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Degenerate scale for multiply connected Laplace problems

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Abstract

The degenerate scale in the boundary integral equation (BIE) or boundary element method (BEM) solution of multiply connected problem is studied in this paper. For the mathematical analysis, we use the null-field integral equation, degenerate kernels and Fourier series to examine the solvability of BIE for multiply connected problem in the discrete system. Two treatments, the method of adding a rigid body term and CHEEF concept (Combined Helmholtz Exterior integral Equation Formulation), are applied to remedy the non-unique solution due to the critical scale. The efficiency and accuracy of the two regularizations are also addressed. For simplicity without loss of generality, the eccentric case is considered to demonstrate the occurring mechanism of degenerate scale.

Keywords: Degenerate scale; Degenerate kernel; Fourier series; Multiply connected problem; Null-field integral equation

1. Introduction

The non-unique solution in BIE or BEM appears in three types such as (a) a rigid body mode for the Neumann problem, (b) the critical size (degenerate scale) of domain and (c) hypersingular formulation for multiply connected problem with constant Dirichlet boundary condition. The singularity occurs physically and mathematically in the sense that the non-unique solution for the singular matrix includes a rigid body solution for the interior Neumann (traction) problem. The second one is not physically realizable but stems from the zero singular value of influence matrix in BIEs. The numerical instability or failure due to the degenerate scale is only imbedded in plane boundary value problems (BVPs). The influence matrix may be singular for the Dirichlet problem when geometry is special. From the view point of linear algebra, the problem also originates from the rank-deficiency in the influence matrices. For example, the non-unique solution of a circle with a unit radius has been noted by Petrovsky (1991) and by Jaswon and Symm (1977). Jaswon and Symm coined the Γ-contours in their book. Some mathematicians coined it the critical value, transfinite boundary, transfinite radius and logarithmic capacity (Yan and Sloan, 1988) since it is mathematically realizable. As follows from

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the classical results summarized in the book by Hille (1962), and also from the review of the paper by Yan and Sloan (1988), it is easy to see that the degenerate scale of any bounded and multiply connected domain is equal to the degenerate scale of the outer boundary contour. A simple proof can also be obtained following the steps of the proof of an analogous statement (Proposition 5) given in the paper by Vodička and Mantič, 2004a,b, in the case of plane elasticity. From the above mentioned classical results in the potential theory it also follows that the degenerate scale of a circle is the inverse of its radius. For the Dirichlet problems, some studies for potential problems (Laplace equation) (Chen et al., 2001), (He et al., 1996) have been done. Also, the degenerate scale of multiply connected problem for the Laplace equation was discussed by Tomlinson et al. (1996). In the recent work, Chen et al. investigated the degenerate scale for the simply connected (circle) and multiply connected problems (annular) (Chen et al., 2002) by using the degenerate kernels and circulant in a discrete system. An annular region has also been considered for the harmonic equation (He et al., 1996) and the possible degenerate scales were studied in both continuous and discrete systems. Regarding to the discrete system, circulant was employed to study the singularity of the influence matrix. However, circulant property fails in the eccentric case. To the authors' best knowledge, proof in continuous system is well documented in the literature. In potential theory, the problem of the degenerate scales has thoroughly been studied theoretically in the continuous system in the book by Hille (1962). However, only annular case was studied analytically using circulant in the discrete system. Additionally, a lot of numerical studies have been carried out by Christiansen and others. This paper extends the proof of annular problem to eccentric case in the discrete system.

In this paper, we focus on the analytical investigation for the phenomenon of degenerate scale in BIE for multiply connected problems. The eccentric case is addressed to derive the occurring mechanism of the degenerate scale appearance by using degenerate kernels and Fourier series in the null-field integral equation. The addition of rigid body term in the fundamental solution can move the original degenerate scale to a new degenerate scale. Besides, the CHEEF technique is proposed to overcome the non-unique solution in the numerical implementation. The constraint of adding a point outside the domain can promote the rank of the singular matrix. A numerical example is considered to demonstrate the numerical failure in case of degenerate scale. The techniques to avoid the numerical failure or instability are verified. The sensitivity, efficiency and accuracy of the regularization methods are also examined. The main contribution of this paper is that we can prove the existence of degenerate scale for eccentric problems in the discrete system in difference to a large amount of literature in the continuous system.

