# SOLUTION OF BIHARMONIC PROBLEMS WITH CIRCULAR BOUNDARIES USING NULL-FIELD INTEGRAL EQUATIONS

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## ABSTRACT

The null-field integral equation method in conjunction with Fourier series and degenerate kernels are proposed to solve the biharmonic equations with circular boundaries. The degenerate kernels in the BIEM are expanded by using the separation of field point and source point. The improper boundary integrals are novelly avoided since the appropriate interior and exterior expansion of degenerate kernels are used. The unknown boundary densities are expressed in terms of Fourier series and the unknown Fourier coefficients can be easily determined by using the collocation method. Finally, the numerical solutions for problems of plate and Stokes flow are compared with the data of finite element solution (ABAQUS) and previous results to demonstrate the validity of the present method.

# 零場積分方程求解含圓形邊界雙諧和問題

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關鍵詞：雙諧和函數、退化核、零場積分方程、邊界積分方程、傅立葉級數、史托克斯流、Kirchhoff 板。

## 摘要

本文採用零場積分方程法結合傅立葉級數與退化核來求解含圓形邊界雙諧和方程問題。其中退化核即為分離核，係將基本解中場、源點分離導得級數形式，藉由使用退化核的內外域表示式可避免主值積分的計算。未知的邊界密度函數以傅立葉級數展開，並配合選點法，未知的傅立葉係數即可輕易求得。針對板與史托克斯流問題，本文所得之數值結果將與有限元素法套裝軟體 (ABAQUS) 結果和前人研究做一比較，以驗證本法的可行性。

## 1. Introduction

The boundary element method (BEM) by discretizing the boundary integral equation (BIE) has been extensively applied for

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solving engineering problems more than domain type methods, *e.g.* finite element methods (FEM) or finite difference methods (FDM). However, it is noted that improper integrals on the boundary should be handled particularly when BEM is used. In the past, many researchers proposed several regularization techniques to deal with the singularity and hypersingularity. To determine the Cauchy principal value (CPV) and the Hadamard principal value (HPV) in the singular and hypersingular integrals is a critical issue in BEM/BIEM. In this paper, instead of using the definitions of CPV and HPV, the kernel function is described in an analytical form on each side (interior and exterior) by employing the separation technique since the potential is discontinuous across the boundary. Therefore, degenerate kernel, namely separable kernel, is a vital tool to study the perforated domain which satisfies the biharmonic equation.

Null-field integral equation in conjunction with degenerate kernel is proposed to solve the biharmonic problems with circular boundaries. It is well known that Fourier series is always incorporated to formulate the solution for problems with circular boundaries [2, 6]. Bird and Steele [2] presented a Fourier series procedure to solve circular plate problems containing multiple circular holes in a similar way of Trefftz method by adopting the interior and exterior T-complete sets. Not only the interior but also exterior bases in the Trefftz method are embedded in degenerate kernels [5]. A bridge to connect the Trefftz method and method of fundamental solution (MFS) was constructed by using the degenerate kernels [5]. However, the null-field equations and degenerate kernels were not employed to fully capture the circular boundaries for the biharmonic problems, although Fourier series expansion was often used in the previous research [2].

According to the degenerate kernels, null-field integral formulation and Fourier series for problems with circular boundaries, the null-field integral equation approach is presented. The present formulation can be extended to solve biharmonic problems regardless of the number, location and size of circular holes. A linear algebraic system is constructed by matching the boundary conditions at the collocation points. In the polar coordinate system, the calculation of potential gradients in the normal and tangential directions for the non-concentric domain must be taken care. Therefore, the technique of vector decomposition is adopted to deal with the problem for the non-concentric domain. Finally, several examples for problems of plate and Stokes flow are presented to show the validity of the present method and some conclusions are made.

