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Revisit of two classical elasticity problems by using the Trefftz method

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ABSTRACT

In this paper, the two classical elasticity cases, Lamé problem and stress concentration factor (SCF), are revisited by using the Trefftz method instead of the inverse or semi-inverse approach in the previous study. First, the Timoshenko and Goodier's approach is reviewed. Based on the superposition principle and the concept of taking free body, the problem of stress concentration factor as well as Lamé problem can be solved without any difficulty in a direct way using the Trefftz method.

1. Introduction

For solving boundary value problems of elasticity, it is usually difficult to find an analytical solution which satisfies the partial differential equations and given boundary conditions at the same time. In some cases, the inverse method or the semi-inverse method can be used. In the inverse method, a solution is found in priori such that it satisfies the governing equation and boundary condition. We can obtain the solution through this way for a luck cases, but it is not a logical way. For the semi-inverse method, certain assumptions about the components of displacement strain are made at the beginning. Then, the solution is confined by satisfying the equations of equilibrium and the boundary conditions. In the classical elasticity, the semi-inverse method was often employed to derive the analytical solution for simple problems. For example, the Saint-Venant torsion solution is a typical case which was obtained by using for the semi-inverse approach.

The Trefftz method was first presented by Trefftz in 1926 [1]. It can be seen as one kind of boundary-type methods. Until now, there are many Trefftz methods developed including direct and indirect formulations. The basic concept of Trefftz method is superimposing the *T*-complete functions which satisfy the governing equation, and the unknown coefficients are determined by matching the boundary conditions. The *T*-complete functions for plane elasticity problems have been already presented by some researchers [2,3]. The solution procedure is easier than other boundary-type methods, e.g. the boundary element method. Moreover, the formulation of the Trefftz method is regular and singular integrals are not required to calculate. Therefore, many applications for the Laplace equation [4], the Helmholtz equation [5], the Navier equation [6] and biharmonic equation [7] were done. More applications were summarized in Refs. [8–10]. Recently, Chen et al. [11] linked the two methods, Trefftz method and method of fundamental solutions, through the degenerate kernel for Laplace and biharmonic equations. They also found that all the Trefftz bases are imbedded in the degenerate kernel for the fundamental solution. Later, Schaback [12] also presented an article to discuss on this issue. Kaw et al. [13] used the finite element method to study the problem and compared with the analytical and experiment results. It is very useful to help students to study and understand the problem. Chen et al. [14] used the null-field boundary integral method to revisit the two classical elasticity problems, Lamé problem and the problem of stress concentration factor (SCF). However, we do not find that problems have been solved by using the Trefftz method in the literature to our best knowledge. Moreover, Timoshenko and Goodier's directly used $\cos(2\theta)$ in their book. Maybe the readers do not know the reason why only $\cos(2\theta)$ is chosen. Based on the Trefftz method, we provided another viewpoint to derive solutions. Therefore, we will attempt to revisit the two classical elasticity problems by using the Trefftz method.

In this paper, we employ the Trefftz method to deal with the Lamé problem and the problem of stress concentration factor. This approach is seen as an analytical method and the solution is derived in a natural and logical way, once the Trefftz base and its coefficient can be determined. For the two problems, they will be revisited by using the Trefftz method. Therefore, a direct way of solution for elasticity problems is our goal.

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2. Methods of solution

2.1. Problem statements

The two classical problems in the Timoshenko and Goodier's book [15] are revisited. One is an infinite plate with a circular hole subject to remote tension (stress concentration factor problem) and another is an annular cylinder subject to uniform pressures (Lamé problem), as shown in Figs. 1 and 2, respectively. The medium is considered as an isotropic, elastic and homogenous body. The governing equation is

$$(\lambda + G)\nabla(\nabla \cdot \underset{\sim}{u}(x)) + G\nabla^2 \underset{\sim}{u}(x) = 0, \quad x \in \Omega, \tag{1}$$

where $u(x)$ is the displacement, Ω is the domain of interest, ∇^2 is the Laplacian operator, and λ and G are the Lamé constants for the isotropic elasticity.