2. Derivation of the occurring mechanism of degenerate scale

2.1. Boundary integral equations for the Laplace problem

The integral formulation for the domain point of Laplace problem can be derived from Green's third identity

$$2\pi u(x) = \int_B T(s,x)u(s)\,\mathrm{d}B(s) - \int_B U(s,x)t(s)\,\mathrm{d}B(s), \quad x \in D, \tag{1}$$

where s and x are the source and field points, respectively, B is the boundary and D is the domain of interest, n_x is the outward normal vector at the field point x, $U(s,x)$ and $T(s,x)$ are the kernel functions which will be elaborated on later by using the degenerate kernel expansion. The kernel function, $U(s,x)$, is the fundamental solution which satisfies

$$\nabla^2 U(x,s) = 2\pi\delta(x-s), \tag{2}$$

where $\delta(x-s)$ denotes the Dirac-delta function. Then, we can obtain the fundamental solution as follows

$$U(s,x) = \ln r, \tag{3}$$

where r is the distance between s and x ($r \equiv |x-s|$). The other kernel functions, $T(s,x)$, is defined by

$$T(s,x) \equiv \frac{\partial U(s,x)}{\partial \mathrm{n}_s}, \tag{4}$$

where n_s denotes the outward normal vector at the source point s. In this paper, we utilize the null-field integral equation to analytically study the degenerate scale problem. Once the field point x is located outside the domain, the null-field integral equations are obtained as shown below

$$0 = \int_B T(s,x)u(s)\,\mathrm{d}B(s) - \int_B U(s,x)t(s)\,\mathrm{d}B(s), \quad x \in D^{\mathrm{c}}, \tag{5}$$

where D^{c} is the complementary domain. Note that the null-field integral equation is not singular since s and x never coincide. By using the degenerate kernels, the BIE for the "boundary point" can be easily derived through the null-field integral equation without the jump and free terms. Based on the separable property, the kernel function $U(s,x)$ can be expanded into series form by separating the source point and field point in the polar coordinate:

$$U(s,x) = \begin{cases} U^{\mathrm{i}}(R,\theta;\rho,\phi) = \ln R - \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{\rho}{R}\right)^m \cos m(\theta-\phi), & R \geqslant \rho \\ U^{\mathrm{e}}(R,\theta;\rho,\phi) = \ln \rho - \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{R}{\rho}\right)^m \cos m(\theta-\phi), & \rho > R \end{cases}, \tag{6}$$

where the superscripts "i" and "e" denote the interior ($R \geqslant \rho$) and exterior ($\rho > R$) cases, respectively. After taking the normal derivative ($\partial/\partial R$) with respect to Eq. (8), the degenerate form of $T(s,x)$ kernel can be easily derived according to Eqs. (4) and (6).

2.2. Proof of the existence of the degenerate scale for a multiply connected problem

For simplicity, an eccentric case in Fig. 1 is utilized to analytically demonstrate the existence of the degenerate scale by using the degenerate kernels and Fourier series in the null-field integral equation. Regarding to

