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New series expansions for fundamental solutions of linear elastostatics in 2D

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Abstract Series expansions of fundamental solutions are essential to algorithms and analysis of the null field method (NFM) and to analysis of the method of fundamental solutions (MFS). For linear elastostatics, new Fourier series expansions of FS are derived, directly from integration. The new expansions of the FS are simpler than those in Chen et al. (J Mech 26(3):393–401, 2010), thus facile to application in NFM and MFS. The new series expansions of FS in this paper are important to both theory and computation of linear elastostatics. Some computation of the MFS for linear elastostatics is provided, where the expansions of fundamental solutions are a basis tool in analysis. Numerical results of a simple example are reported, accompanied with error analysis.

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1 Introduction

Series expansions of fundamental solutions are essential to error and stability analysis of the method of fundamental solutions, and to algorithms and analysis of the null field method (NFM). For linear elastostatics, new Fourier series expansions of FS are derived, directly from integration. The new expansions of the FS are simpler than those in Chen et al. [5], thus facile to analysis of MFS [12–14] and to the NFM [17]. For elasticity problems, when the fundamental solutions (FS) and particular solutions (PS) satisfying PDE are chosen, the Trefftz method (TM) [18] leads to the method of fundamental solutions (MFS) and the method of particular solutions (MPS), respectively. The MFS and MPS can be applied to arbitrary domains. For circular domain with circular holes, the null field method (NFM) is proposed by Chen with his research groups. In NFM, the fundamental solutions (FS) with the source nodes outside of the solution domain are used in the Green formulas. The Fourier expansions of the known and the unknown boundary conditions on the circular boundaries are chosen, so that the explicit algebraic equations are easily obtained by means of orthogonality of Fourier functions. Recently, the explicit linear algebraic equations of NFM are provided in [17], which are easy for real applications. The NFM has been applied to elliptic and eigenvalue problems in circular domains with circular holes, reported in many papers; here we cite Chen's work for Laplace's equation in [2,4], and for elastostatics problems in [3,5]. Extensions to elliptic boundaries can be found in Crouch and Mogilevskaya [8], and Chen et al. [6,7].

For the NFM of elastostatics problems, the expansions of the fundamental solutions (FS) are a must. Although the series expansions of FS for plane elastostatics and basic methods are first given in Chen et al. [5], new and simpler series expansions of the FS are derived directly via integration. On the other hand, for error analysis of the MFS, the expansions of FS are essential. Since the error bounds of harmonic polynomials as PS in TM have been established in [18], the errors between the FS and harmonic polynomials can be found, based on the series expansions of FS, see Li [12] for Laplace's equation. The stability analysis of MFS in [16] also needs the expansions of FS. By following the arguments in [12,16], we may also carry out the analysis for the MFS of linear elastostatics (see [14]), once the series expansions of FS are provided. Hence, the new series expansions of FS in this paper are *important to both theory and computation of linear elastostatics* [10,11,18–21].

This paper is organized as follows. In Sect. 2, basic mathematical description for linear elastostatics problems in 2D is provided, and their fundamental solutions (FS) are introduced. In Sect. 3, preliminary expansion formulas are given, and in Sects. 4 and 5 main integration formulas are derived. In Sect. 6, new expansions of the FS of linear elastostatics in 2D are provided. In the last section, the FS expansions are applied to the MFS with numerical examples.

2 Linear elastostatics problems in 2D

2.1 Basic theory

Consider the linear elastostatics problem in 2D. Denote the displacement vector,

$$\vec{w} = \mathbf{w} = \{w_1(\mathbf{x}), w_2(\mathbf{x})\}^T = \{u(x, y), v(x, y)\}^T, \tag{2.1}$$

where $\vec{x} = \mathbf{x} = (x_1, x_2) = (x, y)$. When there exists no body force $\vec{f} \equiv 0$, we obtain the homogeneous equation, called the Cauchy–Navier equation of linear elastostatics for isotropic body:

$$\mu\Delta\vec{w} + (\lambda + \mu)\nabla(\nabla \cdot \vec{w}) = 0 \quad in \quad S, \tag{2.2}$$

where λ and μ are the Lamé constants. Eq. (2.2) can be written as

$$\Delta\vec{w} + \frac{1}{1-2\nu}\nabla(\nabla \cdot \vec{w}) = 0 \quad in \quad S, \tag{2.3}$$

where the Poisson ratio

$$\nu = \frac{\lambda}{2(\lambda+\mu)}, \quad 0 < \nu < \frac{1}{2}. \tag{2.4}$$

We cite a theorem from Chen and Zhou [1].

Theorem 2.1 *The general solutions of the linear elastostatic equations* (2.2) *in 3D and 2D are given by*

$$\vec{w}(\vec{x}) = \vec{h}(\vec{x}) - \kappa\nabla[\vec{x} \cdot \vec{h}(\vec{x}) + q(\vec{x})], \tag{2.5}$$

where $\vec{h}(\vec{x})$ is the harmonic vector and $q(\vec{x})$ is a harmonic function, and the constant

$$\kappa = \frac{1}{4(1-\nu)}. \tag{2.6}$$

2.2 Fundamental solutions

The principal fundamental solutions of linear elastostatics in 2D are given in [1] as

$$E_2(\mathbf{x}, \xi) = \frac{\lambda+3\mu}{4\pi\mu(\lambda+2\mu)}\left\{-\ln r_{\mathbf{x}\xi} I_2 + \frac{\lambda+\mu}{\lambda+3\mu}\frac{1}{r_{\mathbf{x}\xi}^2}[(\mathbf{x}-\xi)(\mathbf{x}-\xi)^T]\right\}, \tag{2.7}$$

where $r_{\mathbf{x}\xi} = |\mathbf{x} - \xi|$ and I_2 is the identity matrix. The FS in (2.7) satisfy

$$\mu\Delta E_2(\mathbf{x}, \xi) + (\lambda+\mu)\nabla_{\mathbf{x}}[\nabla_{\mathbf{x}} \cdot E_2(\mathbf{x}, \xi)] = -\delta(\mathbf{x}-\xi)I_2. \tag{2.8}$$

Choose the source nodes $Q(\xi_i, \eta_i)$ to be uniformly located on a larger circle of the 2D domain S, and denote the collocation points $P(x, y)$, where

$$x = \rho\cos\theta, \quad y = \rho\sin\theta,$$
$$\xi_i = R\cos\phi_i, \quad \eta_i = R\sin\phi_i, \quad i = 1, 2, \ldots, N, \tag{2.9}$$

where $R > \max_S \rho$, $\rho = \sqrt{x^2 + y^2}$, $R = \sqrt{\xi_i^2 + \eta_i^2}$ and $\phi_i = \frac{i2\pi}{N}$. Then

$$r_i = r_{\mathbf{x}\xi_\mathbf{i}} = \sqrt{R^2 + \rho^2 - 2R\rho\cos(\theta - \phi_i)}. \tag{2.10}$$

Denote $\vec{d}_i = (a_i, b_i)$ with the constants a_i and b_i, and $\vec{u}_i = (u_i, v_i)^T$. We have

$$\vec{u}_i = E_2(\mathbf{x}, \xi_\mathbf{i})\vec{d}_i, \tag{2.11}$$

where

$$u_i = a_i\left(-A\ln r_i + B\frac{(x-\xi_i)^2}{r_i^2}\right) + b_i B\frac{(x-\xi_i)(y-\eta_i)}{r_i^2}, \tag{2.12}$$

$$v_i = a_i B\frac{(x-\xi_i)(y-\eta_i)}{r_i^2} + b_i\left(-A\ln r_i + B\frac{(y-\eta_i)^2}{r_i^2}\right), \tag{2.13}$$

and the constants

$$A = \frac{\lambda + 3\mu}{4\pi\mu(\lambda + 2\mu)}, \quad B = \frac{\lambda + \mu}{4\pi\mu(\lambda + 2\mu)}. \tag{2.14}$$

From Theorem 2.1, by adding the simple fundamental solutions $\nabla(\ln r) = \{\frac{x}{r^2}, \frac{y}{r^2}\}^T$, we obtain the general linear combination of fundamental solutions,

$$u_N = u_N(x, y) = \sum_{i=1}^{N}\left\{a_i\left(-\ln r_i + D\frac{(x-\xi_i)^2}{r_i^2}\right) + b_i D\frac{(x-\xi_i)(y-\eta_i)}{r_i^2} + c_i\frac{(x-\xi_i)}{r_i^2}\right\}, \tag{2.15}$$

$$v_N = v_N(x, y) = \sum_{i=1}^{N}\left\{a_i D\frac{(x-\xi_i)(y-\eta_i)}{r_i^2} + b_i\left(-\ln r_i + D\frac{(y-\eta_i)^2}{r_i^2}\right) + c_i\frac{(y-\eta_i)}{r_i^2}\right\}, \tag{2.16}$$

where a_i, b_i and c_i are the coefficients, the constant

$$D = \frac{B}{A} = \frac{\lambda + \mu}{\lambda + 3\mu} = \frac{1}{3 - 4\nu} = \frac{\kappa}{1 - \kappa}, \tag{2.17}$$

and κ is given in (2.6). By using the FS in (2.15) and (2.16), and by employing the collocation Trefftz techniques in [18], the method of fundamental solutions (MFS) is established. For Laplace's equation, the error analysis of MFS in [14] is made, based on the expansions of FS (see [12]), and the stability analysis of MFS in [16] also needs the expansions of FS. Evidently, the expansions of FS are a basic tool in analysis of the MFS.

