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# On the pseudo-differential operators in the dual boundary integral equations using degenerate kernels and circulants

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**Abstract**

The spectral properties for the six kernels (influence matrices) in the dual boundary integral equations (dual BEM) are investigated for the Laplace and Helmholtz equations of a circular domain. Based on the two-point functions for the six kernels of single layer, double layer, normal derivatives of single and double layer potentials, tangent derivatives of single and double layer potentials, they can be expressed in degenerate kernels. Using the analytical properties of circulants, the spectral properties are studied exactly in a discrete system for a circular cavity when a uniform constant element scheme is adopted. After considering the number of degrees of freedom for the discrete system to be infinite for continuous system, the spectral properties of continuous system can be obtained. The relation for the influence matrices between the interior and exterior problems is addressed. Also, the condition number for the matrices and the orders of the pseudo-differential operators are examined. Finally, the properties of Calderon projector in discrete formulation are derived and are demonstrated analytically by an example of circular domain. Also, numerical results using the dual BEM program are performed to check the identities for the Calderon projector. © 2001 Elsevier Science Ltd. All rights reserved.

*Keywords*: Pseudo-differential operator; Calderon projector; Degenerate kernel; Spectral property

## 1. Introduction

Dual boundary integral equations as well as dual boundary element method (DBEM) was developed by Chen [1]. Many applications for the engineering problems were done, e.g. those with Hermite polynomial element [2], degenerate boundaries [1,3,4], corners [5], the construction of a symmetric matrix [6], the improvement of condition numbers [7], the construction of an image system [7], the tangent flux or hoop stress calculation on the boundary, an error indicator in an adaptive BEM [8], fictitious (irregular) eigenfrequencies in an exterior problem [9], spurious roots in a multiple reciprocity method (MRM) [10,11], real-part BEM [12,13], imaginary-part BEM [14,15], and degenerate scale problem in BEM [16]. Nevertheless, the mathematical aspects for the theory of dual boundary integral equations as well as dual BEM were not studied thoroughly for the researchers in engineering community. For an analytical study, the degenerate kernel is a powerful technique since it can separate the field point and the source point in the kernel. For example, Chen [9] applied the dual series model to study why fictitious frequency occurs for exterior problems using boundary element method. Also, Kuo et al. [12] employed the degenerate kernels and circulants to find the spurious solutions analytically and matched well with numerical results. In the Chen and Zhou's book [17], the theory of pseudo-differential operator and Calderon projector were used to study the kernels for the dual formulation. In mathematical point of view, the dual formulation is nothing new in the concept of Calderon projector [18]. The Calderon projector can construct the relations among the four kernels in the dual formulation. Recently, Amini [19] derived many original and interesting results of the explicit analytical expressions for the elements of boundary integral operators in a continuous system. To the authors' best knowledge, no example in the discrete system has been discussed on the Calderon projector and on the orders of pseudo-differential operators. For a circular geometry problem, it may be possible to examine all the properties since circulant matrices have analytical forms for spectral properties [20,21].

In this paper, we will employ the degenerate kernels in the dual formulation to study the spectral properties for discrete systems analytically and numerically, respectively. An example with a circular geometry will be considered to have the influence matrices in terms of circulants [20,21]. The spectral properties obtained by Amini [19] will be

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examined and some incorrect results will be discussed. Both the Laplace and Helmholtz problems will be taken into considerations. In addition, the condition number will be determined. Finally, the properties of Calderon projector in a discrete system will be demonstrated analytically and numerically.

## 2. Degenerate kernels in dual formulation for the Laplace and the Helmholtz equations

For the one-dimensional Laplace equation, we can express the four kernel functions of dual formulation in the degenerate forms as follows:

$$U(s,x) = \begin{cases} \frac{1}{2}(x-s), & x > s \\ \frac{1}{2}(s-x), & x < s \end{cases} = \begin{cases} \frac{1}{2}\sum_{i=1}^{2} a_i(x)b_i(s), & x > s \\ \frac{1}{2}\sum_{i=1}^{2} a_i(s)b_i(x), & x < s \end{cases} \quad (1)$$

$$T(s,x) = \begin{cases} -\frac{1}{2}, & x > s \\ \frac{1}{2}, & x < s \end{cases} = \begin{cases} \frac{1}{2}\sum_{i=1}^{2} a_i(x)b'_i(s), & x > s \\ \frac{1}{2}\sum_{i=1}^{2} a'_i(s)b_i(x), & x < s \end{cases} \quad (2)$$

$$L(s,x) = \begin{cases} \frac{1}{2}, & x > s \\ -\frac{1}{2}, & x < s \end{cases} = \begin{cases} \frac{1}{2}\sum_{i=1}^{2} a'_i(x)b_i(s), & x > s \\ \frac{1}{2}\sum_{i=1}^{2} a_i(s)b'_i(x), & x < s \end{cases} \quad (3)$$

$$M(s,x) = \begin{cases} 0, & x > s \\ 0, & x < s \end{cases} = \begin{cases} \frac{1}{2}\sum_{i=1}^{2} a'_i(x)b'_i(s), & x > s \\ \frac{1}{2}\sum_{i=1}^{2} a'_i(s)b'_i(x), & x < s \end{cases} \quad (4)$$

where $U$, $T$, $L$ and $M$ are the four kernels in the dual formulation [7], and

$$\begin{cases} a_1(x) = x \\ a_2(x) = -1 \\ b_1(s) = 1 \\ b_2(s) = s \end{cases} \quad (5)$$

It is found that the two-point functions for the four kernels in Eqs. (1)–(4) are separated. Only two terms in the series are required to represent the kernels. Similarly, for the two-dimensional Laplace equation, we have

$$U(s,x) = \ln r$$

$$= \ln\sqrt{(\rho\cos(\phi)) - R\cos(\theta))^2 + (\rho\sin(\phi) - R\sin(\theta))^2}$$

$$= \begin{cases} U^{\mathrm{i}}(s,x) = \ln R - \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{\rho}{R}\right)^m \cos(m(\theta-\phi)), & R > \rho \\ U^{\mathrm{e}}(s,x) = \ln \rho - \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{R}{\rho}\right)^m \cos(m(\theta-\phi)), & \rho > R \end{cases} \quad (6)$$

$$T(s,x) = \frac{\partial U}{\partial R} = \frac{R - \rho\cos(\phi-\theta)}{R^2 + \rho^2 - 2R\rho\cos(\phi-\theta)}$$

$$= \begin{cases} T^{\mathrm{i}}(s,x) = \frac{1}{R} + \sum_{m=1}^{\infty} \frac{\rho^m}{R^{m+1}} \cos(m(\theta-\phi)), & R > \rho \\ T^{\mathrm{e}}(s,x) = -\sum_{m=1}^{\infty} \frac{R^{m-1}}{\rho^m} \cos(m(\theta-\phi)), & \rho > R \end{cases} \quad (7)$$

$$L(s,x) = \frac{\partial U}{\partial \rho} = \frac{\rho - R\cos(\phi-\theta)}{R^2 + \rho^2 - 2R\rho\cos(\phi-\theta)}$$

$$= \begin{cases} L^{\mathrm{i}}(s,x) = -\sum_{m=1}^{\infty} \frac{\rho^{m-1}}{R^m} \cos(m(\theta-\phi)), & R > \rho \\ L^{\mathrm{e}}(s,x) = \frac{1}{\rho} + \sum_{m=1}^{\infty} \frac{R^m}{\rho^{m+1}} \cos(m(\theta-\phi)), & \rho > R \end{cases} \quad (8)$$

$$M(s,x) = \frac{\partial^2 U}{\partial\rho\partial R} = \frac{-2R\rho + (R^2+\rho^2)\cos(\theta-\phi)}{(R^2+\rho^2-2R\rho\cos(\theta-\phi))^2}$$

$$= \begin{cases} M^{\mathrm{i}}(s,x) = \sum_{m=1}^{\infty} \frac{m\rho^{m-1}}{R^{m+1}} \cos(m(\theta-\phi)), & R > \rho \\ M^{\mathrm{e}}(s,x) = \sum_{m=1}^{\infty} \frac{mR^{m-1}}{\rho^{m+1}} \cos(m(\theta-\phi)), & \rho > R \end{cases} \quad (9)$$

where $r$ is the distance between $x$ and $s$, $s = (R, \theta)$ and $x = (\rho, \phi)$ as shown in Fig. 1, the superscripts 'i' and 'e' denote