## 2. Formulation

The boundary integral equations for the domain point can be derived from Rayleigh-Green identity [3, 4] as follows:

$$
\begin{aligned}
8\pi u(x) = & -\int_B U(s,x)v(s)dB(s) + \int_B \Theta(s,x)m(s)dB(s) \\
& -\int_B M(s,x)\theta(s)dB(s) + \int_B V(s,x)u(s)dB(s), \quad x \in \Omega,
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
8\pi \theta(x) = & -\int_B U_\theta(s,x)v(s)dB(s) + \int_B \Theta_\theta(s,x)m(s)dB(s) \\
& -\int_B M_\theta(s,x)\theta(s)dB(s) + \int_B V_\theta(s,x)u(s)dB(s), \quad x \in \Omega,
\end{aligned} \tag{2}
$$

By moving the field point $x$ outside the domain, the null-field integral equations are as follows:

$$0 = -\int_B U(s,x)v(s)dB(s) + \int_B \Theta(s,x)m(s)dB(s) - \int_B M(s,x)\theta(s)dB(s) + \int_B V(s,x)u(s)dB(s), \quad x \in \Omega^C, \tag{3}$$

$$0 = -\int_B U_\theta(s,x)v(s)dB(s) + \int_B \Theta_\theta(s,x)m(s)dB(s) - \int_B M_\theta(s,x)\theta(s)dB(s) + \int_B V_\theta(s,x)u(s)dB(s), \quad x \in \Omega^C, \tag{4}$$

where $\Omega^c$ is the complementary domain of $\Omega$. The kernel functions in Eq.(1)-(4) are expanded to degenerate kernel and can be found in ref. [7] for detail. The unknown boundary densities $u(s)$, $\theta(s)$, $m(s)$ and $v(s)$ are expressed in terms of Fourier series as follows:

$$u(s) = c_0 + \sum_{n=1}^{M}(c_n \cos n\theta + d_n \sin n\theta), \quad s \in B, \tag{5}$$

$$\theta(s) = g_0 + \sum_{n=1}^{M}(g_n \cos n\theta + h_n \sin n\theta), \quad s \in B, \tag{6}$$

$$m(s) = a_0 + \sum_{n=1}^{M}(a_n \cos n\theta + b_n \sin n\theta), \quad s \in B, \tag{7}$$

$$v(s) = p_0 + \sum_{n=1}^{M}(p_n \cos n\theta + q_n \sin n\theta), \quad s \in B, \tag{8}$$

Since the higher-order singular equation Eq.(4) is also one alternative to deal with the problem, potential gradient or higher-order gradients is required to calculate carefully. For the non-concentric case as shown in Fig.1, the field point $x$ and source point $s$ can not locate on the circular boundaries with the same center. Therefore, special treatment for the normal derivative should be taken care.

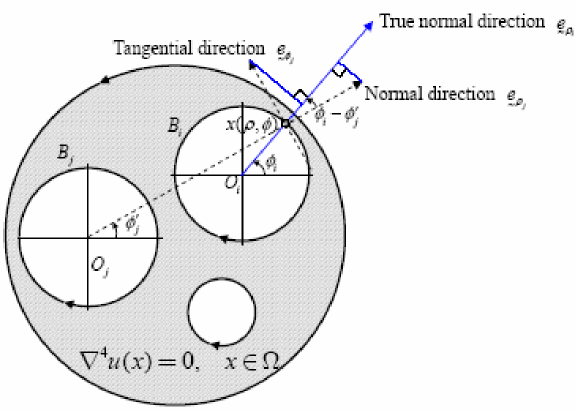



Fig.1 Vector decomposition (Collocation on $x$ and integration on $B_j$)

The degenerate kernels in Eq.(4) are reformulated as follows:

$$U_n(s,x) = \frac{\partial U(s,x)}{\partial n_x}\cos(\phi-\phi') + \frac{\partial U(s,x)}{\partial t_x}\cos(\frac{\pi}{2}-\phi+\phi'), \tag{9}$$

$$\Theta_n(s,x) = \frac{\partial \Theta(s,x)}{\partial n_x}\cos(\phi-\phi') + \frac{\partial \Theta(s,x)}{\partial t_x}\cos(\frac{\pi}{2}-\phi+\phi'), \tag{10}$$

$$M_n(s,x) = \frac{\partial M(s,x)}{\partial n_x}\cos(\phi-\phi') + \frac{\partial M(s,x)}{\partial t_x}\cos(\frac{\pi}{2}-\phi+\phi'), \tag{11}$$

$$V_n(s,x) = \frac{\partial V(s,x)}{\partial n_x}\cos(\phi-\phi') + \frac{\partial V(s,x)}{\partial t_x}\cos(\frac{\pi}{2}-\phi+\phi'). \tag{12}$$

Where $n_x$ and $t_x$ are the normal and tangential direction, respectively. $\phi$ and $\phi'$ are shown in Fig.1. We call this treatment "vector decomposition technique."