The principal FS in (2.7) is expressed in the matrix form

$$\begin{pmatrix} -\ln r(1-\kappa) + \kappa \frac{x^2}{r^2} & \kappa \frac{xy}{r^2} \\ \kappa \frac{xy}{r^2} & -\ln r(1-\kappa) + \kappa \frac{y^2}{r^2} \end{pmatrix} = (1-\kappa) \begin{pmatrix} -\ln r + D\frac{x^2}{r^2} & D\frac{xy}{r^2} \\ D\frac{xy}{r^2} & -\ln r + D\frac{y^2}{r^2} \end{pmatrix}, \tag{2.18}$$

where D is given in (2.17). We cite the Betti–Somigliana formulas with the principal FS [1].

Theorem 2.2 *There exist the Betti–Somigliana formulas for linear elastostatics:*

$$\vec{w}(\mathbf{x}) = \int_{\partial\Omega} \{E_2(\mathbf{x}, \xi) \cdot \vec{\tau}(\vec{w}) - \vec{\tau}(E_2(\mathbf{x}, \xi)) \cdot \vec{w}\} d\sigma_\xi, \quad \mathbf{x} \in \Omega, \tag{2.19}$$

$$\frac{1}{2}\vec{w}(\mathbf{x}) = \int_{\partial\Omega} \{E_2(\mathbf{x}, \xi) \cdot \vec{\tau}(\vec{w}) - \vec{\tau}(E_2(\mathbf{x}, \xi)) \cdot \vec{w}\} d\sigma_\xi, \quad \mathbf{x} \in \partial\Omega, \tag{2.20}$$

$$0 = \int_{\partial\Omega} \{E_2(\mathbf{x}, \xi) \cdot \vec{\tau}(\vec{w}) - \vec{\tau}(E_2(\mathbf{x}, \xi)) \cdot \vec{w}\} d\sigma_\xi, \quad \mathbf{x} \in \bar{\Omega}^c, \tag{2.21}$$

where Ω is an open area, and $\bar{\Omega}^c_\xi$ is the complementary region of the close area $\bar{\Omega}$ including the exterior and the interior boundaries $\partial\Omega$.

The boundary element methods originated from (2.20) and (2.19), but the NFM is developed from (2.21) and (2.19), based on the expansions of FS (see [5]).

3 Preliminary formulas

Denote two nodes by $P(x, y)$ and $Q(\xi, \eta)$ with $x = \rho\cos\theta, y = \rho\sin\theta, \xi = R\cos\phi$, and $\eta = R\sin\phi$. Then $\rho = \sqrt{x^2 + y^2}$, $R = \sqrt{\xi^2 + \eta^2}$ and $r = |\overline{PQ}| = \sqrt{\rho^2 - 2\rho R\cos(\theta - \phi) + R^2}$. In this section, we give the basic integral expansions of $\ln r$ and $\frac{1}{r^2}$. First we obtain a lemma from [9,12].

Lemma 3.1 *For $R > \rho$, there exist the equalities*

$$\frac{1}{2\pi}\int_0^{2\pi}(\ln\sqrt{\rho^2-2\rho R\cos(\theta-\phi)+R^2})d\phi=\ln R,$$

$$\frac{1}{\pi}\int_0^{2\pi}(\ln\sqrt{\rho^2-2\rho R\cos(\theta-\phi)+R^2})\cos m\phi d\phi=-\frac{\rho^m}{mR^m}\cos m\theta,\quad m=1,2,\ldots,$$

$$\frac{1}{\pi}\int_0^{2\pi}(\ln\sqrt{\rho^2-2\rho R\cos(\theta-\phi)+R^2})\sin m\phi d\phi=-\frac{\rho^m}{mR^m}\sin m\theta,\quad m=1,2,\ldots$$

Based on the Fourier coefficients of $\ln\sqrt{\rho^2-2\rho R\cos(\theta-\phi)+R^2}$ from Lemma 3.1, we have the following lemma immediately.

Lemma 3.2 *There exist the expansions*

$$\ln\sqrt{\rho^2-2\rho R\cos(\theta-\phi)+R^2}=\ln R-\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\rho}{R}\right)^n\cos n(\theta-\phi),\quad \rho<R, \tag{3.1}$$

$$\ln\sqrt{\rho^2-2\rho R\cos(\theta-\phi)+R^2}=\ln\rho-\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{R}{\rho}\right)^n\cos n(\theta-\phi),\quad \rho>R. \tag{3.2}$$

Equations (3.1) and (3.2) are the expansions of the FS for Laplace's equation in 2D used in [12,16]. The rest of this paper is devoted to derive new expansions for $\kappa\nabla(\ln r)=\kappa\{\frac{x}{r^2},\frac{y}{r^2}\}^T$ and (2.18) of linear elastostatics in 2D. We give a new lemma.

Lemma 3.3 *There exist the integrals,*

$$\int_0^{2\pi}\frac{\cos nx}{1+a^2-2a\cos x}dx=\frac{2\pi a^n}{1-a^2}(a^2<1)=\frac{2\pi a^{-n}}{a^2-1}(a^2>1), \tag{3.3}$$

$$\int_0^{2\pi}\frac{\sin nx}{1+a^2-2a\cos x}dx=0. \tag{3.4}$$

Proof From Gradsheyan and Ryzhik [9, p. 366]

$$\int_0^{\pi}\frac{\cos nx}{1+a^2-2a\cos x}dx=\frac{\pi a^n}{1-a^2}(a^2<1)=\frac{\pi a^{-n}}{a^2-1}(a^2>1), \tag{3.5}$$

we have

$$\int_0^{2\pi} \frac{\cos nx}{1+a^2-2a\cos x} dx = \int_{-\pi}^{\pi} \frac{\cos nx}{1+a^2-2a\cos x} dx$$
$$= \left(\int_{-\pi}^{0} + \int_0^{\pi} \right) \frac{\cos nx}{1+a^2-2a\cos x} dx. \tag{3.6}$$

Let $x = -t$ and $dx = -dt$. We have

$$\int_{-\pi}^{0} \frac{\cos nx}{1+a^2-2a\cos x} dx = -\int_{\pi}^{0} \frac{\cos nx}{1+a^2-2a\cos x} dx = \int_0^{\pi} \frac{\cos nx}{1+a^2-2a\cos x} dx. \tag{3.7}$$

Combining (3.5)–(3.7) gives

$$\int_0^{2\pi} \frac{\cos nx}{1+a^2-2a\cos x} dx = 2\int_0^{\pi} \frac{\cos nx}{1+a^2-2a\cos x} dx$$
$$= \frac{2\pi a^n}{1-a^2} (a^2 < 1). \tag{3.8}$$

This is the first desired result (3.3) with $a^2 < 1$. The proof for (3.3) with $a^2 > 1$ is similar.

Next we have

$$\int_0^{2\pi} \frac{\sin nx}{1+a^2-2a\cos x} dx = \left(\int_{-\pi}^{0} + \int_0^{\pi} \right) \frac{\sin nx}{1+a^2-2a\cos x} dx = 0, \tag{3.9}$$

since the following equality holds,

$$\int_{-\pi}^{0} \frac{\sin nx}{1+a^2-2a\cos x} dx = -\int_0^{\pi} \frac{\sin nx}{1+a^2-2a\cos x} dx = 0. \tag{3.10}$$

This is the second desired result (3.4), and completes the proof of Lemma 3.3.

