx)\nabla R_m(kx)\}=0. \tag{11}$$

Let the general Wronskian, W, be defined as

$$\begin{aligned} W(R_m(kx),I_m(kx)) &= \{R_m(kx)\nabla I_m(kx)-I_m(kx)\nabla R_m(kx)\} \\ &= \begin{vmatrix} R_m(kx) & \nabla R_m(kx) \\ I_m(kx) & \nabla I_m(kx) \end{vmatrix}, \end{aligned} \tag{12}$$

where $|\;|$ denotes a determinant. W can be reduced to the conventional Wronskian in the one dimensional case and will be elaborated on later. Eq.(11) assures that W is solenoidal(without source); therefore, there exists a field, ψ, such that

$$W(R_m(kx),I_m(kx))=\nabla\times\psi. \tag{13}$$

When one solution $R_m(kx)$ is known, Eqs.(9) and (10) can be used to find a secondary linearly independent solution $I_m(kx)$ using the method of variations of parameters. Noting that

$$W(R_m(kx),I_m(kx))=R_m^2(kx)\;\nabla\{\frac{I_m(kx)}{R_m(kx)}\}, \tag{14}$$

we have

$$I_m(kx)=R_m(kx)\int_{x_0}^{x}\frac{\psi(x)\cdot t_s}{R_m^2(kx)}ds, \tag{15}$$

where x_0 is any reference point, and t_s is the tangential direction along the contour ds.

By using the two bases, we can decompose the kernel functions into

$$U(s,x)=\begin{cases} U^i(s,x)=\sum_{m=0}^{\infty}\frac{i}{c_m}C_m(ks)R_m(kx), & x\in D^i \\ U^e(s,x)=\sum_{m=0}^{\infty}\frac{i}{c_m}C_m(kx)R_m(ks), & x\in D^e \end{cases} \tag{16}$$

$$T(s,x)=\begin{cases} T^i(s,x)=\sum_{m=0}^{\infty}\frac{i}{c_m}\{\nabla_s C_m(ks)\cdot n(s)\}R_m(kx), & x\in D^i \\ T^e(s,x)=\sum_{m=0}^{\infty}\frac{i}{c_m}C_m(kx)\{\nabla_s R_m(ks)\cdot n(s)\}, & x\in D^e \end{cases} \tag{17}$$

$$L(s,x)=\begin{cases} L^i(s,x)=\sum_{m=0}^{\infty}\frac{i}{c_m}C_m(ks)\{\nabla_x R_m(kx)\cdot n(x)\}, & x\in D^i \\ L^e(s,x)=\sum_{m=0}^{\infty}\frac{i}{c_m}\{\nabla_x C_m(kx)\cdot n(x)\}R_m(ks), & x\in D^e \end{cases} \tag{18}$$

$$M(s,x)=\begin{cases} M^i(s,x)=\sum_{m=0}^{\infty}\frac{i}{c_m}\{\nabla_s C_m(ks)\cdot n(s)\}\{\nabla_x R_m(kx)\cdot n(x)\}, & x\in D^i \\ M^e(s,x)=\sum_{m=0}^{\infty}\frac{i}{c_m}\{\nabla_x C_m(kx)\cdot n(x)\}\{\nabla_x R_m(ks)\cdot n(s)\}, & x\in D^e, \end{cases} \tag{19}$$

where c_m can be determined by the jump value using Wronskian, D^i and D^e denote the interior domain and exterior domain enclosed by boundary B, $C_m(kx)$ is a complex function and $R_m(kx)$ is its real part with the physical meaning of the eigenmode if k is a certain eigenvalue. The imaginary part of $C_m(kx)$ is denoted by $-I_m(kx)$. The symmetry and transpose symmetry properties for the four kernels in [11] can be easily verified by the degenerate kernels. The explicit forms for the three special cases, the one-dimensional case, two-dimensional case for circular boundary, and three-dimensional case for spherical boundary, are shown in Table 1. The terms, $cos(ks)$, $J_m(k\bar{\rho})$ and $j_m(k\bar{\rho})$, are the eigenmodes for the interior domain of 1-D duct, 2-D disc and 3-D sphere in case where k is an eigenvalue, respectively.

On acoustic frequencies(interior problem) and fictitious frequencies(exterior problem)

Based on the special one, two and three dimensional cases, the operator of "$\nabla \cdot n$" is reduced to "$'$" for simplicity.

(A). Natural frequency for the internal problem

The eigenvalue, k, for the interior domain using the degenerate U, T and L, M kernels can satisfy the following relations:

$$\lim_{x \to x_B} \begin{bmatrix} C_m(kx)R_m(kx_B) & -kC_m(kx)R'_m(kx_B) \\ kC'_m(kx)R_m(kx_B) & -k^2C'_m(kx_B)R'_m(kx) \end{bmatrix} \begin{Bmatrix} t_m \\ u_m \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \tag{20}$$

where x_B is the boundary point, the first equation results from the UT method while the second results from the LM method. Eq.(20) can be reformulated into

$$\begin{bmatrix} C_m(kx_B) & -kC_m(kx_B) \\ kC'_m(kx_B) & -k^2C'_m(kx_B) \end{bmatrix} \begin{bmatrix} R_m(kx_B) & 0 \\ 0 & R'_m(kx_B) \end{bmatrix} \begin{Bmatrix} t_m \\ u_m \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{21}$$

The second matrix in Eq.(21) is defined as a postmultiplying matrix, $[V]$, where

$$[V] = \begin{bmatrix} R_m(kx_B) & 0 \\ 0 & R'_m(kx_B) \end{bmatrix}. \tag{22}$$