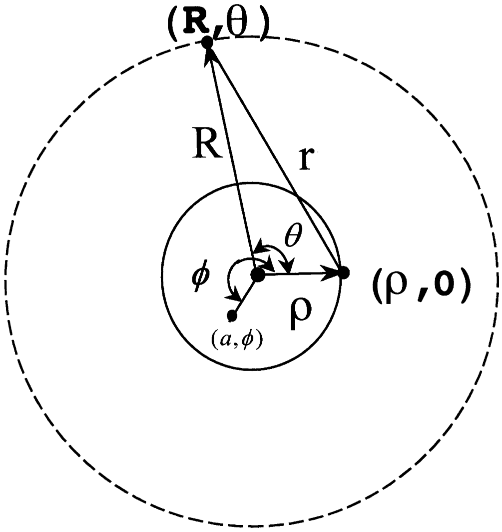


Fig. 1. The definitions of $\rho$, $\theta$, $\phi$, $a$ and $R$.

the interior ($\rho < R$) and exterior ($\rho > R$) domains, respectively. It is also found that the two-point (($\rho$,$\phi$) and ($R$,$\theta$)) functions for the four kernels in the two-dimensional case can be separated since $\cos(m(\theta - \phi)) = \cos(m\theta)\cos(m\phi) + \sin(m\theta)\sin(m\phi)$. Four arguments ($R$,$\theta$) for $s$, and ($\rho$,$\phi$) for $x$ in the four kernels of Eqs. (6)–(9) can be separated. Mathematically speaking, the four kernels can be written in degenerate forms. The number of terms in the series is infinite for the two-dimensional case instead of two terms for the one-dimensional case. For clarity, a potential distribution for the $U$ kernel of point source is shown in Fig. 2 using the degenerate expression in two different coordinate systems, ($P_1$,$\phi_1$) and ($P_2$,$\phi_2$). The two results match well to obey the objectivity, i.e. frame of indifference is satisfied for the expansion expression as shown in Fig. 2.

Similarly, extending the Laplace equation to the Helmholtz equation, the same degenerate kernels can be obtained. For the one-dimensional Helmholtz equation,

Laplace fundational solution : (where s=(R,θ) , x=(ρ,φ) )

$$U(s,x) = \ln r = \ln\sqrt{[\rho\cos(\phi) - R\cos(\theta)]^2 + [\rho\sin(\phi) - R\sin(\theta)]^2}$$

$$= \begin{cases} \mathrm{U^i}(s,x) = \ln R - \sum_{m=1}^{\infty} \frac{1}{m}(\frac{\rho}{R})^m \cos[m(\theta - \phi)] \quad , \quad \mathrm{R} > \rho \\ \mathrm{U^i}(s,x) = \ln \rho - \sum_{m=1}^{\infty} \frac{1}{m}(\frac{R}{\rho})^m \cos[m(\theta - \phi)] \quad , \quad \rho > R \end{cases}$$

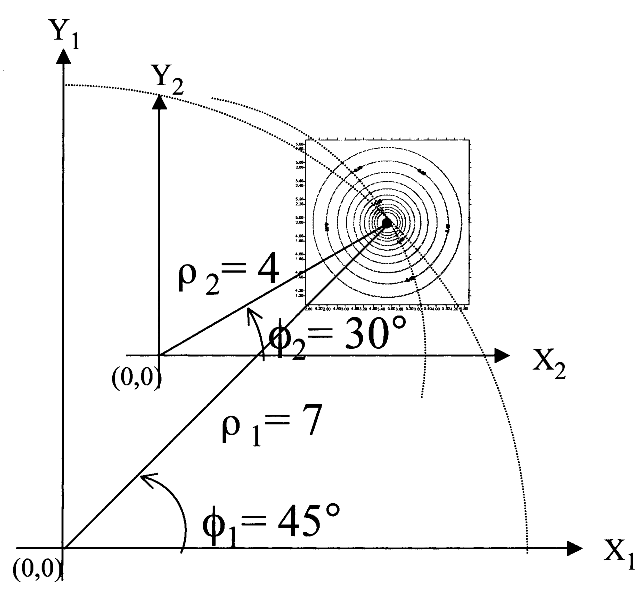


Fig. 2. Potential distribution using degenerate kernels of two different coordinate systems.

we have

$$U(s,x)=\begin{cases}\dfrac{e^{ik(x-s)}}{2ik}, & x>s\\ \dfrac{e^{ik(s-x)}}{2ik}, & x<s\end{cases}=\begin{cases}\dfrac{1}{2ik}\sum_{i=1}^{1}a_i(x)b_i(s), & x>s\\ \dfrac{1}{2ik}\sum_{i=1}^{1}a_i(s)b_i(x), & x<s\end{cases} \tag{10}$$

$$T(s,x)=\begin{cases}-\dfrac{e^{ik(x-s)}}{2}, & x>s\\ \dfrac{e^{ik(s-x)}}{2}, & x<s\end{cases}=\begin{cases}\dfrac{1}{2ik}\sum_{i=1}^{1}a_i(x)b_i'(s), & x>s\\ \dfrac{1}{2ik}\sum_{i=1}^{1}a_i'(s)b_i(x), & x<s\end{cases} \tag{11}$$

$$L(s,x)=\begin{cases}\dfrac{e^{ik(x-s)}}{2}, & x>s\\ -\dfrac{e^{ik(s-x)}}{2}, & x<s\end{cases}=\begin{cases}\dfrac{1}{2ik}\sum_{i=1}^{1}a_i'(x)b_i(s), & x>s\\ \dfrac{1}{2ik}\sum_{i=1}^{1}a_i(s)b_i'(x), & x<s\end{cases} \tag{12}$$

$$M(s,x)=\begin{cases}-\dfrac{ike^{ik(x-s)}}{2}, & x>s\\ -\dfrac{ike^{ik(s-x)}}{2}, & x<s\end{cases}=\begin{cases}\dfrac{1}{2ik}\sum_{i=1}^{1}a_i'(x)b_i'(s), & x>s\\ \dfrac{1}{2ik}\sum_{i=1}^{1}a_i'(s)b_i'(x), & x<s\end{cases} \tag{13}$$

where $k$ is wave number, $i^2=-1$ and

$$\begin{cases}a_1(x)=e^{ikx}\\ b_1(s)=e^{-iks}\end{cases}$$

In this case, only one term in the series is required for the degenerate expressions. Similarly, for the two-dimensional Helmholtz equation, the real-part kernel functions can be expressed as

$$U(s,x)=\mathrm{Re}\left\{\frac{-i\pi H_0^{(1)}(kr)}{2}\right\}=\begin{cases}U^{i}(s,x)=\sum_{m=-\infty}^{\infty}\dfrac{\pi}{2}J_m(k\rho)Y_m(kR)\cos(m(\theta-\phi)), & R>\rho\\ U^{e}(s,x)=\sum_{m=-\infty}^{\infty}\dfrac{\pi}{2}J_m(kR)Y_m(k\rho)\cos(m(\theta-\phi)), & \rho>R\end{cases} \tag{14}$$

$$T(s,x)=\mathrm{Re}\left\{\frac{-i\pi}{2}\frac{\partial H_0^{(1)}(kr)}{\partial R}\right\}=\begin{cases}T^{i}(s,x)=\sum_{m=-\infty}^{\infty}\dfrac{\pi}{2}kJ_m(k\rho)Y_m'(kR)\cos(m(\theta-\phi)), & R>\rho\\ T^{e}(s,x)=\sum_{m=-\infty}^{\infty}\dfrac{\pi}{2}kJ_m'(kR)Y_m(k\rho)\cos(m(\theta-\phi)), & \rho>R\end{cases} \tag{15}$$