Lemma 3.4 *For $a^2 < 1$, there exist the Fourier expansions,*

$$\frac{1}{1+a^2-2a\cos x} = \frac{2}{1-a^2}\left(\frac{1}{2} + \sum_{n=1}^{\infty} a^n \cos nx \right). \tag{3.11}$$

For $a^2 > 1$,

$$\frac{1}{1+a^2-2a\cos x} = \frac{2}{a^2-1}\left(\frac{1}{2} + \sum_{n=1}^{\infty} a^{-n}\cos nx\right). \tag{3.12}$$

Proof Denote the Fourier series

$$\frac{1}{1+a^2-2a\cos x} = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx), \tag{3.13}$$

where the Fourier coefficients a_k and b_k are obtained:

$$a_k = \frac{1}{\pi}\int_0^{2\pi} \frac{\cos kx}{1+a^2-2a\cos x}dx, \quad k = 0, 1, 2, \ldots, \tag{3.14}$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} \frac{\sin kx}{1+a^2-2a\cos x}dx, \quad k = 1, 2, \ldots \tag{3.15}$$

Then from Lemma 3.3 we obtain for $a^2 < 1$

$$a_k = \frac{2}{\pi}\frac{\pi a^k}{1-a^2} = \frac{2a^k}{1-a^2}, \quad k = 0, 1, \ldots, \tag{3.16}$$

$$b_k = 0, \quad k = 1, 2, \ldots \tag{3.17}$$

Substituting the coefficients a_k and b_k into (3.13) gives the desired result (3.11) with $a^2 < 1$. Similarly, Eq. (3.12) with $a^2 > 1$ also holds. This completes the proof of Lemma 3.4.

Theorem 3.1 *There exist the expansions*

$$\frac{1}{r^2} = \frac{2}{R^2-\rho^2}\left(\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{\rho}{R}\right)^n \cos n(\phi-\theta)\right), \quad \rho < R, \tag{3.18}$$

$$\frac{1}{r^2} = \frac{2}{\rho^2-R^2}\left(\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{R}{\rho}\right)^n \cos n(\phi-\theta)\right), \quad \rho > R, \tag{3.19}$$

where $r^2 = R^2 + \rho^2 - 2\rho R\cos(\phi-\theta)$, $\mathbf{x} = (x, y) = (\rho\cos\theta, \rho\sin\theta)$ *and* $\xi = (\xi, \eta) = (R\cos\phi, R\sin\phi)$.

Proof Since $r^2 = R^2(1 + a^2 - 2a\cos(\phi-\theta))$, we obtain (3.18) from Lemma 3.4 by noting $x = \phi - \theta$. Equation (3.19) follows from (3.18) by the symmetry between ρ and R. This completes the proof of Theorem 3.1.

4 Expansions for $\frac{x-\xi}{r^2}$ and $\frac{y-\eta}{r^2}$

The simple FS for $q = \ln r$ is given from (2.5),

$$\nabla q = \left(\frac{x-\xi}{r^2}, \frac{y-\eta}{r^2} \right)^T = (T_1, T_2)^T. \tag{4.1}$$

To obtain the Fourier expansions of (4.1), we need the following lemma.

Lemma 4.1 *Let* $\rho < R$ *and* $r^2 = R^2 + \rho^2 - 2\rho R \cos(\phi - \theta)$. *There exist the integral formulas,*

$$\int_0^{2\pi} \frac{x-\xi}{r^2} d\phi = \int_0^{2\pi} \frac{y-\eta}{r^2} d\phi = 0, \tag{4.2}$$

$$\int_0^{2\pi} \frac{x-\xi}{r^2} \cos n\phi d\phi = -\frac{\pi}{R} \left(\frac{\rho}{R} \right)^{n-1} \cos(n-1)\theta, \tag{4.3}$$

$$\int_0^{2\pi} \frac{x-\xi}{r^2} \sin n\phi d\phi = -\frac{\pi}{R} \left(\frac{\rho}{R} \right)^{n-1} \sin(n-1)\theta, \tag{4.4}$$

$$\int_0^{2\pi} \frac{y-\eta}{r^2} \cos n\phi d\phi = \frac{\pi}{R} \left(\frac{\rho}{R} \right)^{n-1} \sin(n-1)\theta, \tag{4.5}$$

$$\int_0^{2\pi} \frac{y-\eta}{r^2} \sin n\phi d\phi = -\frac{\pi}{R} \left(\frac{\rho}{R} \right)^{n-1} \cos(n-1)\theta. \tag{4.6}$$

Proof Denote $a = \frac{\rho}{R}$. We have from Theorem 3.1 and the orthogonality of trigonometric functions,

$$\begin{aligned}
&\int_0^{2\pi} \frac{\rho\cos\theta - R\cos\phi}{r^2} d\phi \\
&= \rho\cos\theta \int_0^{2\pi} \frac{1}{\rho^2 + R^2 - 2\rho R\cos(\phi-\theta)} d\phi - R\int_0^{2\pi} \frac{\cos\phi}{\rho^2 + R^2 - 2\rho R\cos(\phi-\theta)} d\phi \\
&= \rho\cos\theta \frac{1}{R^2} \frac{2\pi}{1-a^2} - \frac{1}{R^2} \frac{2\pi}{1-a^2} Ra\cos\theta = 0.
\end{aligned} \tag{4.7}$$

This is the left hand side of (4.2), and the proof for the right side of (4.2) is similar.

Next for (4.3), the integrand is given by

$$(x-\xi)\cos n\phi = (\rho\cos\theta - R\cos\phi)\cos n\phi = \rho\cos\theta\cos n\phi - \frac{R}{2}(\cos(n-1)\phi + \cos(n+1)\phi). \tag{4.8}$$

From Theorem 3.1 and the orthogonality of trigonometric functions, we have

$$\begin{aligned}\int_0^{2\pi} \frac{x-\xi}{r^2}\cos n\phi d\phi &= \frac{2\pi}{R^2(1-a^2)}\left\{\rho\cos\theta a^n\cos n\theta - \frac{R}{2}(a^{n-1}\cos(n-1)\theta + a^{n+1}\cos(n+1)\theta)\right\} \\ &= \frac{2\pi}{R^2(1-a^2)}\left\{\frac{\rho a^n}{2}(\cos(n-1)\theta + \cos(n+1)\theta) \right. \\ &\quad \left. - \frac{R}{2}(a^{n-1}\cos(n-1)\theta + a^{n+1}\cos(n+1)\theta)\right\} \\ &= \frac{2\pi}{R^2(1-a^2)}a^{n-1}\left(\frac{\rho a}{2} - \frac{R}{2}\right)\cos(n-1)\theta. \end{aligned} \tag{4.9}$$

Since

$$\frac{\rho a}{2} - \frac{R}{2} = \frac{\rho^2}{2R} - \frac{R}{2} = \frac{1}{2R}(\rho^2 - R^2) = \frac{R}{2}(a^2-1), \tag{4.10}$$

the desired result (4.3) follows from (4.9).

Third for (4.6), the integrand is

$$(y-\eta)\sin n\phi = (\rho\sin\theta - R\sin\phi)\sin n\phi = \rho\sin\theta\sin n\phi - \frac{R}{2}(\cos(n-1)\phi - \cos(n+1)\phi). \tag{4.11}$$

Similarly, we have

$$\begin{aligned}\int_0^{2\pi} \frac{y-\eta}{r^2}\sin n\phi d\phi &= \frac{2\pi}{R^2(1-a^2)}\left\{\rho\sin\theta a^n\sin n\theta - \frac{R}{2}(a^{n-1}\cos(n-1)\theta - a^{n+1}\cos(n+1)\theta)\right\} \\ &= \frac{2\pi}{R^2(1-a^2)}\left\{\frac{\rho a^n}{2}(\cos(n-1)\theta - \cos(n+1)\theta) \right. \\ &\quad \left. - \frac{R}{2}(a^{n-1}\cos(n-1)\theta - a^{n+1}\cos(n+1)\theta)\right\}\end{aligned}$$

$$= \frac{2\pi}{R^2(1-a^2)} a^{n-1} \left(\frac{\rho a}{2} - \frac{R}{2} \right) \cos(n-1)\theta$$
$$= -\frac{\pi}{R} a^{n-1} \cos(n-1)\theta. \tag{4.12}$$

This is the last result (4.6), and the proof for (4.4) and (4.5) is similar. This completes the proof of Lemma 4.1.