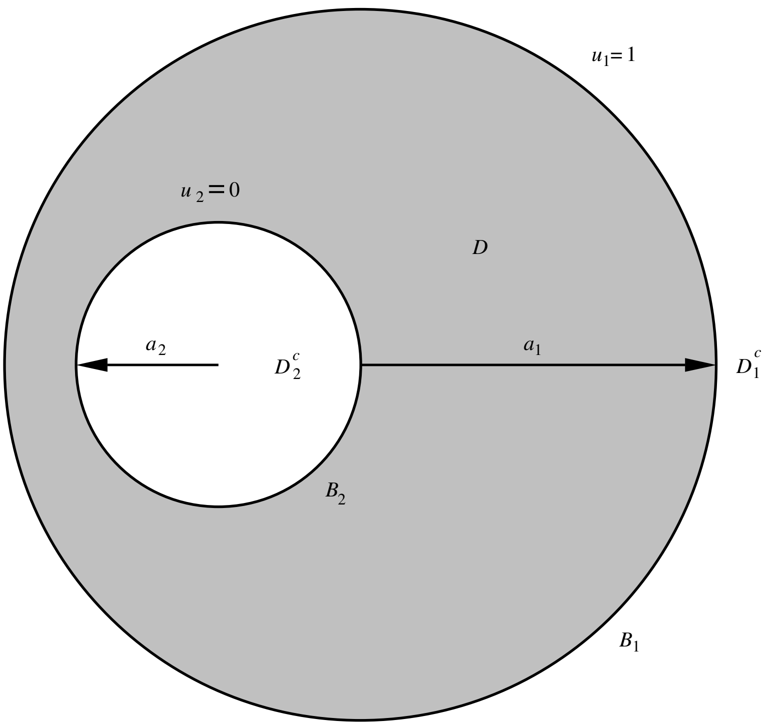

Fig. 1. Laplace problem for the eccentric case.

the boundary integration, we set the origin of the observer system at the center of each circle to fully utilize the degenerate kernels and Fourier series. The null-field integral equation of Eq. (5) can be written as shown below:

$$\begin{aligned}0 &= \int_{B_1} T^{\mathrm{e}}(s,x)u_1(s)\,\mathrm{d}B_1(s) - \int_{B_1} U^{\mathrm{e}}(s,x)t_1(s)\,\mathrm{d}B_1(s)\\ &\quad + \int_{B_2} T^{\mathrm{e}}(s,x)u_2(s)\,\mathrm{d}B_2(s) - \int_{B_2} U^{\mathrm{e}}(s,x)t_2(s)\,\mathrm{d}B_2(s), \quad x \in D_1^{\mathrm{c}}(x \to B_1^+),\end{aligned} \tag{7}$$

$$\begin{aligned}0 &= \int_{B_1} T^{\mathrm{i}}(s,x)u_1(s)\,\mathrm{d}B_1(s) - \int_{B_1} U^{\mathrm{i}}(s,x)t_1(s)\,\mathrm{d}B_1(s)\\ &\quad + \int_{B_2} T^{\mathrm{i}}(s,x)u_2(s)\,\mathrm{d}B_2(s) - \int_{B_2} U^{\mathrm{i}}(s,x)t_2(s)\,\mathrm{d}B_2(s), \quad x \in D_2^{\mathrm{c}}(x \to B_2^-),\end{aligned} \tag{8}$$

where B_1 and B_2 denote the outer and inner boundaries, respectively. Eqs. (7) and (8) are derived from different collocations chosen outside the first circular hole (B_1^+) and inside the second one (B_2^-), respectively. It is noted that the integral path is counterclockwise for the outer boundary. Otherwise, it is clockwise. Eq. (9) can be calculated by employing the orthogonal relations of trigonometric function as follows:

$$\begin{aligned}0 &= \left\{-\pi\sum_{m=1}^{\infty}\left(\frac{R_1}{\rho_1}\right)^m [a_m^1\cos m\phi_1 + b_m^1 \sin m\phi_1]\right\}\\ &\quad - \left\{2\pi R_1 \ln\rho_1 p_0^1 - \sum_{m=1}^{\infty}\frac{\pi R_1}{m}\left(\frac{R_1}{\rho_1}\right)^m [p_m^1\cos m\phi_{\mathrm{o}} + q_m^1\sin m\phi_1]\right\}\\ &\quad - \left\{-\pi\sum_{m=1}^{\infty}\left(\frac{R_2}{\rho_2}\right)^m [a_m^2\cos m\phi_2 + b_m^2 \sin m\phi_2]\right\}\\ &\quad - \left\{2\pi R_2 \ln\rho_2 p_0^2 - \sum_{m=1}^{\infty}\frac{\pi R_2}{m}\left(\frac{R_2}{\rho_2}\right)^m [p_m^2\cos m\phi_2 + q_m^2\sin m\phi_2]\right\},\end{aligned} \tag{9}$$