Since $C_m(kx_B)$ and $C'_m(kx_B)$ are never zero for any k, both the UT and LM methods have the same eigenequation constraint as shown below:

$$R_m(kx_B)t_m - kR'_m(kx_B)u_m = 0. \tag{23}$$

It is never a trivial task to find a non-trivial solution for the eigensystem, as shown below:
For the case of the Dirichlet boundary condition with $u_m = 0$, both methods have the same eigenequation:

$$\lim_{x \to x_B} R_m(kx_B)\, t_m = 0, \tag{24}$$

where a nontrivial solution occurs at $R_m(kx_B) = 0$ when k is a certain eigenvalue. In a similar way, the method can be extended to problems with Neumann and mixed type boundary conditions. It is found that the acoustic frequency for the interior domain is independent of the integral representations, *e.g.*, the UT or LM methods, but is changed by the types of homogeneous boundary conditions as the physical phenomenon shows.

(B). Fictitious eigenvalue for the exterior problem

The algebraic equations for the exterior domain obtained using the dual degenerate kernels can be obtained as follows:

$$\lim_{x\to x_B}\begin{bmatrix} C_m(kx_B)R_m(kx) & -kC'_m(kx_B)R_m(kx) \\ kC_m(kx_B)R'_m(kx) & -k^2C'_m(kx)R'_m(kx_B) \end{bmatrix}\begin{Bmatrix} t_m \\ u_m \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \tag{25}$$

where the first equation results from the UT method while the second results from the LM method. Eq.(25) can be reformulated into

$$\begin{bmatrix} R_m(kx_B) & 0 \\ 0 & R'_m(kx_B) \end{bmatrix}\begin{bmatrix} C_m(kx_B) & -kC'_m(kx_B) \\ kC_m(kx_B) & -k^2C'_m(kx_B) \end{bmatrix}\begin{Bmatrix} t_m \\ u_m \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{26}$$

The first matrix in Eq.(26) is defined as a premultiplying matrix, $[S]$, where

$$[S] = \begin{bmatrix} R_m(kx_B) & 0 \\ 0 & R'_m(kx_B) \end{bmatrix}. \tag{27}$$

For radiation problems with any boundary conditions, the UT and LM methods have different trivial constraints as show below:

$$R_m(k_{UT}\,x_B) = 0, \; UT \; method$$

$$R'_m(k_{LM}\,x_B) = 0, \; LM \; method,$$

where k_{UT} and k_{LM} denote different fictitious eigenvalues for the UT and LM methods, respectively. This means that the UT or LM method provides trivial information when k_{UT} or k_{LM} is a fictitious eigenvalue which renders $[S]$ singular. In comparison with the acoustic frequency in the interior domain, the results indicate that the boundary condition type can not change the positions of fictitious eigenvalues, but that the integral representations can.

Concluding remarks

The mechanism of fictitious eigenvalues in direct BEM has been discussed in detail using dual series representations. The fictitious eigenvalues for the three special cases are summarized in Table 1. Also, the fictitious eigenvalues embedded in the indirect method by single layer potential or double layer potential occur at the same positions obtained by the UT or LM methods, respectively. It is found that the irregular values depend on the first or second equation used in the dual integral equations. The theoretical proof of the independence of boundary conditions has been shown in this paper by using degenerate kernels, and three examples have been given. All three examples show that the first UT equation produces fictitious eigenvalues which are associated with the interior acoustic frequency with essential homogeneous boundary conditions, while the second LM equation produces fictitious eigenvalues which are associated with the interior eigenfrequency with natural homogeneous boundary conditions.

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Table 1: Fictitious eigenvalues for one, two and three dimensional problems.

	duct 1-D	circular disc 2-D	spherical body 3-D
single layer using U, L	$cos(k\bar{a}) = 0$	$J_m(k\bar{a}) = 0$	$j_m(k\bar{a}) = 0$
double layer using T, M	$sin(k\bar{a}) = 0$	$J'_m(k\bar{a}) = 0$	$j'_m(k\bar{a}) = 0$
direct BEM using U, T	$cos(ka) = 0$	$J_m(ka) = 0$	$j_m(ka) = 0$
direct BEM using L, M	$sin(ka) = 0$	$J'_m(ka) = 0$	$j'_m(ka) = 0$
$R_m(ks)$	$cos(ks) = 0$	$J_m(k\bar{\rho})e^{in\bar{\theta}}$	$j_m(k\bar{\rho})P^l_m(cos(\bar{\theta}))cos(l\bar{\phi})$
$C_m(ks)$	e^{-iks}	$H^{(2)}_m(k\bar{\rho})e^{-in\bar{\theta}}$	$h^{(2)}_m(k\bar{\rho})P^l_m(cos(\bar{\theta}))cos(l\bar{\phi})$
$I_m(ks)$	$sin(ks) = 0$	$Y_m(k\bar{\rho})e^{in\bar{\theta}}$	$y_m(k\bar{\rho})P^l_m(cos(\bar{\theta}))cos(l\bar{\phi})$
c_m	k	4	$4\pi/k$
Wronskian	$W(cos(ka), sin(ka)) = 1$	$W(J_m(ka), Y_m(ka)) = \frac{2}{\pi ka}$	$W(j_m(ka), y_m(ka)) = \frac{1}{k^2a^2}$

where $H^{(2)}_m(k\bar{\rho})$ is the Hankel function of the second kind with the m^{th} order, $h^{(2)}_m(k\bar{\rho})$ is the spherical Hankel function of the second kind with the m^{th} order, and $P^l_m(cos(\bar{\theta}))$ is the associated Lengendre function.