Theorem 4.1 *Let $\rho < R$ and $r^2 = R^2 + \rho^2 - 2\rho R\cos(\phi - \theta)$. There exist the expansions,*[1]

$$T_1^i = \frac{x-\xi}{r^2} = -\frac{1}{R}\sum_{n=1}^{\infty}\left(\frac{\rho}{R}\right)^{n-1}(\cos(n-1)\theta\cos n\phi + \sin(n-1)\theta\sin n\phi)$$
$$= -\frac{1}{R}\sum_{n=0}^{\infty}\left(\frac{\rho}{R}\right)^{n}\cos(n(\theta-\phi)-\phi), \tag{4.13}$$
$$T_2^i = \frac{y-\eta}{r^2} = \frac{1}{R}\sum_{n=1}^{\infty}\left(\frac{\rho}{R}\right)^{n-1}(\sin(n-1)\theta\cos n\phi - \cos(n-1)\theta\sin n\phi)$$
$$= \frac{1}{R}\sum_{n=0}^{\infty}\left(\frac{\rho}{R}\right)^{n}\sin(n(\theta-\phi)-\phi). \tag{4.14}$$

Proof We have the Fourier expansion,

$$\frac{x-\xi}{r^2} = \frac{\alpha_0}{2} + \sum_{k=1}^{\infty}(\alpha_k \cos k\phi + \beta_k \sin k\phi), \tag{4.15}$$

where

$$\alpha_k = \frac{1}{\pi}\int_0^{2\pi}\frac{x-\xi}{r^2}\cos k\phi, \quad \beta_k = \frac{1}{\pi}\int_0^{2\pi}\frac{x-\xi}{r^2}\sin k\phi. \tag{4.16}$$

From Lemma 4.1, we obtain the first equality of (4.13). Let $n-1=k$ we have from (4.13)

$$\frac{x-\xi}{r^2} = -\frac{1}{R}\sum_{k=1}^{\infty}\left(\frac{\rho}{R}\right)^{k}(\cos k\theta\cos(k+1)\phi + \sin k\theta\sin(k+1)\phi)$$
$$= -\frac{1}{R}\sum_{k=0}^{\infty}\left(\frac{\rho}{R}\right)^{k}\cos(k(\theta-\phi)-\phi). \tag{4.17}$$

The proof for (4.14) is similar, and completes the proof of Theorem 4.1.

[1] The superscript "i" denotes the case of $\rho < R$, and the superscript "e" in Theorem 6.4 denotes the case of $\rho > R$.

5 Expansions for $\frac{(x-\xi)^2}{r^2}$, $\frac{(y-\eta)^2}{r^2}$ and $\frac{(x-\xi)(y-\eta)}{r^2}$

For expansions of the principal FS, we need the following lemma.

Lemma 5.1 *Let $\rho < R$, $a = \frac{\rho}{R}$ and $r^2 = R^2 + \rho^2 - 2\rho R\cos(\phi - \theta)$. There exist the integral formulas*

$$\int_0^{2\pi} \frac{(x-\xi)^2}{r^2} d\phi = \int_0^{2\pi} \frac{(y-\eta)^2}{r^2} d\phi = \pi, \tag{5.1}$$

$$\int_0^{2\pi} \frac{(x-\xi)(y-\eta)}{r^2} d\phi = 0, \tag{5.2}$$

$$\int_0^{2\pi} \frac{(x-\xi)^2}{r^2} \cos\phi d\phi = -\frac{\pi}{2} a\cos\theta, \tag{5.3}$$

$$\int_0^{2\pi} \frac{(x-\xi)^2}{r^2} \sin\phi d\phi = \frac{\pi}{2} a\sin\theta, \tag{5.4}$$

$$\int_0^{2\pi} \frac{(y-\eta)^2}{r^2} \cos\phi d\phi = \frac{\pi}{2} a\cos\theta, \tag{5.5}$$

$$\int_0^{2\pi} \frac{(y-\eta)^2}{r^2} \sin\phi d\phi = -\frac{\pi}{2} a\sin\theta, \tag{5.6}$$

$$\int_0^{2\pi} \frac{(x-\xi)(y-\eta)}{r^2} \cos\phi d\phi = -\frac{\pi}{2} a\sin\theta, \tag{5.7}$$

$$\int_0^{2\pi} \frac{(x-\xi)(y-\eta)}{r^2} \sin\phi d\phi = -\frac{\pi}{2} a\cos\theta, \tag{5.8}$$

$$\int_0^{2\pi} \frac{(x-\xi)^2}{r^2} \cos n\phi d\phi = \frac{\pi}{2}(1-a^2)\left(\frac{\rho}{R}\right)^{n-2} \cos(n-2)\theta, \quad n \geq 2, \tag{5.9}$$

$$\int_0^{2\pi} \frac{(x-\xi)^2}{r^2} \sin n\phi d\phi = \frac{\pi}{2}(1-a^2)\left(\frac{\rho}{R}\right)^{n-2} \sin(n-2)\theta, \quad n \geq 2, \tag{5.10}$$

$$\int_0^{2\pi} \frac{(y-\eta)^2}{r^2} \cos n\phi d\phi = -\frac{\pi}{2}(1-a^2)\left(\frac{\rho}{R}\right)^{n-2} \cos(n-2)\theta, \quad n \geq 2, \tag{5.11}$$

$$\int_0^{2\pi} \frac{(y-\eta)^2}{r^2} \sin n\phi d\phi = -\frac{\pi}{2}(1-a^2)\left(\frac{\rho}{R}\right)^{n-2} \sin(n-2)\theta, \quad n \geq 2, \tag{5.12}$$

$$\int_0^{2\pi} \frac{(x-\xi)(y-\eta)}{r^2} \cos n\phi d\phi = -\frac{\pi}{2}(1-a^2)\left(\frac{\rho}{R}\right)^{n-2} \sin(n-2)\theta, \quad n \geq 2, \tag{5.13}$$

$$\int_0^{2\pi} \frac{(x-\xi)(y-\eta)}{r^2} \sin n\phi d\phi = \frac{\pi}{2}(1-a^2)\left(\frac{\rho}{R}\right)^{n-2} \cos(n-2)\theta, \quad n \geq 2. \tag{5.14}$$

Proof We only prove a few of them, since the proof of the others is similar. First, since

$$\begin{aligned}(x-\xi)^2 &= (\rho\cos\theta - R\cos\phi)^2 = \rho^2\cos^2\theta + R^2\cos^2\phi - 2\rho R\cos\theta\cos\phi \\ &= \rho^2\frac{1+\cos 2\theta}{2} + R^2\frac{1+\cos 2\phi}{2} - 2\rho R\cos\theta\cos\phi,\end{aligned} \tag{5.15}$$

we obtain the integral on the left side of (5.1):

$$\begin{aligned}\int_0^{2\pi} \frac{(x-\xi)^2}{r^2} &= \frac{2\pi}{R^2(1-a^2)}\left\{\rho^2\frac{1+\cos 2\theta}{2} + R^2\frac{1+a^2\cos 2\theta}{2} - 2\rho Ra\cos^2\theta\right\} \\ &= \frac{2\pi}{R^2(1-a^2)}\left\{\frac{R^2+\rho^2}{2} - \rho^2\right\} = \pi.\end{aligned} \tag{5.16}$$

Below we will show (5.9)–(5.14) first, and (5.3)–(5.8) afterwards. For (5.9), the integrand is given from (5.15)

$$\begin{aligned}(x-\xi)^2\cos n\phi &= \rho^2\left(\frac{1+\cos 2\theta}{2}\right)\cos n\phi + R^2\left(\frac{1+\cos 2\phi}{2}\right)\cos n\phi \\ &\quad -2\rho R\cos\theta\cos\phi\cos n\phi \\ &= \frac{\rho^2+R^2}{2}\cos n\phi + \rho^2\left(\frac{\cos 2\theta}{2}\right)\cos n\phi \\ &\quad + \frac{R^2}{4}[\cos(n+2)\phi + \cos(n-2)\phi] \\ &\quad -\rho R\cos\theta[\cos(n+1)\phi + \cos(n-1)\phi].\end{aligned} \tag{5.17}$$

Then the left side integral in (5.9) gives

$$\int_0^{2\pi} \frac{(x-\xi)^2}{r^2} \cos n\phi d\phi = \frac{2\pi}{R^2(1-a^2)} \left\{ \frac{\rho^2 + R^2}{2} a^n \cos n\theta + \rho^2 \left(\frac{\cos 2\theta}{2} \right) a^n \cos n\theta \right.$$
$$+ \frac{R^2}{4} [a^{n+2} \cos(n+2)\theta + a^{n-2} \cos(n-2)\theta]$$
$$\left. - \rho R \cos\theta [a^{n+1} \cos(n+1)\theta + a^{n-1} \cos(n-1)\theta] \right\}$$
$$= \frac{2\pi}{R^2(1-a^2)} \left\{ \frac{\rho^2 + R^2}{2} a^n \cos n\theta + \frac{\rho^2}{4} a^n [\cos(n+2)\theta + \cos(n-2)\theta] \right.$$
$$+ \frac{R^2}{4} [a^{n+2} \cos(n+2)\theta + a^{n-2} \cos(n-2)\theta]$$
$$\left. - \frac{\rho R}{2} a^{n+1} [\cos(n+2)\theta + \cos n\theta] - \frac{\rho R}{2} a^{n-1} [\cos n\theta + \cos(n-2)\theta] \right\}. \tag{5.18}$$

In the above equations, the final coefficients in front of $\cos n\theta$ and $\cos(n+2)\theta$ are just zero. Hence we obtain

$$\int_0^{2\pi} \frac{(x-\xi)^2}{r^2} \cos n\phi d\phi = \frac{\pi a^{n-2}}{2R^2(1-a^2)} \left[R^2 + \frac{\rho^4}{R^2} - 2\rho^2 \right] \cos(n-2)\theta. \tag{5.19}$$

There exists the equality,

$$R^2 + \frac{\rho^4}{R^2} - 2\rho^2 = R^2 - \rho^2 + \rho^2 \left(\frac{\rho^2}{R^2} - 1 \right)$$
$$= (R^2 - \rho^2) \left\{ 1 - \frac{\rho^2}{R^2} \right\} = R^2(1-a^2)^2. \tag{5.20}$$

Combining (5.19) and (5.20) gives the desired result (5.9).