## 3. Numerical examples

*Case 1: circular plate with three holes*

A circular plate with the three circular holes which had been solved by Bird and Steele [2] is considered. The geometric data and the essential boundary conditions are shown in Fig.2. The contour plots of displacement by using different numbers of terms in the Fourier series (M=10, 20, 30, 40, 50) are shown in Fig.3 (a), (b), (c), (d) and (e). The case was also solved by using the ABAQUS with 6,606 triangle elements as shown in Fig.3 (g). Good agreement among the data of Bird and Steele [2] (Fig.3 (f)), ABAQUS results [1] (Fig.3 (h)) and present solutions is obtained.

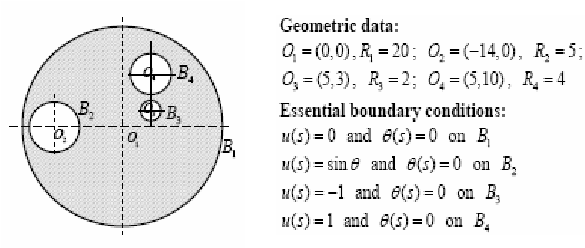
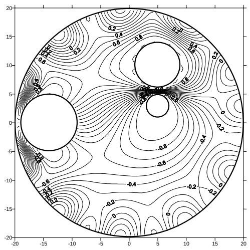
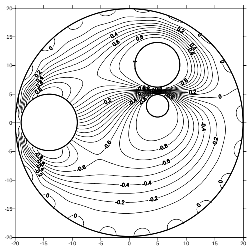
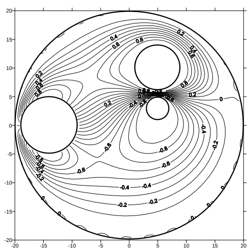
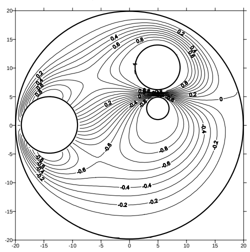


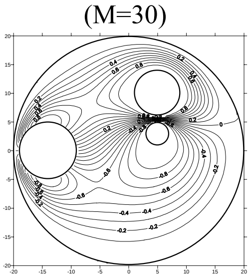

Fig.2 A circular plate containing three circular holes subject to the essential boundary conditions [2]

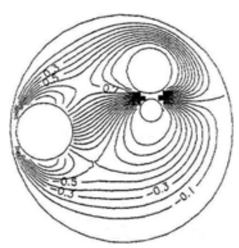

(a) Present method (M=10)

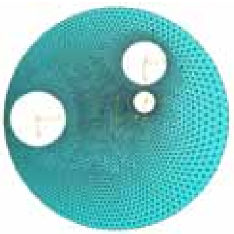

(b) Present method (M=20)


(c) Present method (M=30)


(d) Present method (M=40)


(e) Present method (M=50)


(f) Bird and Steele [2]


(g) Finite element mesh

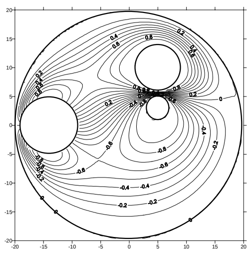

(h) FEM (ABAQUS)

Fig.3 Contour plots of circular plate with three holes

*Case 2: Stokes flow of eccentric cylinder*

The present method is also proposed to solve the slow flow between eccentric cylinders since the stream function satisfy the biharmonic equation. The inner cylinder is rotating with a constant angular velocity and the outer one is stationary as shown in Fig.4. The following parameters are defined: $R_1$, radius of inner cylinder; $R_2$, radius of outer cylinder; $c=R_2-R_1$, the clearance; $\varepsilon=e/c$, the eccentricity; $e$, separation of centers of cylinders; $\omega_1$ for the anticlockwise angular velocity of inner cylinder. The essential boundary conditions are specified as follows:

$$u(s)=u_1,\quad \theta(s)=\frac{\partial u(s)}{\partial n}=\omega_1 r_1=0.5,\quad s \text{ on } B_1 \tag{11}$$

$$u(s)=0,\quad \theta(s)=\frac{\partial u(s)}{\partial n}=0,\quad s \text{ on } B_2 \tag{12}$$