where the boundary data are expressed in term of Fourier series and kernels are expanded into degenerate (separable) forms. Similarly, Eq. (8) yields

$$\begin{aligned}0 &= \left\{2\pi a_0^1 + \pi\sum_{m=1}^{\infty}\left(\frac{\rho_1}{R_1}\right)^m [a_m^1\cos m\phi_1 + b_m^1 \sin m\phi_1]\right\}\\ &\quad - \left\{2\pi R_1 \ln R_1 p_0^1 - \sum_{m=1}^{\infty}\frac{\pi R_1}{m}\left(\frac{\rho_1}{R_1}\right)^m [p_m^1\cos m\phi_1 + q_m^1\sin m\phi_1]\right\}\\ &\quad - \left\{2\pi a_0^2 + \pi\sum_{m=1}^{\infty}\left(\frac{\rho_2}{R_2}\right)^m [a_m^2\cos m\phi_2 + b_m^2 \sin m\phi_2]\right\}\\ &\quad - \left\{2\pi R_2 \ln R_2 p_0^2 - \sum_{m=1}^{\infty}\frac{\pi R_2}{m}\left(\frac{\rho_2}{R_2}\right)^m [p_m^2\cos m\phi_2 + q_m^2\sin m\phi_2]\right\}.\end{aligned} \tag{10}$$

Eqs. (9) and (10) can be assembled to a linear algebraic equation

$$[\mathbf{U}]\{\mathbf{t}\} = [\mathbf{T}]\{\mathbf{u}\}, \tag{11}$$

and after substituting the boundary conditions as shown in Fig. 1, e.g., Eq. (11)

$$\left[\begin{array}{cc:cc} -2\pi a_1 \ln a_1 & \cdots & -2\pi a_2 \ln \rho_2 & \cdots \\ -2\pi a_1 \ln a_1 & \cdots & -2\pi a_2 \ln \rho_2 & \cdots \\ \vdots & \ddots & \vdots & \ddots \\ -2\pi a_1 \ln a_1 & \cdots & -2\pi a_2 \ln \rho_2 & \cdots \\ \hdashline -2\pi a_1 \ln a_1 & \cdots & -2\pi a_2 \ln a_2 & \cdots \\ -2\pi a_1 \ln a_1 & \cdots & -2\pi a_2 \ln a_2 & \cdots \\ \vdots & \ddots & \vdots & \ddots \\ -2\pi a_1 \ln a_1 & \cdots & -2\pi a_2 \ln a_2 & \cdots \end{array}\right] \left\{\begin{array}{c} p_0^1 \\ \vdots \\ p_M^1 \\ q_M^1 \\ \hdashline p_0^2 \\ \vdots \\ p_M^2 \\ q_M^2 \end{array}\right\} = \left\{\begin{array}{c} 0 \\ \vdots \\ 0 \\ 0 \\ \hdashline 2\pi \\ \vdots \\ 2\pi \\ 2\pi \end{array}\right\}. \tag{12}$$

When the outer boundary has a radius of one ($a_1 = 1.0$), Eq. (12) results in

$$\left[\begin{array}{cc:cc} 0 & \cdots & -2\pi a_2 \ln \rho_2 & \cdots \\ 0 & \cdots & -2\pi a_2 \ln \rho_2 & \cdots \\ \vdots & \ddots & \vdots & \ddots \\ 0 & \cdots & -2\pi a_2 \ln \rho_2 & \cdots \\ \hdashline 0 & \cdots & -2\pi a_2 \ln a_2 & \cdots \\ 0 & \cdots & -2\pi a_2 \ln a_2 & \cdots \\ \vdots & \ddots & \vdots & \ddots \\ 0 & \cdots & -2\pi a_2 \ln a_2 & \cdots \end{array}\right] \left\{\begin{array}{c} p_0^1 \\ \vdots \\ p_M^1 \\ q_M^1 \\ \hdashline p_0^2 \\ \vdots \\ p_M^2 \\ q_M^2 \end{array}\right\} = \left\{\begin{array}{c} 0 \\ \vdots \\ 0 \\ 0 \\ \hdashline 2\pi \\ \vdots \\ 2\pi \\ 2\pi \end{array}\right\}. \tag{13}$$