Next for (5.11), there exist the equalities,

$$(y-\eta)^2 = (\rho \sin\theta - R \sin\phi)^2 = \rho^2 \sin^2\theta + R^2 \sin^2\phi - 2\rho R \sin\theta \sin\phi$$
$$= \rho^2 \frac{1-\cos 2\theta}{2} + R^2 \frac{1-\cos 2\phi}{2} - 2\rho R \sin\theta \sin\phi, \tag{5.21}$$

and

$$(y-\eta)^2 \cos n\phi = \frac{\rho^2 + R^2}{2} \cos n\phi - \frac{\rho^2}{2} \cos 2\theta \cos n\phi$$
$$- \frac{R^2}{2} \cos 2\phi \cos n\phi - 2\rho R \sin\theta \sin\phi \cos n\phi$$

$$
\begin{aligned}
&= \frac{\rho^2 + R^2}{2} \cos n\phi - \frac{\rho^2}{2} \cos 2\theta \cos n\phi \\
&\quad - \frac{R^2}{4} [\cos(n+2)\phi + \cos(n-2)\phi] \\
&\quad - \rho R \sin\theta [\sin(n+1)\phi - \sin(n-1)\phi]. \qquad (5.22)
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
\int_0^{2\pi} \frac{(y-\eta)^2}{r^2} \cos n\phi &= \frac{2\pi}{R^2(1-a^2)} \left\{ \frac{\rho^2 + R^2}{2} a^n \cos n\theta - \frac{\rho^2}{2} \cos 2\theta a^n \cos n\theta \right. \\
&\quad - \frac{R^2}{4} [a^{n+2} \cos(n+2)\theta + a^{n-2} \cos(n-2)\theta] \\
&\quad \left. - \rho R \sin\theta [a^{n+1} \sin(n+1)\theta - a^{n-1} \sin(n-1)\theta] \right\} \\
&= \frac{2\pi}{R^2(1-a^2)} \left\{ \frac{\rho^2 + R^2}{2} a^n \cos n\theta \right. \\
&\quad - \frac{\rho^2}{4} a^n [\cos(n+2)\theta + \cos(n-2)\theta] \\
&\quad - \frac{R^2}{4} [a^{n+2} \cos(n+2)\theta + a^{n-2} \cos(n-2)\theta] \\
&\quad - \frac{\rho R}{2} [a^{n+1} (\cos n\theta - \cos(n+2)\theta) \\
&\quad \left. - a^{n-1} (\cos(n-2)\theta - \cos n\theta)] \right\} \\
&= -\frac{\pi}{2R^2(1-a^2)} a^{n-2} \left(R^2 + \frac{\rho^4}{R^2} - 2\rho^2 \right) \cos(n-2)\theta \\
&= -\frac{\pi}{2} (1-a^2) a^{n-2} \cos(n-2)\theta, \qquad (5.23)
\end{aligned}
$$

where we have used (5.20). This is the desired result (5.11).

Moreover for (5.14), we have

$$
\begin{aligned}
(x-\xi)(y-\eta) &= (\rho\cos\theta - R\cos\phi)(\rho\sin\theta - R\sin\phi) \\
&= \rho^2 \cos\theta \sin\theta + R^2 \cos\phi \sin\phi - \rho R[\cos\theta \sin\phi + \sin\theta \cos\phi] \\
&= \frac{\rho^2}{2} \sin 2\theta + \frac{R^2}{2} \sin 2\phi - \rho R[\cos\theta \sin\phi + \sin\theta \cos\phi],
\end{aligned}
$$

and

$$
\begin{aligned}
(x-\xi)(y-\eta) \sin n\phi &= \frac{\rho^2}{2} \sin 2\theta \sin n\phi + \frac{R^2}{2} \sin 2\phi \sin n\phi \\
&\quad - \rho R[\cos\theta \sin\phi + \sin\theta \cos\phi] \sin n\phi
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\rho^2}{2}\sin 2\theta \sin n\phi + \frac{R^2}{4}[-\cos(n+2)\phi + \cos(n-2)\phi] \\
&\quad - \frac{\rho R}{2}[\cos\theta(-\cos(n+1)\phi + \cos(n-1)\phi) \\
&\quad + \sin\theta(\sin(n+1)\phi + \sin(n-1)\phi)].
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\int_0^{2\pi} \frac{(x-\xi)(y-\eta)}{r^2}\sin n\phi &= \frac{2\pi}{R^2(1-a^2)}\left\{\frac{\rho^2}{2}\sin n\theta a^n \sin 2\theta\right. \\
&\quad + \frac{R^2}{4}[-a^{n+2}\cos(n+2)\theta + a^{n-2}\cos(n-2)\theta] \\
&\quad - \frac{\rho R}{2}[\cos\theta(-a^{n+1}\cos(n+1)\theta + a^{n-1}\cos(n-1)\theta) \\
&\quad \left. + \sin\theta(a^{n+1}\sin(n+1)\theta + a^{n-1}\sin(n-1)\theta)]\right\} \\
&= \frac{2\pi}{R^2(1-a^2)}\left\{\frac{\rho^2}{4}a^n[-\cos(n+2)\theta + \cos(n-2)\theta]\right. \\
&\quad + \frac{R^2}{4}[-a^{n+2}\cos(n+2)\theta + a^{n-2}\cos(n-2)\theta] \\
&\quad - \frac{\rho R}{4}\{-a^{n+1}[\cos(n+2)\theta + \cos n\theta] \\
&\quad + a^{n-1}[\cos n\theta + \cos(n-2)\theta] \\
&\quad + a^{n+1}[-\cos(n+2)\theta + \cos n\theta] \\
&\quad \left. + a^{n-1}[-\cos n\theta + \cos(n-2)]\}\right\} \\
&= \frac{\pi}{2R^2(1-a^2)}\left(R^2 + \frac{\rho^4}{R^2} - 2\rho^2\right)\left(\frac{\rho}{R}\right)^{n-2}\cos(n-2)\theta \\
&= \frac{\pi}{2}(1-a^2)\left(\frac{\rho}{R}\right)^{n-2}\cos(n-2)\theta. \qquad (5.24)
\end{aligned}
$$

This is the desired result (5.14).

Last, we show (5.3)–(5.8). Although we may follow the above approaches to derive integrals directly, we will solicit different arguments by means of a remedy for (5.9)–(5.14) with $n = 1$. First we have from (5.9) with $n = 1$

$$
\widehat{\int_0^{2\pi}} \frac{(x-\xi)^2}{r^2}\cos\phi d\phi = \frac{\pi}{2}(1-a^2)\left(\frac{\rho}{R}\right)^{-1}\cos(-\theta). \qquad (5.25)
$$

Let us scrutinize its derivation in (5.18). Note that in the expansions (3.18), there exist no negative powers as $(\frac{\rho}{R})^{-1}$. In the last but one line in (5.18), we have found an improper term

$$\frac{2\pi}{R^2(1-a^2)}\frac{R^2a^{-1}}{4}\cos\theta = \frac{\pi}{1-a^2}\frac{a^{-1}}{2}\cos\theta, \tag{5.26}$$

which should be replaced by the correct term

$$\frac{\pi}{1-a^2}\frac{a}{2}\cos\theta. \tag{5.27}$$

Therefore, we find the error in (5.18) and so in (5.25) with $n = 1$

$$\begin{aligned} E_1 &= \frac{\pi}{1-a^2}\frac{a}{2}\cos\theta - \frac{\pi}{1-a^2}\frac{a^{-1}}{2}\cos\theta \\ &= \frac{\pi}{2}\frac{1}{1-a^2}(a-a^{-1})\cos\theta = \frac{\pi}{2}\frac{1}{1-a^2}a^{-1}(a^2-1)\cos\theta \\ &= -\frac{\pi}{2}a^{-1}\cos\theta. \end{aligned} \tag{5.28}$$

By removing this error, the true integral is obtained from (5.25) and (5.28)

$$\begin{aligned} \int_0^{2\pi} \frac{(x-\xi)^2}{r^2}\cos\phi d\phi &= \widehat{\int_0^{2\pi}} \frac{(x-\xi)^2}{r^2}\cos\phi d\phi + E_1 \\ &= \frac{\pi}{2}(1-a^2)\left(\frac{\rho}{R}\right)^{-1}\cos\theta - \frac{\pi}{2}a^{-1}\cos\theta \\ &= -\frac{\pi}{2}a\cos\theta. \end{aligned} \tag{5.29}$$

This is the desired result (5.3).