2.2. Review of Timoshenko and Goodier's solution

2.2.1. Lamé problem

This problem was first solved by Lamé [16]. According to the axial symmetry property, Timoshenko and Goodier represented as Ref. [15], assumed the airy stress function, ϕ, as

$$\phi(r,\theta) = A\ln r + Br^2 \ln r + Cr^2 + D. \tag{2}$$

Since $\phi(r,\theta)$ is symmetric, it is dependent on angle only in this case. The stress fields can be yielded as

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = \frac{A}{r^2} + B(1 + 2\ln r) + 2C, \tag{3}$$

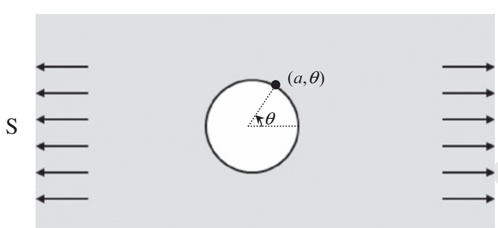

Fig. 1. An infinite plate with a circular hole subject to remote tension.

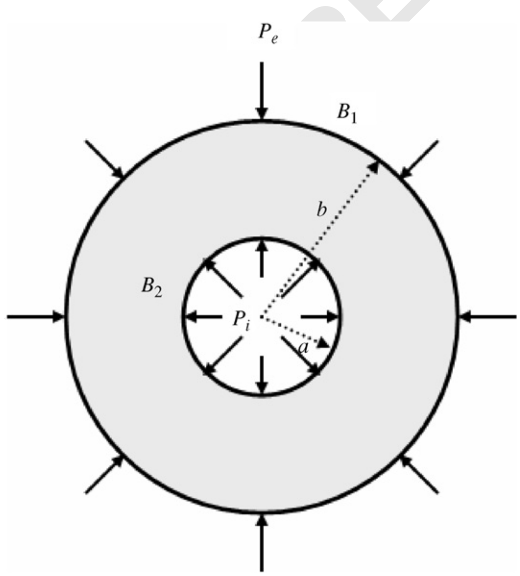

Fig. 2. An annular cylinder subject to uniform pressures.

$$\sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} = -\frac{A}{r^2} + B(3 + 2\ln r) + 2C, \tag{4}$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) = 0. \tag{5}$$

In order to ensure that the displacement field is a single-valued function, the coefficient B must be zero in the general solution. Eqs. (3) and (4) become

$$\sigma_{rr} = \frac{A}{r^2} + 2C, \tag{6}$$

$$\sigma_{\theta\theta} = -\frac{A}{r^2} + 2C. \tag{7}$$

Two boundary conditions ($\sigma_{rr}|_{r=b} = -P_o$ and $\sigma_{rr}|_{r=a} = -P_i$) for outer and inner boundaries, respectively, are needed to be satisfied. The two coefficients, A and C, are obtained by

$$A = \frac{a^2b^2(P_o - P_i)}{b^2 - a^2}, \tag{8}$$

$$C = \frac{1}{2}\frac{a^2P_i - b^2P_o}{b^2 - a^2}. \tag{9}$$

After obtaining the coefficients, the Airy stress function is obtained as

$$\phi(r,\theta) = \frac{a^2b^2(P_o - P_i)}{b^2 - a^2}\ln r + \frac{1}{2}\frac{a^2P_i - b^2P_o}{b^2 - a^2}r^2 + D, \tag{10}$$

where D is a constant which can be interpreted as a rigid body term. The stresses in Eqs. (3) and (4) are obtained

$$\sigma_{rr} = \frac{a^2b^2(P_o - P_i)}{b^2 - a^2}\frac{1}{r^2} + \frac{a^2P_i - b^2P_o}{b^2 - a^2}, \tag{11}$$

$$\sigma_{\theta\theta} = -\frac{a^2b^2(P_o - P_i)}{b^2 - a^2}\frac{1}{r^2} + \frac{a^2P_i - b^2P_o}{b^2 - a^2}. \tag{12}$$