No matter what the null-field point is collocated (B_1^+ and B_2^-) due to the property of degenerate kernel in Eq. (6), one column of the influence matrix [**U**] is a zero vector. The influence matrix is singular for the Dirichlet problem as the radius a_1 is one. This finding extends the proof of annular case where the outer radius of one is a degenerate scale (Ingham and Kelmanson, 1984; Liu and Lean, 1990). It is straightforward to extend the result to multiple holes bounded by an outer circle. No matter how many inner holes are randomly distributed inside the outer boundary, the Dirichlet problem with radius ($a_1 = 1.0$) of the outer boundary is not solvable due to the rank-deficiency matrix of [**U**] in Eq. (13). This matches the comment of Vodička and Mantič as quoted from (Vodička and Mantič, 2004a,b). "*Not only the BIE associated to the exterior BVP is not-invertible but also that associated to the interior BVP defined on the domain, possibly with cavities, which outer contour is a boundary of critical domain*". The zero determinant of [**U**] results in a degenerate scale. By detecting the zero eigenvalue of matrix, we can determine the degenerate scale. The main concern of this paper is the mathematical proof of degenerate scale from an annular region to an eccentric case in the discrete system where circulant property cannot be applied.

3. Two regularization techniques to solve the non-uniqueness problem

The special scale of outer boundary results in numerical failure and/or instability. A suitable treatment of unstable system is required to solve the rank-deficiency problem and find a unique solution. Two methods, adding a rigid body term and CHEEF concept, are adopted to suppress the occurrence of the degenerate scale.

3.1. Method of adding a rigid body term

Since the [**U**] matrix is singular in the case of degenerate scale, one alternative to treat the problem is to superimpose a rigid body term in the fundamental solution. The fundamental solution can be modified by adding a rigid body term c,

$$U^{\mathrm{m}}(s,x) = U(s,x) + c, \tag{14}$$

where $U^{\mathrm{m}}(s,x)$ means the modified kernel function. The element on the first column of influence matrix [U] in Eq. (14) is added by $2\pi a_1 c$ where a_1 is the radius of the outer circle. Then, the zero singular value in [U] is shifted to a nonzero singular value for $[\mathbf{U}^{\mathrm{m}}]$. To demonstrate the effectiveness, the determinant versus radius a_1 after using the modified fundamental solution is examined in the following example.

3.2. CHEEF concept

Since the outer radius a_1 is equal to "one", the influence matrix is singular. In order to promote the rank of [U] matrix, the CHEEF concept by collocating the point outside the domain as an auxiliary constraint is applied to deal with this problem. By adding the CHEEF points outside the domain for the null-field BIE, the additional constraint is

$$\langle \mathbf{w} \rangle \{\mathbf{t}\} = \langle \mathbf{v} \rangle \{\mathbf{u}\}, \tag{15}$$