Finally, we show (5.8). From (5.14) with $n = 1$, we have

$$\widehat{\int_0^{2\pi}} \frac{(x-\xi)(y-\eta)}{r^2}\sin n\phi d\phi = \frac{\pi}{2}(1-a^2)\left(\frac{\rho}{R}\right)^{-1}\cos\theta. \tag{5.30}$$

In its derivation (5.24) with $n = 1$, there also exists the error

$$E_2 = \frac{\pi}{2}\frac{1}{1-a^2}(-a^{-1}+a)\cos\theta = -\frac{\pi}{2}a^{-1}\cos\theta. \tag{5.31}$$

By removing this error we have the true integral

$$\int_0^{2\pi} \frac{(x-\xi)(y-\eta)}{r^2} \sin n\phi d\phi = \widehat{\int_0^{2\pi}} \frac{(x-\xi)(y-\eta)}{r^2} \sin n\phi d\phi + E_2$$
$$= \frac{\pi}{2}(1-a^2)\left(\frac{\rho}{R}\right)^{-1} \cos\theta - \frac{\pi}{2} a^{-1} \cos\theta$$
$$= -\frac{\pi}{2} a \cos\theta. \tag{5.32}$$

This is the desired result (5.8). The proof of other formulas is similar, and this completes the proof of Lemma 5.1.

Note that for the integration computation of Fourier functions by means of orthogonality, we should also derive those formulas with $n = 2$ due to $\cos(n-2)\phi = 1$. However, since in (3.18), the constant is just $\frac{1}{2}$, fortunately, the final integral values in (5.18) are exactly the same, so that we do not need extra-evaluation of (5.9). For the same reason, we do not need an extra-evaluation for the expansions (4.17).

From Lemma 5.1 we have the Fourier expansions immediately.

Theorem 5.1 *Let $\rho < R$, $a = \frac{\rho}{R}$ and $r^2 = R^2 + \rho^2 - 2\rho R \cos(\phi - \theta)$. There exist the expansions,*

$$\frac{(x-\xi)^2}{r^2} = \frac{1}{2} - \frac{a}{2}\cos(\theta+\phi)$$
$$+ \frac{1-a^2}{2} \sum_{n=2}^{\infty} \left(\frac{\rho}{R}\right)^{n-2} (\cos(n-2)\theta \cos n\phi + \sin(n-2)\theta \sin n\phi)$$
$$= \frac{1}{2} - \frac{a}{2}\cos(\theta+\phi) + \frac{1-a^2}{2} \sum_{n=0}^{\infty} \left(\frac{\rho}{R}\right)^{n} \cos(n(\theta-\phi) - 2\phi),$$

$$\frac{(y-\eta)^2}{r^2} = \frac{1}{2} + \frac{a}{2}\cos(\theta+\phi)$$
$$- \frac{1-a^2}{2} \sum_{n=2}^{\infty} \left(\frac{\rho}{R}\right)^{n-2} (\cos(n-2)\theta \cos n\phi + \sin(n-2)\theta \sin n\phi)$$
$$= \frac{1}{2} + \frac{a}{2}\cos(\theta+\phi) - \frac{1-a^2}{2} \sum_{n=0}^{\infty} \left(\frac{\rho}{R}\right)^{n} \cos(n(\theta-\phi) - 2\phi),$$

$$\frac{(x-\xi)(y-\eta)}{r^2} = -\frac{a}{2}\sin(\theta+\phi)$$
$$+ \frac{1-a^2}{2} \sum_{n=2}^{\infty} \left(\frac{\rho}{R}\right)^{n-2} (-\sin(n-2)\theta \cos n\phi + \cos(n-2)\theta \sin n\phi)$$
$$= -\frac{a}{2}\sin(\theta+\phi) - \frac{1-a^2}{2} \sum_{n=0}^{\infty} \left(\frac{\rho}{R}\right)^{n} \sin(n(\theta-\phi) - 2\phi).$$

6 The expansions of FS

Based on Sects. 4 and 5, we provide new expansions of the FS for linear elastostatics in 2D. For the simple FS with $\rho < R$ we have their expansions from Theorem 4.1.

Theorem 6.1 *Let $\rho < R$ and $r^2 = R^2 + \rho^2 - 2\rho R\cos(\phi - \theta)$. There exist the expansions of the simple FS,*

$$\kappa \begin{pmatrix} \frac{x-\xi}{r^2} \\ \frac{y-\eta}{r^2} \end{pmatrix} = \frac{\kappa}{R} \sum_{n=0}^{\infty} \left(\frac{\rho}{R}\right)^n \begin{pmatrix} -\cos(n(\theta-\phi)-\phi) \\ \sin(n(\theta-\phi)-\phi) \end{pmatrix}.$$

Next for the principal FS with $\rho < R$, we have their expansions from Theorem 5.1.

Theorem 6.2 *Let $\rho < R, a = \frac{\rho}{R}$ and $r^2 = R^2 + \rho^2 - 2\rho R\cos(\phi - \theta)$. There exist the expansions of the principal FS,*

$$E_2^i(\mathbf{x} - \xi) = \begin{pmatrix} T_{11}^i & T_{12}^i \\ T_{12}^i & T_{22}^i \end{pmatrix},$$

where the entries

$$\begin{aligned} T_{11}^i &= (1-\kappa)\left\{-\left[\ln R - \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{\rho}{R}\right)^n \cos n(\theta-\phi)\right]\right. \\ &\quad \left. + D\left[\frac{1}{2} - \frac{a}{2}\cos(\theta+\phi) + \frac{1-a^2}{2}\sum_{n=0}^{\infty}\left(\frac{\rho}{R}\right)^n \cos(n(\theta-\phi)-2\phi)\right]\right\}, \\ T_{22}^i &= (1-\kappa)\left\{-\left[\ln R - \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{\rho}{R}\right)^n \cos n(\theta-\phi)\right]\right. \\ &\quad \left. + D\left[\frac{1}{2} + \frac{a}{2}\cos(\theta+\phi) - \frac{1-a^2}{2}\sum_{n=0}^{\infty}\left(\frac{\rho}{R}\right)^n \cos(n(\theta-\phi)-2\phi)\right]\right\}, \\ T_{12}^i &= (1-\kappa)D\left[-\frac{a}{2}\sin(\theta+\phi) - \frac{1-a^2}{2}\sum_{n=0}^{\infty}\left(\frac{\rho}{R}\right)^n \sin(n(\theta-\phi)-2\phi)\right]. \end{aligned}$$

In the NFM, we do need the Fourier expansions for both $\rho > R$ and $\rho < R$. We have the following theorem.

Theorem 6.3 *Let $\rho > R$ and $r^2 = R^2 + \rho^2 - 2\rho R\cos(\phi - \theta)$. There exist the expansions of the simple FS,*

$$\kappa \begin{pmatrix} \frac{x-\xi}{r^2} \\ \frac{y-\eta}{r^2} \end{pmatrix} = \frac{\kappa}{\rho} \sum_{n=0}^{\infty} \left(\frac{R}{\rho}\right)^n \begin{pmatrix} \cos(n(\phi-\theta)-\theta) \\ -\sin(n(\phi-\theta)-\theta) \end{pmatrix}. \tag{6.1}$$

Proof We will use Theorem 6.1 via symmetry. When $\rho > R$, we have

$$\frac{x-\xi}{r^2} = \frac{\rho\cos\theta - R\cos\phi}{R^2+\rho^2-2\rho R\cos(\phi-\theta)} = -\frac{R\cos\phi-\rho\cos\theta}{R^2+\rho^2-2\rho R\cos(\theta-\phi)}. \tag{6.2}$$

If switching (ρ, θ) and (R, ϕ), from Theorem 6.1 and (6.2), we obtain the first component of the vector in (6.1):

$$\kappa\frac{x-\xi}{r^2} = \frac{\kappa}{\rho}\sum_{n=0}^{\infty}\left(\frac{R}{\rho}\right)^n \cos(n(\phi-\theta)-\theta). \tag{6.3}$$

The proof for the second component of the vector in (6.1) is similar, and this completes the proof of Theorem 6.3.