$$L(s,x)=\mathrm{Re}\left\{\frac{-i\pi}{2}\frac{\partial H_0^{(1)}(kr)}{\partial \rho}\right\}=\begin{cases}L^{i}(s,x)=\sum_{m=-\infty}^{\infty}\dfrac{\pi}{2}kJ_m'(k\rho)Y_m(kR)\cos(m(\theta-\phi)), & R>\rho\\ L^{e}(s,x)=\sum_{m=-\infty}^{\infty}\dfrac{\pi}{2}kJ_m(kR)Y_m'(k\rho)\cos(m(\theta-\phi)), & \rho>R\end{cases} \tag{16}$$

$$M(s,x)=\mathrm{Re}\left\{\frac{-i\pi}{2}\frac{\partial^2 H_0^{(1)}(kr)}{\partial \rho\partial R}\right\}=\begin{cases}M^{i}(s,x)=\sum_{m=-\infty}^{\infty}\dfrac{\pi}{2}k^2J_m'(k\rho)Y_m'(kR)\cos(m(\theta-\phi)), & R>\rho\\ M^{e}(s,x)=\sum_{m=-\infty}^{\infty}\dfrac{\pi}{2}k^2J_m'(kR)Y_m'(k\rho)\cos(m(\theta-\phi)), & \rho>R\end{cases} \tag{17}$$

where Re means the real part, $J_m$ and $Y_m$ are the $m$th order Bessel functions of the first and second kind, and $H_0^{(1)}$ is the zeroth order Hankel function of the first kind.

For the imaginary-part kernels, we can decompose the four kernels into

$$U(s,x)=\mathrm{Im}\left\{\frac{-i\pi H_0^{(1)}(kr)}{2}\right\}=\begin{cases}U^{i}(s,x)=-\sum_{m=-\infty}^{\infty}\dfrac{\pi}{2}J_m(k\rho)J_m(kR)\cos(m(\theta-\phi)), & R>\rho\\ U^{e}(s,x)=-\sum_{m=-\infty}^{\infty}\dfrac{\pi}{2}J_m(kR)J_m(k\rho)\cos(m(\theta-\phi)), & \rho>R\end{cases} \tag{18}$$

$$T(s,x) = \mathrm{Im}\left\{ \frac{-\mathrm{i}\pi}{2} \frac{\partial H_0^{(1)}(kr)}{\partial R} \right\}$$

$$= \begin{cases} T^{\mathrm{i}}(s,x) = -\sum_{m=-\infty}^{\infty} \frac{\pi}{2} kJ_m(k\rho)J'_m(kR)\cos(m(\theta-\phi)), & R > \rho \\ T^{\mathrm{e}}(s,x) = -\sum_{m=-\infty}^{\infty} \frac{\pi}{2} kJ'_m(kR)J_m(k\rho)\cos(m(\theta-\phi)), & \rho > R \end{cases} \quad (19)$$

$$L(s,x) = \mathrm{Im}\left\{ \frac{-\mathrm{i}\pi}{2} \frac{\partial H_0^{(1)}(kr)}{\partial \rho} \right\}$$

$$= \begin{cases} L^{\mathrm{i}}(s,x) = -\sum_{m=-\infty}^{\infty} \frac{\pi}{2} kJ'_m(k\rho)J_m(kR)\cos(m(\theta-\phi)), & R > \rho \\ L^{\mathrm{e}}(s,x) = -\sum_{m=-\infty}^{\infty} \frac{\pi}{2} kJ_m(kR)J'_m(k\rho)\cos(m(\theta-\phi)), & \rho > R \end{cases} \quad (20)$$

$$M(s,x) = \mathrm{Im}\left\{ \frac{-\mathrm{i}\pi}{2} \frac{\partial^2 H_0^{(1)}(kr)}{\partial\rho\partial R} \right\}$$

$$= \begin{cases} M^{\mathrm{i}}(s,x) = -\sum_{m=-\infty}^{\infty} \frac{\pi}{2} k^2 J'_m(k\rho)J'_m(kR)\cos(m(\theta-\phi)), & R > \rho \\ M^{\mathrm{e}}(s,x) = -\sum_{m=-\infty}^{\infty} \frac{\pi}{2} k^2 J'_m(kR)J'_m(k\rho)\cos(m(\theta-\phi)), & \rho > R \end{cases} \quad (21)$$

where Im denotes the imaginary part. By combining the real with imaginary parts together, the complex-valued kernels can be expressed in degenerate forms.

## 3. Spectral properties for the influence matrices in the dual formulation for the Laplace and Helmholtz equations using constant element scheme of a circular boundary

Suppose the boundary is circular. Then, the influence matrix for the $U$ kernel can be expressed as

$$[\mathbf{U}] = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{2N-2} & a_{2N-1} \\ a_{2N-1} & a_0 & a_1 & \cdots & a_{2N-3} & a_{2N-2} \\ a_{2N-2} & a_{2N-1} & a_0 & \cdots & a_{2N-4} & a_{2N-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_{2N-1} & a_0 \end{bmatrix} \quad (22)$$

if $2N$ constant elements scheme is adopted for the circular problem [12,14]. This matrix [**U**] is found to be a circulant due to rotational symmetry. By employing the mean-value theorem, the element in [**U**] matrix is

$$a_m = \int_{(m-(1/2))\Delta\theta}^{(m+(1/2))\Delta\theta} U(R,\theta;\rho,0)R\,\mathrm{d}\theta \approx U(R,\theta;\rho,0)R\Delta\theta \quad (23)$$

where $\phi$ has been set to be zero ($\phi = 0$) for the source point, $\theta_m = m\Delta\theta$, and $\Delta\theta = (2\pi)/(2N)$. By employing the eigen properties for circulants [20], the eigenvalue $\lambda_l$ can be derived as follows:

$$\lambda_l = \sum_{m=0}^{2N-1} a_m \alpha_l^m = \sum_{m=0}^{2N-1} a_m\, \mathrm{e}^{\mathrm{i}\Delta\theta\, ml}, \quad (24)$$

$l = 0, 1, 2, \ldots, (N-1), N$

Since the rotation symmetry for the circle can be obtained, i.e. $a_m = a_{2N-m}$, the eigenvalue can be simplified into

$$\lambda_l = a_0 + (-1)^l a_N + \sum_{m=1}^{N-1} (\alpha_l^m + \alpha_l^{2N-m}) a_m$$

$$= \int_0^{2\pi} \cos(l\theta) U(\theta, 0) R\,\mathrm{d}\theta \quad (25)$$

By employing the properties of symmetric circulants [20,21], we can derive the eigenvalues of the four kernels using $U^{\mathrm{e}}$, $T^{\mathrm{e}}$, $L^{\mathrm{e}}$ and $M^{\mathrm{e}}$ kernels to avoid the jump terms for interior problems. Therefore, we have

$U^{\mathrm{e}}$ kernel:

$$\lambda_l^U = \int_0^{2\pi} \cos(l(\theta-0)) U^{\mathrm{e}}(\theta,0) R\,\mathrm{d}\theta$$

$$= \begin{cases} 2\pi R \ln\rho, & l = 0 \\ -\pi \dfrac{R^{l-1}}{l\,\rho^l}, & l = 1,2,3,\ldots,(N-1),N \end{cases} \quad (26)$$

$T^{\mathrm{e}}$ kernel:

$$\lambda_l^T = \int_0^{2\pi} \cos(l(\theta-0)) T^{\mathrm{e}}(\theta,0) R\,\mathrm{d}\theta$$

$$= \begin{cases} 0, & l = 0 \\ -\pi \left(\dfrac{R}{\rho}\right)^l, & l = 1,2,3,\ldots,(N-1),N \end{cases} \quad (27)$$

$L^{\mathrm{e}}$ kernel:

$$\lambda_l^L = \int_0^{2\pi} \cos(l(\theta-0)) L^{\mathrm{e}}(\theta,0) R\,\mathrm{d}\theta$$

$$= \begin{cases} 2\pi \dfrac{R}{\rho}, & l = 0 \\ \pi \left(\dfrac{R}{\rho}\right)^{l+1}, & l = 1,2,3,\ldots,(N-1),N \end{cases} \quad (28)$$