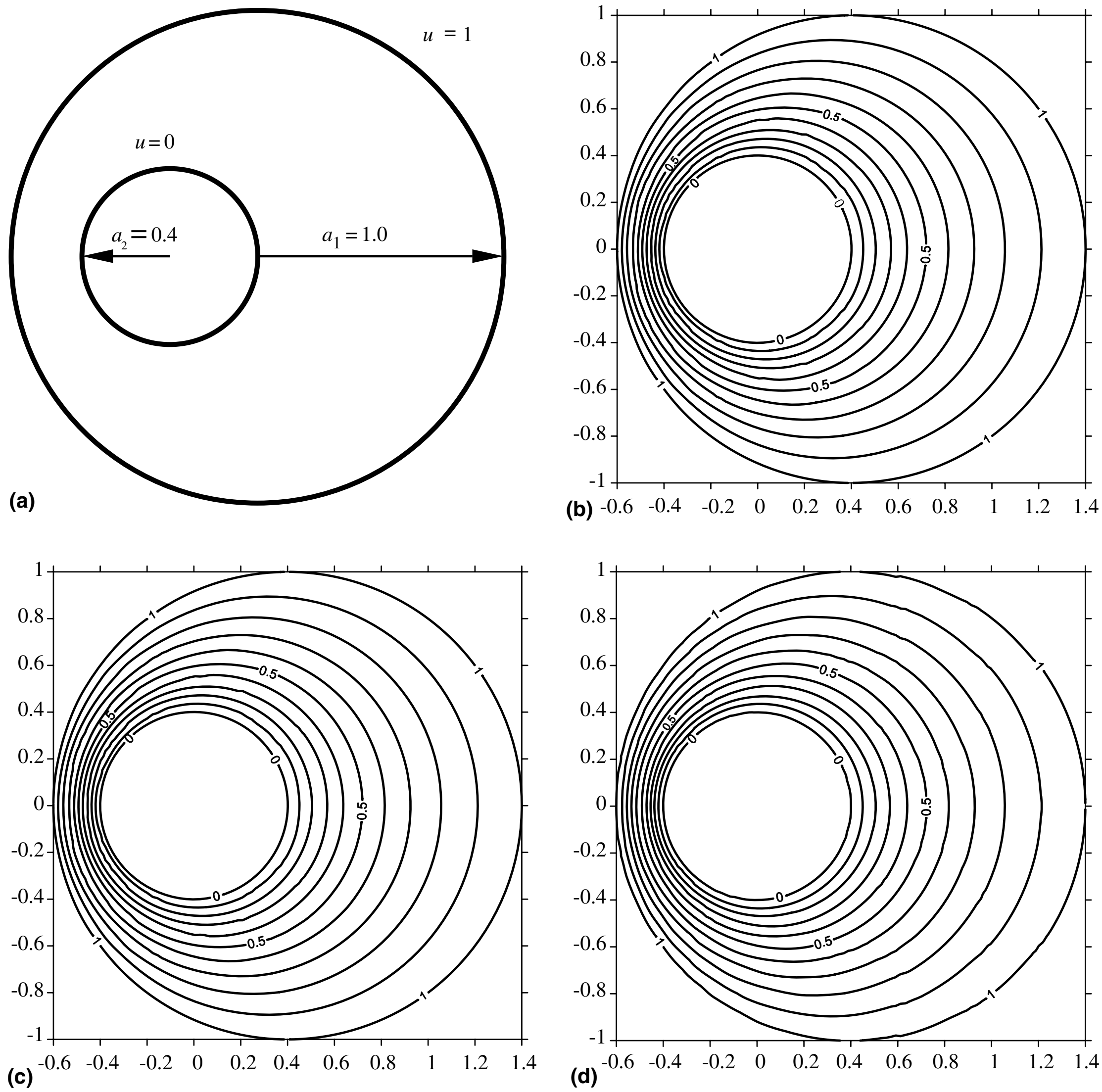

Fig. 2. (a) Problem statements; (b) contour plot for the method of adding a rigid body term; (c) contour plot for the CHEEF concept; (d) contour plot for the exact solution.

where $\langle \mathbf{w} \rangle$ and $\langle \mathbf{v} \rangle$ are the influence row vectors. By combining Eq. (11) with Eq. (15), an over-determined system is acquired

$$\begin{bmatrix} [\mathbf{U}] \\ \langle \mathbf{w} \rangle \end{bmatrix} \{\mathbf{t}\} = \begin{bmatrix} [\mathbf{T}] \\ \langle \mathbf{v} \rangle \end{bmatrix} \{\mathbf{u}\} \tag{16}$$

The zero singular value of $[\mathbf{U}]$ is changed to the nonzero singular value of $\begin{bmatrix} [\mathbf{U}] \\ \langle \mathbf{w} \rangle \end{bmatrix}$.