It is noted that the boundary condition $u_1$ is an unknown constant along the inner boundary. An additional constraint is required to ensure a unique solution.

$$\int_{B_1}\frac{\partial \omega}{\partial n}dB_1=\int_{B_1}\omega_n dB_1=0 \tag{13}$$

where vorticity, $\omega=\nabla^2 u$ . The results $u_1$ with different eccentricities are calculated and are listed in Table 1. By using the fewer degrees of freedom than BIE [8,9], the present results are more accurate after comparing with the analytical solution as follows:

$$u_1=\frac{A\omega_1 r_1(\sinh\delta-\delta\cosh\delta)(\sinh\alpha_2\sinh\delta-\delta\sinh\alpha_1)}{2[(\delta+\sinh\alpha_1\cosh\alpha_1-\cosh\alpha_2\sinh\alpha_2)(\sinh\delta-\delta\cosh\delta)+\cosh\delta(\delta^2-\sinh\delta^2)]} \tag{14}$$

where

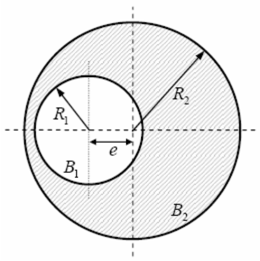
$$A=\frac{c}{\varepsilon}\left[(1-\varepsilon^2)[(\frac{r_1+r_2}{c})^2-\varepsilon^2]\right]^{1/2}, \tag{15}$$

$$\alpha_1 = -\sinh^{-1}\left(\frac{A}{2r_1}\right), \quad (16)$$

$$\alpha_2 = -\sinh^{-1}\left(\frac{A}{2r_2}\right), \quad (17)$$

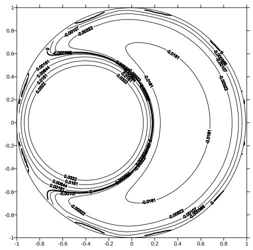
$$\delta = \alpha_1 - \alpha_2 . \quad (18)$$

The contour plots of streamlines for $\varepsilon = 0.8$ solved by employing the present method are compared with the results [8, 9] obtained by using the 160 boundary nodes as shown in Fig.5 (a), (b), (c) and (d). The case was also solved by using the ABAQUS with 2,162 triangle elements as shown in Fig.5 (e) and (f). Good agreement was also obtained.

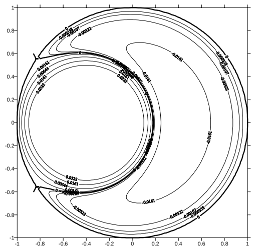



**Geometric data:**
$R_1 = 0.5$, $R_2 = 1$, $e = 0.4$
**Boundary conditions:**
$u_1(s) = 0.0322$ and $\theta_1(s) = 0.5$ on $B_1$
$u_2(s) = 0$ and $\theta_2(s) = 0$ on $B_2$

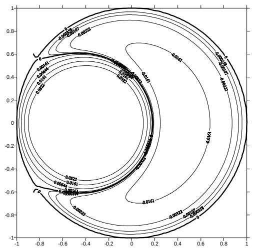
Fig.4 Eccentric bearing



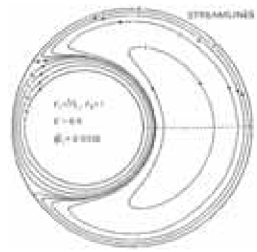
(a) Present method (M=10)



(b) Present method (M=20)

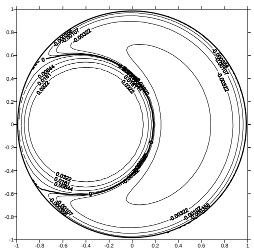


(c) Present method (M=40)



(d) Kelmanson & Ingham [8, 9]

(e) FEM mesh



(f) FEM result

Fig.5 Streamlines contour plots for $\varepsilon = 0.8$.