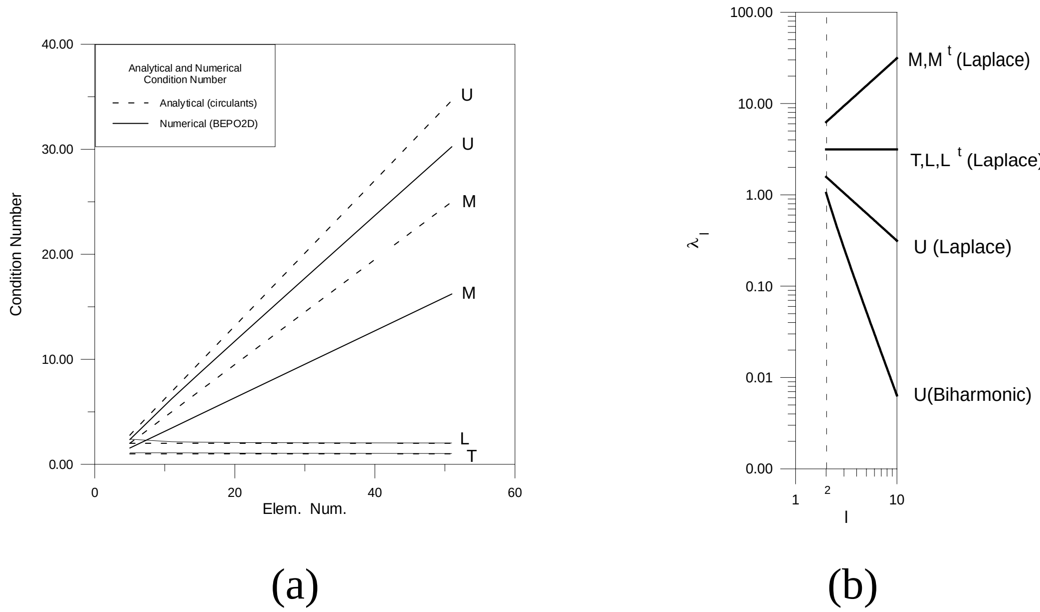


Fig. 3. The condition numbers for the four influence matrices.

$M^e$ kernel:

$$\lambda_l^M = \int_0^{2\pi} \cos(l(\theta - 0))M^e(\theta, 0)R \, d\theta$$

$$= \begin{cases} 0, & l = 0 \\ \pi l \dfrac{R^l}{\rho^{l+1}}, & l = 1, 2, 3, \ldots, (N-1), N \end{cases} \quad (29)$$

All the above spectral properties are useful for the direct BEM with a fictitious boundary since $R$ is greater than $p$ for the exterior problems as shown in Fig. 1.

For the direct BEM with $\rho = R$, the spectral properties for the four kernels reduce to

$$\lambda_l^U = \begin{cases} 2\pi R \ln R, & l = 0 \\ -\pi \dfrac{R}{|l|}, & l = \pm 1, \pm 2, \pm 3, \ldots, \pm(N-1), N \end{cases} \quad (30)$$

$$\lambda_l^T = \begin{cases} 0, & l = 0 \\ -\pi, & l = \pm 1, \pm 2, \pm 3, \ldots, \pm(N-1), N \end{cases} \quad (31)$$

$$\lambda_l^L = \begin{cases} 2\pi, & l = 0 \\ \pi, & l = \pm 1, \pm 2, \pm 3, \ldots, \pm(N-1), N \end{cases} \quad (32)$$

$$\lambda_l^M = \begin{cases} 0, & l = 0 \\ \pi \dfrac{|l|}{R}, & l = \pm 1, \pm 2, \pm 3, \ldots, \pm(N-1), N \end{cases} \quad (33)$$

where the superscripts on $\lambda_l$ denote the corresponding kernels, respectively. It must be noted that the double roots occur when $\lambda_{-l} = \lambda_l$, $l = 1, 2, \ldots, (N-1)$. Therefore, the determinant for the influence matrices can be derived as

$$\det[\mathbf{K}] = \lambda_0 \lambda_N (\lambda_1 \lambda_2 \ldots \lambda_{N-1})(\lambda_{-1}\lambda_{-2}\ldots\lambda_{-(N-1)})$$
$$= \lambda_0 \lambda_N (\lambda_1 \lambda_2 \ldots \lambda_{N-1})^2 \quad (34)$$

where $\mathbf{K}$ may be $\mathbf{U}$, $\mathbf{T}$, $\mathbf{L}$ or $\mathbf{M}$. The condition numbers for the matrices are shown in Fig. 3. Also, the tangent derivative properties for the single and double layer potentials, $L^t$ and $M^t$ [5] are discussed as follows:

$$L^t(s, x) = \frac{\partial U(s, x)}{\partial \phi} = \sum_{m=1}^{\infty} \sin(m(\theta - \phi)) \quad (35)$$

The eigenvalues for $L^t(s,x)$ are derived, we have

$$\lambda_l = \int_0^{2\pi} [\cos(l(\theta - \phi)) + i \sin(l(\theta - \phi))] L^t(s, x) R \, d(\theta - \phi)$$

$$= \begin{cases} \pi R i, & l > 0 \\ 0, & l = 0 \\ -\pi R i, & l < 0 \end{cases} \quad (36)$$

For the tangent derivative of double layer potential, we have

$$M^t(s, x) = \frac{\partial^2 U(s, x)}{\partial \phi \partial \theta} - \sum_{m=1}^{\infty} m \cos(m(\theta - \phi)) \quad (37)$$

Table 1
The orders of pseudo-differential operators in the dual formulation (all the influence matrices have the dimensions of $(2N+1)$ by $(2N+1)$)

| | Order of pseudo-differential operator | $\lambda_l$ | Rank |
|---|---|---|---|
| $U(s,x)$ | $-1$ | $2\pi R \ln R,\ l=0;\ -\pi(R/l),\ l=1,2,3,\dots,N$ | $2N,\ R=1;\ 2N+1,\ R\neq 1$ |
| $T(s,x)$ | 0 | $0,\ l=0;\ -\pi,\ l=1,2,3,\dots,N$ | $2N$ |
| $L(s,x)$ | 0 | $2\pi,\ l=0;\ \pi,\ l=1,2,3,\dots,N$ | $2N+1$ |
| $M(s,x)$ | 1 | $0,\ l=0;\ \pi(l/R),\ l=1,2,3,\dots,N$ | $2N$ |
| $L^t(s,x)$ | 0 | $\pi R\mathrm{i},\ l>0;\ 0,\ l=0;\ -\pi R\mathrm{i},\ l<0$ | $2N$ |
| $M^t(s,x)$ | 1 | $0,\ l=0;\ -l\pi R,\ l=1,2,3,\dots,N$ | $2N$ |

The eigenvalues can be obtained as shown below

$$\lambda_l = \int_0^{2\pi} [\cos(l(\theta-\phi)) + \mathrm{i}\sin(l(\theta-\phi))]M^t(s,x)R\,\mathrm{d}(\theta-\phi) = \begin{cases} 0, & l=0 \\ -l\pi R, & l=1,2,3,\dots,N \end{cases} \tag{38}$$

According to the eigenvalues for the six kernels, we have the orders of pseudo-differential operators as shown below:

$$\mathrm{O_p}(U) = -1, \quad H^r(B) \to H^{r+1}(B), \quad \lambda_m \to \mathrm{O}\left(\frac{1}{m}\right) \tag{39}$$

$$\mathrm{O_p}(T) = 0, \quad H^r(B) \to H^r(B), \quad \lambda_m \to \mathrm{O}(1) \tag{40}$$

$$\mathrm{O_p}(L) = 0, \quad H^r(B) \to H^r(B), \quad \lambda_m \to \mathrm{O}(1) \tag{41}$$

$$\mathrm{O_p}(M) = 1, \quad H^r(B) \to H^{r-1}(B), \quad \lambda_m \to \mathrm{O}(m) \tag{42}$$

$$\mathrm{O_p}(L^t) = 0, \quad H^r(B) \to H^r(B), \quad \lambda_m \to \mathrm{O}(1) \tag{43}$$

$$\mathrm{O_p}(M^t) = 1, \quad H^r(B) \to H^{r-1}(B), \quad \lambda_m \to \mathrm{O}(m) \tag{44}$$

where $H^r(B)$ denotes the Sobolev space of $r$th order on the boundary $B$ [17]. The results are summarized in Table 1. After comparing the results with those of Ref. [19], some errors in his paper for $\lambda_l^T$ in Eq. (31) can be found.