To obtain a squared system in the CHEEF method, either SVD or least squares techniques can be employed to solve Eq. (16).

4. Illustrative example and discussions

An eccentric case in Fig. 2(a) is examined which has the outer radius of 1 m ($a_1 = 1.0$ m) and the inner radius of 0.4 m ($a_2 = 0.4$ m). The essential boundary conditions on B_1 and B_2 are $u_1 = 1$ and $u_2 = 0$, respectively. Twenty-one collocation points are both chosen on the outer and inner boundaries. After introducing the two regularization techniques, the non-uniqueness problem due to the critical scale is solved. Fig. 2(b) and (c) show the contour plots of potential after adding a rigid body mode and CHEEF point for comparison with the exact solution of Fig. 2(d). Good agreement is made. The minimum singular value versus the radius a_1 is plotted in Fig. 3 by using the singular formulation in conjunction with the method of adding a rigid body term $c = 1.0$ and adding a CHEEF point outside the domain $(5.0, 5.0)$. The boundary flux by using the two regularization methods is shown in Fig. 4. Besides, Fig. 5 shows the relative error. According to Figs. 4 and 5, it is found that the addition of rigid body term is more accurate than choosing the CHEEF point. Also, the selection of CHEEF point is more sensitive than the constant rigid body term. The far CHEEF point improves the solution. The sensitivity of the rigid body term is not evident. Numerically speaking, addition of rigid body term is easier since one CHEEF point makes the number of equation larger in the linear algebraic system.

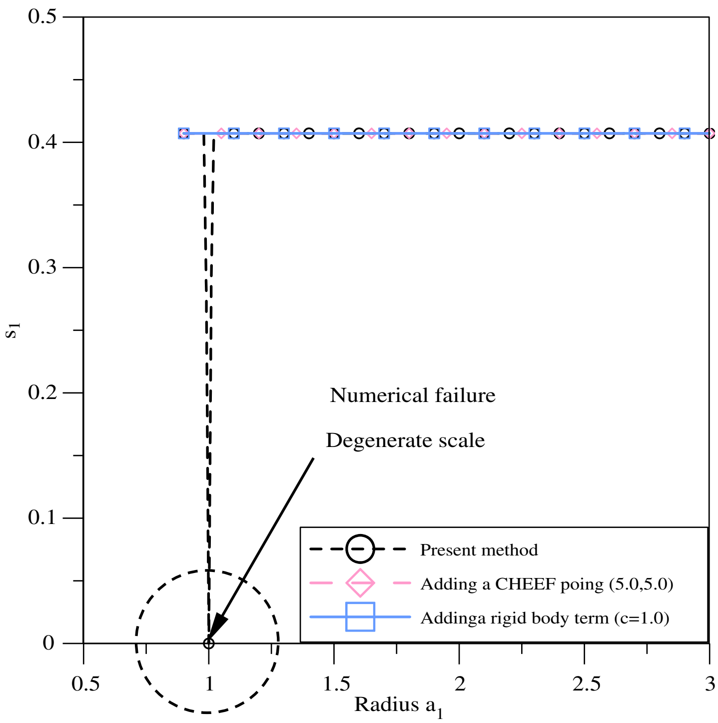

Fig. 3. The minimum singular value versus radius a_1 using different methods for the eccentric case.