Table 1. Comparison of analytical and numerical results of $u_1$ for the eccentric bearing.

| | Kelmanson & Ingham[8, 9] | | | | Analytical solution | Present method |
|---|---|---|---|---|---|---|
| $\varepsilon$ | n=80 | n=160 | n=320 | Limit n→∞ | | |
| 0.0 | 0.1066 | 0.1062 | 0.1061 | 0.1061 | 0.1060 | 0.1060 (N=5) |
| 0.1 | 0.1052 | 0.1048 | 0.1047 | 0.1047 | 0.1046 | 0.1046 (N=7) |
| 0.2 | 0.1011 | 0.1006 | 0.1005 | 0.1005 | 0.1005 | 0.1005 (N=7) |
| 0.3 | 0.0944 | 0.0939 | 0.0938 | 0.0938 | 0.0938 | 0.0938 (N=7) |
| 0.4 | 0.0854 | 0.0850 | 0.0848 | 0.0846 | 0.0848 | 0.0848 (N=9) |
| 0.5 | 0.0748 | 0.0740 | 0.0739 | 0.0739 | 0.0738 | 0.0738 (N=11) |
| 0.6 | 0.0622 | 0.0615 | 0.0613 | 0.0612 | 0.0611 | 0.0611 (N=17) |
| 0.7 | 0.0484 | 0.0477 | 0.0474 | 0.0472 | 0.0472 | 0.0472 (N=17) |
| 0.8 | 0.0347 | 0.0332 | 0.0326 | 0.0322 | 0.0322 | 0.0322 (N=21) |
| 0.9 | 0.0191 | 0.0175 | 0.0168 | 0.0163 | 0.0164 | 0.0164 (N=31) |

(**n:** number of boundary nodes. **N:** number of collocation points on the inner cylinder.)

## 4. Conclusions

For biharmonic problems with circular boundaries, a semi-analytical solution by using degenerate kernels, null-field integral equation and Fourier series has been obtained. The main advantage of the present method over BEM is that all the improper integrals are easily calculated when degenerate kernels are used. Also, discretzation of boundaries are not required. Case studies of biharmonic problems with circular boundaries were performed, namely a circular plate containing three circular holes subject to the essential boundary conditions and the Stokes flow of eccentric cylinder with the inner cylinder of a constant angular velocity and the outer one stationary. Validity of the present method is confirmed by good agreement made among available analytical results [2, 8, 9] and numerical results acquired by using the finite element code, ABAQUS [1].

## References


1. ABAQUS/CAE 6.5, 2004. Hibbitt, Karlsson and Sorensen, Inc., RI.
2. Bird, M. D. and Steele, C. R. (1991), Separated solution procedure for bending of circular plates with circular holes, *Applied Mechanics Reviews*, **44**, 27-35.
3. Chen, J. T., Wu, C. S. and Chen, K. H. (2005), A study of free terms for plate problems in the dual boundary integral equations, *Engineering Analysis with Boundary Elements*, **29**, 435-446.
4. Chen, J. T., Wu, C. S., Chen, K. H. and Lee, Y. T. (2005), Degenerate scale for plate analysis using the boundary integral equation method and boundary element method, *Computational Mechanics*, Accepted.
5. Chen, J. T., Wu, C. S., Lee, Y. T. and Chen, K. H. (2005), On the equivalence of the Trefftz method and method of fundamental solutions for Laplace and biharmonic equations, *Computers and Mathematics with Applications*, Accepted.
6. Chen, J. T., Shen, W. C. and Wu, A. C. (2005), Null-field integral equation for stress field around circular holes under antiplane shear, *Engineering Analysis with Boundary Elements*, Revised.
7. Hsiao, C. C. (2005), A semi-analytical approach for Stokes flow and plate problems with circular boundaries, Master thesis, Department of Harbor and River Engineering, National Taiwan Ocean University.
8. Ingham, Derek B. and Kelmanson, Mark A., 1984. Boundary Integral Equation Analyses of Singular, Potential, and Biharmonic Problems (Lecture notes in engineering; 7), Springer-Verlag Berlin, Heidelberg.
9. Kelmanson, M. A. (1984), A boundary integral equation method for the study of slow flow in bearings with arbitrary geometries, *Journal of Tribology*, **106**, 260-264.