Similarly, for the fundamental solution ($r^2 \ln r$) of biharmonic operator, we have the order of pseudo-differential operator as shown below

$$U(s,x) = r^2 \ln r = \begin{cases} U^{\mathrm{i}}(s,x) = [(\rho\cos\phi - R\cos\theta)^2 + (\rho\sin\phi - R\sin\theta)^2][\ln R - \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{\rho}{R}\right)^m \cos(m(\theta-\phi))], & R > \rho \\ U^{\mathrm{e}}(s,x) = [(\rho\cos\phi - R\cos\theta)^2 + (\rho\sin\phi - R\sin\theta)^2][\ln \rho - \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{R}{\rho}\right)^m \cos(m(\theta-\phi))], & \rho > R \end{cases} \tag{45}$$

By moving the field point to the boundary, we have

$$U(s,x) = r^2 \ln r = R^2 \ln R[2 - 2\cos(\theta-\phi)] + R^2[2-2\cos(\theta-\phi)]\left[-\sum_{n=1}^{\infty}\frac{1}{n}\cos(n(\theta-\phi))\right] \tag{46}$$

After lengthy derivation, the eigenvalues for the influence matrix can be derived as shown below

$$\lambda_l = \begin{cases} 4\pi R^3 \ln R + 2\pi R^3, & l=0 \\ -2\pi R^3 \ln R - \frac{3}{2}\pi R^3, & l=\pm 1 \\ \frac{2\pi R^3}{l(l^2-1)}, & l=\text{otherwise} \end{cases} \tag{47}$$

The order of pseudo-differential operator for the $U$ kernel of the circular plate is

$$\mathrm{O_p}(U) = -3, \quad H^r(B) \to H^{r+3}(B), \quad \lambda_m \to \mathrm{O}\left(\frac{1}{m^3}\right)$$

## 4. Calderon projector in the dual formulation for the Laplace and Helmholtz equations — discrete system for a circle

In order to verify the property of the Calderon projector for the circular Laplace problem, we redefine the normalized fundamental solution as

$$U(s,x) = \frac{1}{2\pi}\ln r$$

When $R$ approaches $\rho$, the four kernels reduce to

$$U(s,x) = \frac{1}{2\pi}\ln \;= \frac{1}{2\pi}\ln r(R\sqrt{2-2\cos(\theta-\phi)}) = \begin{cases} U^{i}(s,x) = \dfrac{\ln R}{2\pi} - \dfrac{1}{2\pi}\sum_{m=1}^{\infty}\dfrac{1}{m}\cos(m(\theta-\phi)), & \rho\to R^- \\ U^{e}(s,x) = \dfrac{\ln R}{2\pi} - \dfrac{1}{2\pi}\sum_{m=1}^{\infty}\dfrac{1}{m}\cos(m(\theta-\phi)), & \rho\to R^+ \end{cases} \tag{48}$$

$$T(s,x) = \frac{\partial U}{\partial R} = \frac{1}{4\pi R}\ \ (\text{if } \phi\neq\theta) = \begin{cases} T^{i}(s,x) = \dfrac{1}{2\pi R} + \dfrac{1}{2\pi R}\sum_{m=1}^{\infty}\cos(m(\theta-\phi)) = \dfrac{1}{4\pi R} + \dfrac{1}{2R}\sum_{m=-\infty}^{\infty}\delta((\theta-\phi)-2\pi m), & \rho\to R^- \\ T^{e}(s,x) = -\dfrac{1}{2\pi R}\sum_{m=1}^{\infty}\cos(m(\theta-\phi)) = \dfrac{1}{4\pi R} - \dfrac{1}{2R}\sum_{m=-\infty}^{\infty}\delta((\theta-\phi)-2\pi m), & \rho\to R^+ \end{cases} \tag{49}$$

$$L(s,x) = \frac{\partial U}{\partial \rho} = \frac{1}{4\pi R}\ \ (\text{if } \phi\neq\theta)$$
$$= \begin{cases} L^{i}(s,x) = -\dfrac{1}{2\pi R}\sum_{m=1}^{\infty}\cos(m(\theta-\phi)) = \dfrac{1}{4\pi R} - \dfrac{1}{2R}\sum_{m=-\infty}^{\infty}\delta((\theta-\phi)-2\pi m), & \rho\to R^- \\ L^{e}(s,x) = \dfrac{1}{2\pi R} + \dfrac{1}{2\pi R}\sum_{m=1}^{\infty}\cos(m(\theta-\phi)) = \dfrac{1}{4\pi R} + \dfrac{1}{2R}\sum_{m=-\infty}^{\infty}\delta((\theta-\phi)-2\pi m), & \rho\to R^+ \end{cases} \tag{50}$$

$$M(s,x) = \frac{\partial^2 U}{\partial\rho\partial R} = \frac{-1}{8\pi R^2\sin^2\left(\dfrac{\phi-\theta}{2}\right)} = \begin{cases} M^{i}(s,x) = \dfrac{1}{2\pi R^2}\sum_{m=1}^{\infty} m\cos(m(\theta-\phi)), & \rho\to R^- \\ M^{e}(s,x) = \dfrac{1}{2\pi R^2}\sum_{m=1}^{\infty} m\cos(m(\theta-\phi)), & \rho\to R^+ \end{cases} \tag{51}$$

By observing the difference between the $T$ and $L$ kernels, the diagonal terms for the influence matrix have the jump terms $\pm 1/2$. For the influence matrix of $T^{i}$ kernel, we have

$$[\mathbf{T}_{jj}]^{i} = \int_{-(1/2)\Delta\phi}^{(1/2)\Delta\phi}\left\{\frac{1}{4\pi R} + \frac{1}{2R}\sum_{m=-\infty}^{\infty}\delta((\theta-0) - 2\pi m)R\right\}\mathrm{d}\theta\ \ (j\text{ no sum}) = \frac{\Delta\phi}{4\pi} + \frac{1}{2} \tag{52}$$

where $\Delta\phi = \pi/N$. The explicit form for the element of $[\mathbf{T}]$ matrix is

$$T_{ij} = \begin{cases} \dfrac{\Delta\phi}{4\pi} + \dfrac{1}{2}, & i = j \\ \dfrac{\Delta\phi}{4\pi}, & i\neq j \end{cases} \tag{53}$$

This result for a circular boundary is different from that of a straight element where the Cauchy principal value is zero instead of $\Delta\phi/4\pi$. The reason is that the approximation of boundary contour is different. According to the explicit forms for the four kernels in the dual frame, it is obvious to find [7,22]

$$[\mathbf{U}^{i}] = [\mathbf{U}^{e}] \tag{54}$$

$$[\mathbf{M}^{i}] = [\mathbf{M}^{e}] \tag{55}$$

$$[\mathbf{T}^{i}] = [\mathbf{L}^{e}] \tag{56}$$

$$[\mathbf{T}^{e}] = [\mathbf{L}^{i}] \tag{57}$$

By defining the two matrices, $[\mathbf{T}]$ and $[\mathbf{L}]$

$$[\mathbf{T}] = [\mathbf{L}] = [\mathbf{T}^{i}] - \frac{1}{2}[\mathbf{I}] = [\mathbf{T}^{e}] + \frac{1}{2}[\mathbf{I}] = [\mathbf{L}^{i}] + \frac{1}{2}[\mathbf{I}] = [\mathbf{L}^{e}] - \frac{1}{2}[\mathbf{I}] \tag{58}$$

we have

$$[\mathbf{T}] = [\mathbf{L}] = \frac{\Delta\phi}{4\pi}\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}_{2N\times 2N} \tag{59}$$

Since a symmetric circulant for the influence matrices can be reserved for a circular problem, we can prove the four properties of the Calderon projector as folows:

$$-\frac{1}{4}[\mathbf{I}] + [\mathbf{T}]^2 - [\mathbf{U}][\mathbf{M}] = 0 \tag{60}$$

$$[\mathbf{U}][\mathbf{L}] = [\mathbf{T}][\mathbf{U}] \tag{61}$$

$$[\mathbf{M}][\mathbf{T}] = [\mathbf{L}][\mathbf{M}] \tag{62}$$

$$-\frac{1}{4}[\mathbf{I}] + [\mathbf{L}]^2 - [\mathbf{M}][\mathbf{U}] = 0 \tag{63}$$