Similarly, we have the following expansions from Theorem 6.2 via the symmetry.

Theorem 6.4 *Let $\rho > R$, $a = \frac{\rho}{R}$ and $r^2 = R^2 + \rho^2 - 2\rho R\cos(\phi-\theta)$. There exist the exapansions of the principal FS,*

$$E_2^e(\mathbf{x}-\xi) = \begin{pmatrix} T_{11}^e & T_{12}^e \\ T_{12}^e & T_{22}^e \end{pmatrix},$$

where the entries are given by

$$\begin{aligned} T_{11}^e &= (1-\kappa)\left\{-\left[\ln\rho - \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{R}{\rho}\right)^n \cos n(\theta-\phi)\right]\right. \\ &\quad \left. + D\left[\frac{1}{2} - \frac{a^{-1}}{2}\cos(\theta+\phi) + \frac{1-a^{-2}}{2}\sum_{n=0}^{\infty}\left(\frac{R}{\rho}\right)^n \cos(n(\phi-\theta)-2\theta)\right]\right\}, \\ T_{22}^e &= (1-\kappa)\left\{-\left[\ln\rho - \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{R}{\rho}\right)^n \cos n(\theta-\phi)\right]\right. \\ &\quad \left. + D\left[\frac{1}{2} + \frac{a^{-1}}{2}\cos(\theta+\phi) - \frac{1-a^{-2}}{2}\sum_{n=0}^{\infty}\left(\frac{R}{\rho}\right)^n \cos(n(\phi-\theta)-2\theta)\right]\right\}, \\ T_{12}^e &= (1-\kappa)D\left[-\frac{a^{-1}}{2}\sin(\theta+\phi) - \frac{1-a^{-2}}{2}\sum_{n=0}^{\infty}\left(\frac{R}{\rho}\right)^n \sin(n(\phi-\theta)-2\theta)\right]. \end{aligned}$$

All series expansions in this paper have been verified by numerical computation.

Remark 6.1 The principal FS is more important than the simple FS, because we may not need the simple FS in computation from the numerical experiments in the next section. By series operations, some expansions for $\frac{(x-\xi)^2}{r^2}$, $\frac{(y-\eta)^2}{r^2}$ and $\frac{(x-\xi)(y-\eta)}{r^2}$ are given in Chen et al. [5], which are more complicated than those in Theorem 5.1. The simplicity of expansions of FS in this paper is imperative to real application of linear elastostatics, such as error and stability analysis of MFS in [13,14,16], and algorithms and analysis of the NFM in [17].

7 Applications to method of fundamental solutions

The series expansions of the FS are employed for the NFM in [5], and they may also be applied to the method of fundamental solutions (MFS). By following the error analysis of MFS for Laplace's equations based on (3.1) in [12], the errors between the FS and the particular solutions (PS) are first evaluated, based on the FS expansions in Sect. 6. Then the final errors of the solutions by the MFS for linear elastostatics can be derived. This is our motivation to develop the series expansions of the FS in this paper.

7.1 Algorithms

For simplicity, we consider only the displacement conditions

$$\vec{w}(\mathbf{x}) = \vec{f}(\mathbf{x}) \ \ on \ \ \Gamma, \tag{7.1}$$

where $\partial S = \Gamma$, and $\vec{f} = (f_1, f_2)^T$. Also assume that the solution $\vec{w}$ is smooth enough so that the fundamental solutions in Sect. 2.2 can be used directly. Hence the admissible functions satisfy the Cauchy–Navier equation exactly. Define the boundary energy

$$I(u, v) = \int_\Gamma [(u - f_1)^2 + (v - f_2)^2], \tag{7.2}$$

Denote by V_N the set of all linear combinations of fundamental solutions given in (2.15) and (2.16). The method of fundamental solutions (MFS) reads: To seek $(u_N, v_N) \in V_N$ such that

$$I(u_N, v_N) = \min_{(u,v)\in V_N} I(u, v). \tag{7.3}$$

When the integrals in (7.2) involve numerical approximation

$$\hat{I}(u, v) = \widehat{\int_\Gamma} [(u - f_1)^2 + (v - f_2)^2], \tag{7.4}$$

where $\widehat{\int}$ is the numerical approximation of $\int$ by some rule, such as the central or Gaussian rule. Equation (7.3) leads to

$$\hat{I}(u_N, v_N) = \min_{(u,v)\in V_N} \hat{I}(u, v). \tag{7.5}$$

We may establish the collocation equations directly from the boundary conditions (7.1)

$$\sqrt{\Delta h_j} u_N(Q_j) = \sqrt{\Delta h_j} f_1(Q_j), \quad \sqrt{\Delta h_j} v_N(Q_j) = \sqrt{\Delta h_j} f_2(Q_j), \quad Q_j \in \Gamma, \quad (7.6)$$

Let Γ be divided into small section $\Delta\Gamma_j$, i.e., $\Gamma = \cup_j \Delta\Gamma_j$. If choosing the boundary collocation nodes Q_j as the middle nodes of $\Delta\Gamma_j$, Eqs. (7.6) are just equivalent to (7.5) with the central rule. We choose the number of collocation nodes being equal or larger than that of unknown coefficients, and obtain an over-determined system

$$\mathbf{A}\vec{\mathbf{y}} = \mathbf{b}, \quad (7.7)$$

where $\mathbf{A} \in R^{p\times q}(p \geq q)$ and $\vec{\mathbf{y}} = \{\ldots a_i, b_i, c_i, \ldots\}^T$. We use the singular value decomposition method or the QR method to solve (7.7).

We may choose the particular solutions [13]

$$u_n = \sum_{k=1}^{n} r^k\{a_k[-\sin k\theta + Dk\sin(k-2)\theta] + b_k[\cos k\theta - Dk\cos(k-2)\theta] + c_k \sin k\theta - d_k \cos k\theta\} - d_0, \quad (7.8)$$

$$v_n = \sum_{k=1}^{n} r^k\{a_k[\cos k\theta + Dk\cos(k-2)\theta] + b_k[\sin k\theta + Dk\sin(k-2)\theta] + c_k \cos k\theta + d_k \sin k\theta\} + c_0, \quad (7.9)$$

where D is given in (2.17), and a_k, b_k, c_k and d_k are the coefficients to be determined. When the FS in (2.15) and (2.16) are replaced by the PS in (7.8) and (7.9), Eq. (7.5) (or (7.6)) is called the collocation Trefftz method (CTM). In the CTM, the number of unknown coefficients in (7.8) and (7.9) is $4n + 2$.

Suppose that for harmonic solutions, there exist the regularities

$$u^H, v^H, q^H \in H^{t+1}(S) \left(t \geq \frac{3}{2}\right), \quad (7.10)$$

where $\vec{h}(\vec{x}) = (u^H, v^H)^T$ and $q^H = q(\vec{x})$ in (2.5). Denote $r_{max} = \max r|_S$ and $r_{min} = \max r|_{S_{in}}$, where $S_{in} \subseteq S$ is a disk inside of S. Also choose $R > r_{max}$.

Theorem 7.1 *Let the following inequality be given for $q \geq 0$,*

$$2^{2q+1}\left(\frac{R}{r_{max}}\right)^{-2N} \leq 1, \quad (7.11)$$

Suppose that (7.10) hold. Then for the solutions $\vec{w}$ by the MFS, there exists the bound,

$$\|\vec{w} - \vec{w}_N\|_{0,\Gamma} \leq C\left\{\frac{1}{n^{t-\frac{1}{2}}} + n\left(\frac{R}{r_{max}}\right)^{2n-N}\left(\frac{r_{max}}{r_{min}}\right)^n\right\}, \quad (7.12)$$

where C is a constant independent of N in (2.15) *and* (2.16)*, and n in* (7.8) *and* (7.9).

From Theorem 7.1 we have the following theorem.

Corollary 7.1 *Let the conditions in Theorem 7.1 hold. Choose N such that*

$$\left(\frac{R}{r_{max}}\right)^{2n-N}\left(\frac{r_{max}}{r_{min}}\right)^{n} = O\left(\frac{1}{n^{t+\frac{1}{2}}}\right). \tag{7.13}$$

Then

$$\|u - u_N\|_{0,\Gamma} + \|v - v_N\|_{0,\Gamma} \leq C\frac{1}{N^{t-\frac{1}{2}}}. \tag{7.14}$$

In (7.12), the first term $O(\frac{1}{n^{t-\frac{1}{2}}})$ results from the errors of the CTM, and the second term $O(n(\frac{R}{r_{max}})^{2n-N}(\frac{r_{max}}{r_{min}})^{n})$ results from the errors between the PS and the FS, based on the FS expansions in Theorems 6.1 and 6.2. Detailed proof is given in Li et al. [14]. Corollary 7.1 indicates that the convergence rates of the MFS may reach those of the CTM in [18].