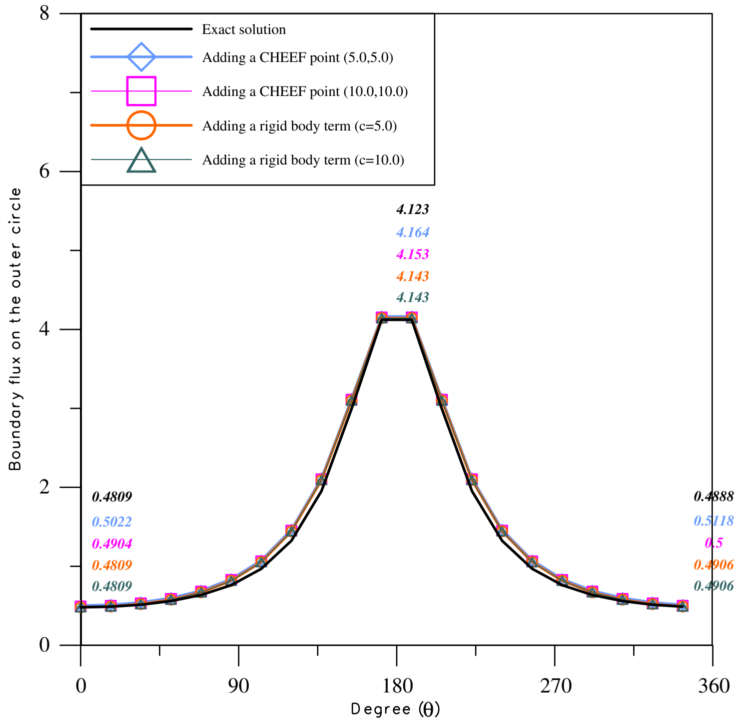

Fig. 4. Boundary flux using different methods.

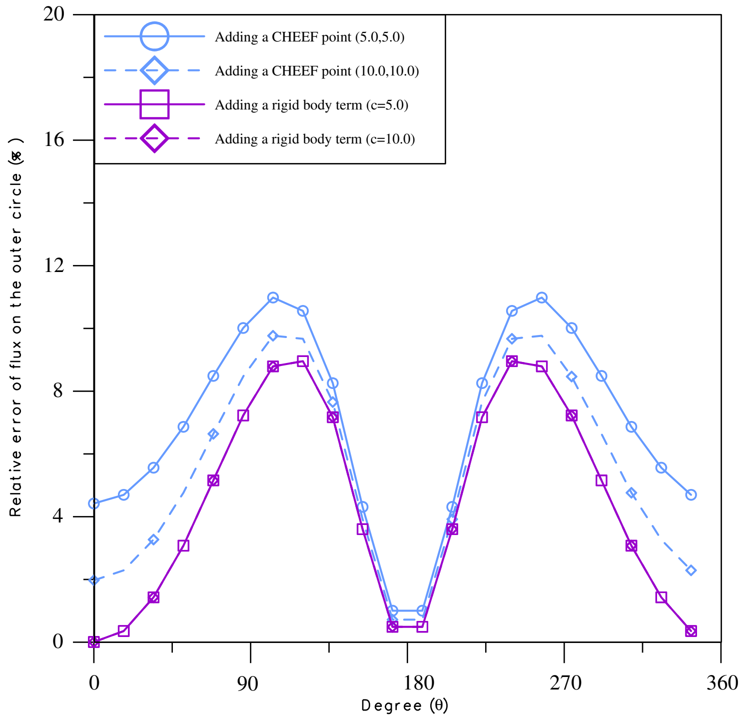

Fig. 5. Relative error of boundary flux using different methods.

5. Concluding remarks

The paper dealt with the Dirichlet problem for Laplace equation solved by the singular BIE in a special case of bounded and multiply connected domains in plane given by circular boundary curves. The contribution of the work is to show in an explicit analytic way, by means of an expansion of the integral kernel functions using degenerate kernels, how this degenerate scale appears if the unknown boundary density is approximated by a Fourier series of trigonometric functions. Also two methods how to avoid the singularity of the linear system matrix for a truncated Fourier series were proposed. We have proved the existence of degenerate scale for multiply connected problem subject to the Dirichlet boundary condition through an eccentric case in the discrete system. It can be easily extended the result to the problem containing multiple circles. The unit radius for the outer boundary is a degenerate scale if the singular equation is used. For the degenerate scale problem, the method of adding a rigid body term and CHEEF concept, have been successfully adopted to regularize the solution. The CHEEF technique can promote the rank while the method of adding a rigid body term introduces another degenerate scale. The sensitivity of the parameters of rigid body term and location of CHEEF point on the solution is also addressed. The numerical experiment of the eccentric problem was performed to demonstrate the validity of the remedies.

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