Using Eq. (58), the first and fourth identities can be rewritten as

$$[\mathbf{T}^{\mathrm{i}}][\mathbf{T}^{\mathrm{e}}] = [\mathbf{U}][\mathbf{M}] \tag{64}$$

$$[\mathbf{L}^{\mathrm{i}}][\mathbf{L}^{\mathrm{e}}] = [\mathbf{M}][\mathbf{U}] \tag{65}$$

Based on the circulant property, we can decompose the matrices into

$$[\mathbf{U}^{\mathrm{i}}] = [\mathbf{U}^{\mathrm{e}}] = [\mathbf{\Phi}][\mathbf{D}_U][\mathbf{\Phi}]^{-1} \tag{66}$$

$$[\mathbf{M}^{\mathrm{i}}] = [\mathbf{M}^{\mathrm{e}}] = [\mathbf{\Phi}][\mathbf{D}_M][\mathbf{\Phi}]^{-1} \tag{67}$$

$$[\mathbf{T}^{\mathrm{i}}] = [\mathbf{L}^{\mathrm{e}}] = [\mathbf{\Phi}][\mathbf{D}_{T^{\mathrm{i}}}][\mathbf{\Phi}]^{-1} = [\mathbf{\Phi}][\mathbf{D}_{L^{\mathrm{e}}}][\mathbf{\Phi}]^{-1} \tag{68}$$

$$[\mathbf{T}^{\mathrm{e}}] = [\mathbf{L}^{\mathrm{i}}] = [\mathbf{\Phi}][\mathbf{D}_{T^{\mathrm{e}}}][\mathbf{\Phi}]^{-1} = [\mathbf{\Phi}][\mathbf{D}_{L^{\mathrm{i}}}][\mathbf{\Phi}]^{-1} \tag{69}$$

where **D** is the diagonal matrix composed by its eigenvalues and the transformation matrix, $\mathbf{\Phi}$, is

$$[\mathbf{\Phi}] = \frac{1}{\sqrt{2N}}\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & \mathrm{e}^{(\pi \mathrm{i})/N} & \cdots & \mathrm{e}^{\pi \mathrm{i}(2N-1)/N} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \mathrm{e}^{\pi \mathrm{i}(2N-1)/N} & \cdots & \mathrm{e}^{\pi \mathrm{i}(2N-1)^2/N} \end{bmatrix} \tag{70}$$

The diagonal elements for the four matrices, $D$, are

$$\lambda_l^{[\mathbf{U}]} = \begin{cases} R\ln R, & l=0 \\ -\dfrac{R}{2l}, & l=1,2,3,\dots,(N-1),N \end{cases} \tag{71}$$

$$\lambda^{[\mathbf{T}]} = \begin{cases} \lambda_l^{[\mathbf{T}^{\mathrm{i}}]} = \begin{cases} 1, & l=0 \\ \dfrac{1}{2}, & l=1,2,3,\dots,(N-1),N \end{cases} \\ \lambda_l^{[\mathbf{T}^{\mathrm{e}}]} = \begin{cases} 0, & l=0 \\ -\dfrac{1}{2}, & l=1,2,3,\dots,(N-1),N \end{cases} \end{cases} \tag{72}$$

$$\lambda^{[\mathbf{L}]} = \begin{cases} \lambda_l^{[\mathbf{L}^{\mathrm{i}}]} = \begin{cases} 0 & l=0 \\ -\dfrac{1}{2}, & l=1,2,3,\cdots,(N-1),N \end{cases} \\ \lambda_l^{[\mathbf{L}^{\mathrm{e}}]} = \begin{cases} 1, & l=0 \\ \dfrac{1}{2}, & l=1,2,3,\dots,(N-1),N \end{cases} \end{cases} \tag{73}$$

$$\lambda_l^{[\mathbf{M}]} = \begin{cases} 0, & l=0 \\ \dfrac{l}{2R}, & l=1,2,3,\dots,(N-1),N \end{cases} \tag{74}$$

It is found that Eqs. (71)–(74) show the adjoint properties of the influence matrices for the interior Neumann and exterior Dirichlet problems as mentioned by Kupradze [23]. For the Laplace problem, the four identities for Calderon projector can be easily proved as shown below:

First identity:

$$[\mathbf{\Phi}][\mathbf{D}_{T^{\mathrm{i}}}][\mathbf{D}_{T^{\mathrm{e}}}][\mathbf{\Phi}]^{-1} = [\mathbf{\Phi}][\mathbf{D}_U][\mathbf{D}_M][\mathbf{\Phi}]^{-1}$$

$$= [\mathbf{\Phi}]\begin{bmatrix} 0 & & & & \\ & \frac{-1}{4} & & & \\ & & \frac{-1}{4} & & \\ & & & \ddots & \\ & & & & \frac{-1}{4} \end{bmatrix}[\mathbf{\Phi}]^{-1} \tag{75}$$

Second identity:

$$[\mathbf{\Phi}][\mathbf{D}_U][\mathbf{D}_L][\mathbf{\Phi}]^{-1} = [\mathbf{\Phi}][\mathbf{D}_T][\mathbf{D}_U][\mathbf{\Phi}]^{-1}$$

$$= [\mathbf{\Phi}]\begin{bmatrix} \frac{1}{2}R\ln R & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{bmatrix}[\mathbf{\Phi}]^{-1} \tag{76}$$

Third identity:

$$[\mathbf{\Phi}][\mathbf{D}_M][\mathbf{D}_T][\mathbf{\Phi}]^{-1} = [\mathbf{\Phi}][\mathbf{D}_L][\mathbf{D}_M][\mathbf{\Phi}]^{-1}$$

$$= [\mathbf{\Phi}]\begin{bmatrix} \frac{1}{2}R\ln R & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{bmatrix}[\mathbf{\Phi}]^{-1} \tag{77}$$

Fourth identity:

$$[\mathbf{\Phi}][\mathbf{D}_{L^{\mathrm{i}}}][\mathbf{D}_{L^{\mathrm{e}}}][\mathbf{\Phi}]^{-1} = [\mathbf{\Phi}][\mathbf{D}_M][\mathbf{D}_U][\mathbf{\Phi}]^{-1}$$

$$= [\mathbf{\Phi}]\begin{bmatrix} 0 & & & & \\ & \frac{-1}{4} & & & \\ & & \frac{-1}{4} & & \\ & & & \ddots & \\ & & & & \frac{-1}{4} \end{bmatrix}[\mathbf{\Phi}]^{-1} \quad (78)$$

To demonstrate the validity, an error index using the Frobenius norm

$$\|A\|_F = \left[\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2\right]^{1/2}$$

where $\|A\|$ is the measure of the residuals for the identities. Based on the numerical experiments using the dual BEM, the four residues are shown in Fig. 4.

Similarly, we can extend the Laplace equation to the Helmholtz equation. Redefining the fundamental solution for the Helmholtz equation, we have

$$U(s,x) = \frac{-\mathrm{i}H_0^{(1)}(kr)}{4} \quad (79)$$

The real-part kernels can be expressed in terms of degenerate forms as follows:

$$U(s,x) = \mathrm{Re}\left\{\frac{-\mathrm{i}H_0^{(1)}(kr)}{4}\right\}$$
$$= \begin{cases} U^{\mathrm{i}}(s,x) = \sum_{m=-\infty}^{\infty}\frac{1}{4}J_m(k\rho)Y_m(kR)\cos(m(\theta-\phi)), & R>\rho \\ U^{\mathrm{e}}(s,x) = \sum_{m=-\infty}^{\infty}\frac{1}{4}J_m(kR)Y_m(k\rho)\cos(m(\theta-\phi)), & \rho>R \end{cases} \quad (80)$$