For the special case of zero external pressure ($P_o = 0$), Eqs. (11) and (12) give

$$\sigma_{rr} = \frac{a^2P_i}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right), \tag{13}$$

$$\sigma_{\theta\theta} = \frac{a^2P_i}{b^2 - a^2}\left(1 + \frac{b^2}{r^2}\right). \tag{14}$$

2.2.2. Stress concentration factor problem

In this case, it can be seen as the extension from the Lamé problem, when the outer radius b approaches infinity. In the procedure of solution, the annular case is the considered domain to analyze the problem. For the far field at infinity, the stresses are

$$\sigma_{rr}|_{r=b} = S\cos^2\theta = \frac{1}{2}S(1 + \cos 2\theta), \tag{15}$$

$$\sigma_{\theta\theta}|_{r=b} = S\sin^2\theta = \frac{1}{2}S(1 - \cos 2\theta), \tag{16}$$

$$\sigma_{r\theta}|_{r=b} = -\frac{1}{2}S\sin 2\theta. \tag{17}$$

The traction free boundary condition is satisfied on the boundary of the hole. The stresses on the outer boundary can be decomposed into two parts. One is the constant normal stress ($S/2$), and it can be calculated by using Eqs. (11) and (12). The Airy stress function of this part is

$$\phi(r,\theta) = -\frac{S}{2}a^2 \ln r + \frac{S}{4}r^2. \tag{18}$$

The other part consists of the normal stress $(1/2)S\cos 2\theta$ and the shear stress $-(1/2)S\sin 2\theta$. The Airy stress function was assumed [11]

$$\phi(r,\theta) = \left(Ar^2 + Br^4 + C\frac{1}{r^2} + D\right)\cos 2\theta. \tag{19}$$

The stress components are

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = -\left(2A + \frac{6C}{r^4} + \frac{4D}{r^2}\right)\cos 2\,\theta. \tag{20}$$

$$\sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} = \left(2A + 12Br^2 + \frac{6C}{r^4}\right)\cos 2\theta \tag{21}$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) = \left(2A + 6Br^2 - \frac{6C}{r^4} - \frac{2D}{r^2}\right)\sin 2\theta. \tag{22}$$

By employing Eqs. (20)–(22) to satisfy the boundary condition Eqs. (15)–(17) and setting $a/b \approx 0$, we obtain coefficients

$$A = -\frac{S}{4}, \quad B = 0, \quad C = -\frac{a^4}{4}S, \quad D = \frac{a^2}{2}S, \tag{23}$$

by using the Mathematica manipulation. The Airy stress function is obtained.

$$\phi(r,\theta) = \left(-\frac{S}{4}r^2 - \frac{a^4S}{4}\frac{1}{r^2} + \frac{a^2S}{2}\right)\cos 2\theta. \tag{24}$$

Therefore, we have the total stress function by superimposing two parts

$$\phi(r,\theta) = -\frac{a^2S}{2}\ln r + \frac{S}{4}r^2 + \left(-\frac{S}{4}r^2 - \frac{a^4S}{4}\frac{1}{r^2} + \frac{a^2S}{2}\right)\cos 2\theta. \tag{25}$$

The stress components are

$$\sigma_{rr} = \frac{S}{2}\left(1 - \frac{a^2}{r^2}\right) + \frac{S}{2}\left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right)\cos 2\theta, \tag{26}$$

$$\sigma_{\theta\theta} = \frac{S}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{S}{2}\left(1 + \frac{3a^4}{r^4}\right)\cos 2\theta, \tag{27}$$

$$\sigma_{r\theta} = -\frac{S}{2}\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\sin 2\theta. \tag{28}$$

By substituting $r = a$ in Eq. (27), we find the hoop stress

$$\sigma_{\theta\theta} = S - 2S\cos 2\theta, \quad \sigma_{rr} = \sigma_{r\theta} = 0. \tag{29}$$

It is noted that the hoop stress ($\sigma_{\theta\theta}$) reaches the maximum of 3S when $\theta = \pi/2$ or $\theta = 3\pi/2$.