7.2 Numerical experiments

For the unit square $S = \{(x, y), 0 \leq x \leq 1, 0 \leq y \leq 1\}$, (Fig. 1) choose the following true solutions of (2.3)

$$\begin{pmatrix} u \\ v \end{pmatrix} = (1-\kappa)\begin{pmatrix} e^x \cos y \\ e^x \sin y \end{pmatrix} - \kappa\left[\begin{pmatrix} xe^x \cos y + ye^x \sin y \\ -xe^x \sin y + ye^x \cos y \end{pmatrix} + \begin{pmatrix} -\pi \sin(\pi x)\cosh(\pi y) \\ \pi \cos(\pi x)\sinh(\pi y) \end{pmatrix}\right]. \tag{7.15}$$

Consider the displacement boundary condition $u = \bar{u}$ and $v = \bar{v}$ on ∂S, where $\bar{u}$ and $\bar{v}$ are given from (7.15). Choose the general fundamental solutions (2.15) and (2.16). Let

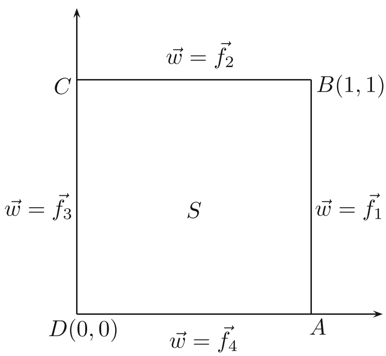

Fig. 1 The solutions domain S

Table 1 The errors and the condition numbers for displacement conditions by the MFS using FS in (2.15) and (2.16)

N	$\|\varepsilon\|_{\infty,\Gamma}$	$\|\varepsilon\|_{0,S}$	$\|\varepsilon\|_{1,S}$	Cond	Cond_eff	$\|\mathbf{y}\|$
6	0.914	0.259	3.25	114	7.99	66.1
10	0.175	3.99 (−2)	0.769	1.80 (3)	103	64.3
14	5.98 (−2)	9.45 (−3)	0.265	2.50 (4)	294	264
18	4.31 (−3)	5.07 (−4)	1.82 (−2)	4.08 (5)	5.83 (3)	193
22	3.18 (−4)	3.04 (−5)	1.35 (−3)	6.29 (6)	1.05 (5)	149
26	2.31 (−5)	1.82 (−6)	9.35 (−5)	8.62 (7)	1.48 (6)	134
30	1.26 (−6)	1.02 (−7)	6.19 (−6)	9.62 (8)	1.47 (7)	140

$R = 1.2$, $\lambda = \mu = 1$ and $D = 0.5$. Denote by m the number of collocation nodes of each edge of ∂S, and choose $m = 50$ in computation. We obtain the over-determined system (7.7). The traditional condition number is defined

$$\text{Cond} = \frac{\sigma_{max}}{\sigma_{min}}, \tag{7.16}$$

where σ_{max} and σ_{min} are the maximal and the minimal singular values of the matrix $\mathbf{A}$ in (7.7), respectively. The new effective condition number is defined in [15] by

$$\text{Cond\_eff} = \frac{\|\mathbf{b}\|}{\sigma_{min}\|\mathbf{y}\|}, \tag{7.17}$$

where $\|\mathbf{b}\|$ and $\|\mathbf{y}\|$ are the 2-norms. Since the effective condition number is smaller and even much smaller than the condition number, the effective condition number is a better criterion for numerical stability.

Denote the errors $\varepsilon = \vec{w} - \vec{w}_N = \left\{(u-u_N)^2 + (v-v_N)^2\right\}^{\frac{1}{2}}$, and

$$\|\varepsilon\|_{0,S} = \left\{\iint_S \varepsilon^2 ds\right\}^{\frac{1}{2}}, \quad |\varepsilon|_{1,S} = \left\{\iint_S ({\varepsilon_x}^2 + {\varepsilon_y}^2) ds\right\}^{\frac{1}{2}}, \tag{7.18}$$

where ${\varepsilon_x}^2 = [u_x - (u_N)_x]^2 + [v_x - (v_N)_x]^2$ and ${\varepsilon_y}^2 = [u_y - (u_N)_y]^2 + [v_y - (v_N)_y]^2$. On ∂S, denote the maximal boundary errors,

$$\|\varepsilon\|_{\infty,\Gamma} = \max\left\{(u-u_N)^2 + (v-v_N)^2\right\}^{\frac{1}{2}} \quad \text{on } \Gamma. \tag{7.19}$$

The errors and condition numbers are given in Table 1. All computations are carried out performed by using 50 decimal working digits. From Table 1, we can see

$$\begin{aligned} &\|\varepsilon\|_{\infty,\Gamma} = O(0.83^{3N}), \quad \|\varepsilon\|_{0,S} = O(0.81^{3N}), \quad |\varepsilon|_{1,S} = O(0.83^{3N}), \\ &\quad \text{Cond} = O(1.25^{3N}), \quad \text{Cond\_eff} = O(1.23^{3N}). \end{aligned} \tag{7.20}$$

Table 2 The errors and the condition numbers for displacement conditions by the MFS using FS in (2.15) and (2.16) with $c_i = 0$

N	$\|\varepsilon\|_{\infty,\Gamma}$	$\|\varepsilon\|_{0,S}$	$\|\varepsilon\|_{1,S}$	Cond	Cond_eff	$\|\mathbf{y}\|$
9	2.86	0.548	6.96	103	6.19	112
15	0.156	2.57 (−2)	0.555	860	70.4	63.6
21	1.03 (−2)	9.99 (−4)	2.80 (−2)	9.98 (3)	645	68.0
27	2.08 (−4)	1.49 (−5)	6.07 (−4)	9.18 (4)	5.77 (3)	61.7
33	8.38 (−6)	4.09 (−7)	2.23 (−5)	8.84 (5)	5.57 (4)	55.7
39	3.44 (−7)	1.33 (−8)	8.40 (−7)	9.85 (6)	6.20 (5)	51.2
45	1.21 (−8)	13.54 (−10)	2.76 (−8)	9.58 (7)	6.03 (6)	47.7

Table 3 The errors and the condition numbers for displacement conditions by the CTM using PS in (7.8) and (7.9)

n	$\|\varepsilon\|_{\infty,\Gamma}$	$\|\varepsilon\|_{0,S}$	$\|\varepsilon\|_{1,S}$	Cond	Cond_eff	$\|\mathbf{y}\|$
3	2.71	0.645	6.60	7.85	2.79	19.3
5	0.234	5.52 (−2)	0.882	24.3	7.07	23.4
7	1.61 (−2)	2.78 (−3)	5.87 (−2)	75.0	21.5	23.7
9	5.24 (−4)	8.81 (−5)	2.36 (−3)	229	65.9	23.7
11	1.34 (−5)	1.98 (−6)	6.27 (−5)	691	198	23.7
13	2.29 (−7)	3.25 (−8)	1.20 (−6)	2.05 (3)	590	23.7
15	3.04 (−9)	4.09 (−10)	1.70 (−8)	6.06 (3)	1.74 (3)	23.7

Next, we may choose a_i and b_i only, i.e., $c_i = 0$ in the general (2.15) and (2.16), the numerical results are listed in Table 2. We can see

$$\begin{aligned} \|\varepsilon\|_{\infty,\Gamma} = O(0.76^{2N}), \quad \|\varepsilon\|_{0,S} = O(0.74^{2N}), \quad |\varepsilon|_{1,S} = O(0.76^{2N}), \\ \text{Cond} = O(1.21^{2N}), \quad \text{Cond\_eff} = O(1.21^{2N}). \end{aligned} \tag{7.21}$$

Compared (7.21) with (7.20), using two coefficients a_i and b_i yields slightly better accuracy and stability. Hence we may simply choose a_i and b_i in real application.

Third, we choose the particular solutions (PS) in (7.8) and (7.9), where a_k, b_k, c_k and d_k are the unknown coefficients to be sought. This is the CTM using PS, and the numerical results are listed in Table 3. We can see

$$\begin{aligned} \|\varepsilon\|_{\infty,\Gamma} = O(0.65^{4n}), \quad \|\varepsilon\|_{0,S} = O(0.64^{4n}), \quad |\varepsilon|_{1,S} = O(0.66^{4n}), \\ \text{Cond} = O(1.15^{4n}), \quad \text{Cond\_eff} = O(1.15^{4n}). \end{aligned} \tag{7.22}$$

Compared (7.20) and (7.21) with (7.22), the convergence rates of the MFS are, basically, equivalent to those of the CTM, thus to coincide with Corollary 7.1.

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