$$T(s,x) = \mathrm{Re}\left\{\frac{-\mathrm{i}}{4}\frac{\partial H_0^{(1)}(kr)}{\partial R}\right\}$$
$$= \begin{cases} T^{\mathrm{i}}(s,x) = \sum_{m=-\infty}^{\infty}\frac{1}{4}kJ_m(k\rho)Y'_m(kR)\cos(m(\theta-\phi)), & R>\rho \\ T^{\mathrm{e}}(s,x) = \sum_{m=-\infty}^{\infty}\frac{1}{4}kJ'_m(kR)Y_m(k\rho)\cos(m(\theta-\phi)), & \rho>R \end{cases} \quad (81)$$

$$L(s,x) = \mathrm{Re}\left\{\frac{-\mathrm{i}}{4}\frac{\partial H_0^{(1)}(kr)}{\partial \rho}\right\}$$
$$= \begin{cases} L^{\mathrm{i}}(s,x) = \sum_{m=-\infty}^{\infty}\frac{1}{4}kJ'_m(k\rho)Y_m(kR)\cos(m(\theta-\phi)), & R>\rho \\ L^{\mathrm{e}}(s,x) = \sum_{m=-\infty}^{\infty}\frac{1}{4}kJ_m(kR)Y'_m(k\rho)\cos(m(\theta-\phi)), & \rho>R \end{cases} \quad (82)$$

$$M(s,x) = \mathrm{Re}\left\{\frac{-\mathrm{i}}{4}\frac{\partial^2 H_0^{(1)}(kr)}{\partial\rho\partial R}\right\}$$
$$= \begin{cases} M^{\mathrm{i}}(s,x) = \sum_{m=-\infty}^{\infty}\frac{1}{4}k^2J'_m(k\rho)Y'_m(kR)\cos(m(\theta-\phi)), & R>\rho \\ M^{\mathrm{e}}(s,x) = \sum_{m=-\infty}^{\infty}\frac{1}{4}k^2J'_m(kR)Y'_m(k\rho)\cos(m(\theta-\phi)), & \rho>R \end{cases} \quad (83)$$

The series in Eqs. (80)–(83) becomes uniformly convergent ($U$), piecewise convergent ($T$,$L$) and divergent ($M$) when the field point moves to the boundary with $\rho \to \mathrm{R}^+$ from the exterior domain or $\rho \to \mathrm{R}^-$ from the interior domain. If the constant element scheme is used, the diagonal terms of the influence matrix also have the jump terms of $\pm 1/2$. For the coefficient matrix of $T^{\mathrm{i}}$ kernel, we have

$$T^{\mathrm{i}}(s,x) = \sum_{m=-\infty}^{\infty}\frac{1}{4}kJ_m(k\rho)Y'_m(kR)\cos(m(\theta-\phi))$$
$$= \frac{k}{8}\sum_{m=-\infty}^{\infty}(J_mY'_m + J'_mY_m)\cos(m(\theta-\phi))$$
$$+ \frac{k}{8}\sum_{m=-\infty}^{\infty}(J_mY'_m - J'_mY_m)\cos(m(\theta-\phi))$$
$$= T_r^*(\theta,\phi) + \frac{1}{2R}\delta(\theta-\phi)$$
$$= \frac{T^{\mathrm{i}}+T^{\mathrm{e}}}{2} + \frac{1}{2R}\delta(\theta-\phi), \quad (84)$$

and

$$[\mathbf{T}^{\mathrm{i}}_{jj}] = \int_{-\Delta\phi/2}^{\Delta\phi/2}\left\{T_r^*(\theta,0) + \frac{1}{2R}\delta(0-\theta)R\right\}\mathrm{d}\theta$$
$$= \frac{1}{2}([\mathbf{T}^{\mathrm{i}}_{jj}] + [\mathbf{T}^{\mathrm{e}}_{jj}]) + \frac{1}{2} \quad (j \text{ no sum}) \quad (85)$$

where $\Delta\phi = \pi/N$, and $T_r^*(\theta,\phi)$ is a regular function. The Wronskian of the Bessel functions has been

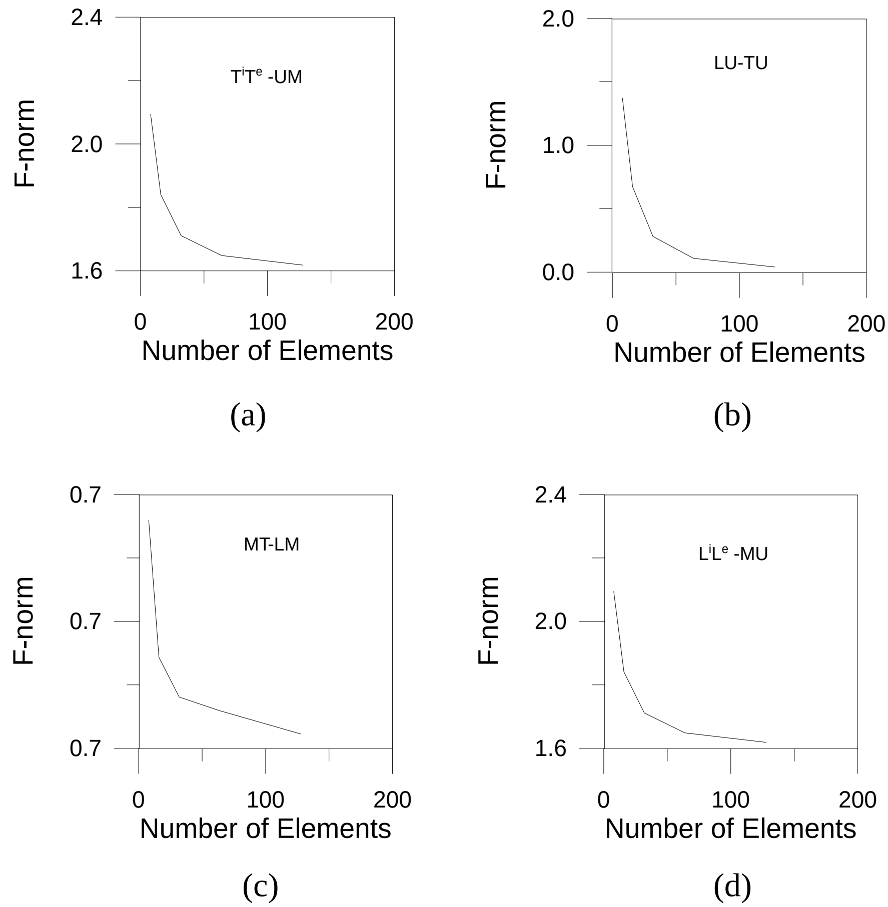


Fig. 4. The norms for the four identities of Calderon projector.

employed as shown below

$$W(J_m, Y_m) = Y'_m J_m - Y_m J'_m = \frac{2}{\pi k R} \tag{86}$$

in deriving Eq. (84). Also the identity

$$\sum_{m=-\infty}^{\infty} \cos(m\phi) = 2\pi \sum_{m=-\infty}^{\infty} \delta(\phi - 2m\pi) \tag{87}$$

in the generalized function [24] was used. For the Helmholtz equation, the influence matrices for the four kernels also satisfy Eqs.(54)–(57). By observing the difference between $[\mathbf{T}^{\mathrm{i}}]$ and $[\mathbf{T}^{\mathrm{e}}]$, we redefine

$$[\mathbf{T}_r^*] = [\mathbf{T}] = [\mathbf{L}] = [\mathbf{T}^{\mathrm{i}}] - \frac{1}{2}[\mathbf{I}] = [\mathbf{T}^{\mathrm{e}}] + \frac{1}{2}[\mathbf{I}]$$

$$= [\mathbf{L}^{\mathrm{i}}] + \frac{1}{2}[\mathbf{I}] = [\mathbf{L}^{\mathrm{e}}] - \frac{1}{2}[\mathbf{I}] \tag{88}$$

In a similar way for the Laplace equation, the four matrices also satisfy the four indentities of Calderon projector as shown in Eqs.(60)–(63). The first and fourth identities can also be reduced to

$$[\mathbf{T}^{\mathrm{i}}][\mathbf{T}^{\mathrm{e}}] = [\mathbf{U}][\mathbf{M}] \tag{89}$$

$$[\mathbf{L}^{\mathrm{i}}][\mathbf{L}^{\mathrm{e}}] = [\mathbf{M}][\mathbf{U}] \tag{90}$$

Since all the matrices are circulants, they can be decomposed into

$$[\mathbf{U}^{\mathrm{i}}] = [\mathbf{U}^{\mathrm{e}}] = [\mathbf{\Phi}][\mathbf{D}_U][\mathbf{\Phi}]^{-1} \tag{91}$$

$$[\mathbf{M}^{\mathrm{i}}] = [\mathbf{M}^{\mathrm{e}}] = [\mathbf{\Phi}][\mathbf{D}_M][\mathbf{\Phi}]^{-1} \tag{92}$$

$$[\mathbf{T}^{\mathrm{i}}] = [\mathbf{L}^{\mathrm{e}}] = [\mathbf{\Phi}][\mathbf{D}_{T^{\mathrm{i}}}][\Phi]^{-1} = [\mathbf{\Phi}][\mathbf{D}_{L^{\mathrm{e}}}][\mathbf{\Phi}]^{-1} \tag{93}$$

$$[\mathbf{T}^{\mathrm{e}}] = [\mathbf{L}^{\mathrm{i}}] = [\mathbf{\Phi}][\mathbf{D}_{T^{\mathrm{e}}}][\mathbf{\Phi}]^{-1} = [\mathbf{\Phi}][\mathbf{D}_{L^{\mathrm{i}}}][\mathbf{\Phi}]^{-1} \tag{94}$$

$$[\mathbf{T}_r^*] = \frac{1}{2}([\mathbf{T}^{\mathrm{i}}] + [\mathbf{T}^{\mathrm{e}}]) = \frac{1}{2}[\mathbf{\Phi}]([\mathbf{D}_{T^{\mathrm{i}}}] + [\mathbf{D}_{T^{\mathrm{e}}}])[\mathbf{\Phi}]^{-1} \tag{95}$$

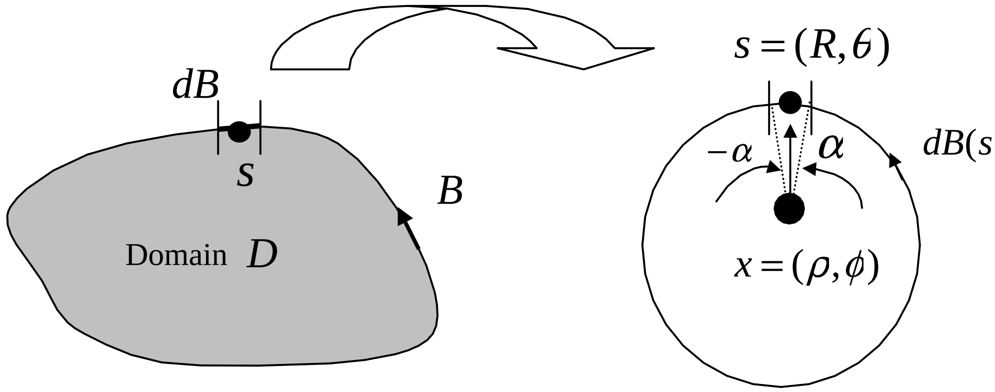


Fig. 5. The infinitesimal arc length with a center of curvature for a general boundary.

where the elements for the diagonal matrices, $D$, of the four kernels are

$$\lambda_l^{[\mathbf{U}]} = \frac{\pi R}{2} J_l(kR) Y_l(kR), \quad l = 0, 1, 2, 3, \ldots, (N-1), N \tag{96}$$

$$\lambda^{[\mathbf{T}]} = \begin{cases} \lambda_l^{[\mathbf{T}^{\mathrm{i}}]} = \dfrac{\pi kR}{2} J_l(kR) Y_l'(kR), & l = 0, 1, 2, 3, \ldots, (N-1), N \\ \lambda_l^{[\mathbf{T}^{\mathrm{e}}]} = \dfrac{\pi kR}{2} J_l'(kR) Y_l(kR), & l = 0, 1, 2, 3, \ldots, (N-1), N \end{cases} \tag{97}$$

$$\lambda^{[\mathbf{L}]} = \begin{cases} \lambda_l^{[\mathbf{L}^{\mathrm{i}}]} = \dfrac{\pi kR}{2} J_l'(kR) Y_l(kR), & l = 0, 1, 2, 3, \ldots, (N-1), N \\ \lambda_l^{[\mathbf{L}^{\mathrm{e}}]} = \dfrac{\pi kR}{2} J_l(kR) Y_l'(kR), & l = 0, 1, 2, 3, \ldots, (N-1), N \end{cases} \tag{98}$$

$$\lambda_l^{[\mathbf{M}]} = \frac{\pi k^2 R}{2} J_l'(kR) Y_l'(kR), \quad l = 0, 1, 2, 3, \ldots, (N-1), N \tag{99}$$

The first and fourth identities can be proved as shown below

$$[\boldsymbol{\Phi}][\mathbf{D}_{T^{\mathrm{i}}}][\mathbf{D}_{T^{\mathrm{e}}}][\boldsymbol{\Phi}]^{-1} = [\boldsymbol{\Phi}][\mathbf{D}_U][\mathbf{D}_M][\boldsymbol{\Phi}]^{-1} = \frac{\pi^2 k^2 R^2}{4} [\boldsymbol{\Phi}] \begin{bmatrix} J_0 Y_0 J_0' Y_0' & & & & & \\ & J_1 Y_1 J_1' Y_1' & & & & \\ & & \ddots & & & \\ & & & J_N Y_N J_N' Y_N' & & \\ & & & & \ddots & \\ & & & & & J_1 Y_1 J_1' Y_1' \end{bmatrix} [\boldsymbol{\Phi}]^{-1} \tag{100}$$

$$[\boldsymbol{\Phi}][\mathbf{D}_{L^{\mathrm{i}}}][\mathbf{D}_{L^{\mathrm{e}}}][\boldsymbol{\Phi}]^{-1} = [\boldsymbol{\Phi}][\mathbf{D}_M][\mathbf{D}_U][\boldsymbol{\Phi}]^{-1} = \frac{\pi^2 k^2 R^2}{4} [\boldsymbol{\Phi}] \begin{bmatrix} J_0 Y_0 J_0' Y_0' & & & & & \\ & J_1 Y_1 J_1' Y_1' & & & & \\ & & \ddots & & & \\ & & & J_N Y_N J_N' Y_N' & & \\ & & & & \ddots & \\ & & & & & J_1 Y_1 J_1' Y_1' \end{bmatrix} [\boldsymbol{\Phi}]^{-1} \tag{101}$$

For the second identity, we have

$$[\mathbf{U}][\mathbf{L}] = [\mathbf{U}][\mathbf{T}_r^*] = \frac{1}{2}[\boldsymbol{\Phi}]([\mathbf{D}_U][\mathbf{D}_{T^{\mathrm{i}}}] + [\mathbf{D}_U][\mathbf{D}_{T^{\mathrm{e}}}])[\boldsymbol{\Phi}]^{-1} = \frac{1}{2}[\boldsymbol{\Phi}]([\mathbf{D}_{T^{\mathrm{e}}}][\mathbf{D}_U] + [\mathbf{D}_{T^{\mathrm{i}}}][\mathbf{D}_U])[\boldsymbol{\Phi}]^{-1} = [\mathbf{T}][\mathbf{U}] \tag{102}$$

In the same way, the third identity can be proved.

Through a circular example, the theory of pseudo-differential operator in the dual formulation can be easily understood by engineers. Although this paper deals with the circular case, it can be extended to an arbitrary boundary. For the general boundary, the center of curvature can be constructed for the infinitesimal arc length as shown in Fig. 5. Degenerate kernels can be employed in the local region of circular boundary for the analytical study. If the boundary is straight, the curvature is zero. Based on the degenerate kernels, the fast multipole BEM was successfully applied to deal with some problems [25,26].

## 5. Conclusions

In this paper, the spectral properties for the boundary integral equations were studied for the two-dimensional circular problems. Both the Laplace and Helmholtz equations were considered. The six kernels for the single layer, double layer, their normal derivatives and tangent derivatives were all addressed. The orders of pseudo-differential operators, eigenvalues and eigenvectors were all determined using the analytical properties for the circulants. Also, the properties of Calderon projector in the dual formulation are examined numerically in discrete system using constant element scheme.

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