2.3. *Trefftz formulation—the present approach*

In the Trefftz method, the field solution $u(x)$ is

$$u(x) = \sum_{j=1}^{N_T} c_j u_j(x), \tag{30}$$

where N_T is the number of *T*-complete functions, c_j is the *j*th unknown coefficient and $u_j(x)$ is the *j*th *T*-complete function which satisfies the governing equation. The *T*-complete functions of biharmonic problem of the Airy stress function for interior and exterior cases can be found in Table 1 [2].

Table 1
T-complete functions of the Trefftz method and degenerate kernels of the MFS for the biharmonic problem.

Method of fundamental solution (MFS)			Trefftz method	
	Fundamental solution	Degenerate kernel	Interior basis	Exterior basis
Basis functions and degenerate kernels				
1D	$(1/12)r^3$	$U(s,x) = \begin{cases} \frac{1}{12}(x^3 - 3x^2s + 3xs^2 - s^3), & x > s \\ \frac{1}{12}(s^3 - 3s^2x + 3sx^2 - x^3), & x < s \end{cases}$	$1, x, x^2, x^3$	$1, x, x^2, x^3$
2D	$r^2\ln(r)$	$U(s,x) = \begin{cases} U^I(s,x) = \rho^2(1+\ln R) + R^2\ln R - R\rho(1+2\ln R)\cos(\theta-\phi) - \sum_{m=1}^{\infty}\frac{1}{m(m+1)}\frac{\rho^{m+2}}{R^m}\cos[m(\theta-\phi) + \sum_{m=2}^{\infty}\frac{1}{m(m-1)}\frac{\rho^m}{R^{m-2}}\cos[m(\theta-\phi)], & R > \rho \\ U^E(s,x) = R^2(1+\ln\rho) + \rho^2\ln\rho - \rho R(1+2\ln\rho)\cos(\theta-\phi) - \sum_{m=1}^{\infty}\frac{1}{m(m+1)}\frac{R^{m+2}}{\rho^m}\cos[m(\theta-\phi)] + \sum_{m=2}^{\infty}\frac{1}{m(m-1)}\frac{R^m}{\rho^{m-2}}\cos[m(\theta-\phi)], & \rho > R \end{cases}$	$1, \rho^2, \rho^m\cos(m\phi), \rho^m\sin(m\phi), \rho^{m+2}\cos(m\phi), \rho^{m+2}\sin(m\phi)$	$\ln(\rho), \rho^2\ln(\rho), \rho^{-m}\cos(m\phi), \rho^{-m}\sin(m\phi), \rho^{2-m}\cos(m\phi), \rho^{2-m}\sin(m\phi)$
The basis function which satisfy the equation				
1D		$(\partial U^4(x,s)/\partial x^4) = \delta(x-s)$	$d^4u(x)/dx^4 = 0$	
2D		$\nabla^4 U(x,s) = 8\pi\delta(x-s)$	$\nabla^4 U(\rho,\phi) = 0$	

Where $m = 0,1,2,3,\ldots$.

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2.3.1. Lamé problem

For the Lamé problem, the Airy stress function is obtained

$$\begin{aligned}\phi(r,\theta) = {} & \bar{a}_0 + \sum_{m=1}^{N} \bar{a}_m r^m \cos m\theta + \sum_{m=1}^{N} \bar{b}_m r^m \sin m\theta \\ & + \bar{c}_0 r^2 + \sum_{m=1}^{N} \bar{c}_m r^{m+2} \cos m\theta + \sum_{m=1}^{N} \bar{d}_m r^{m+2} \sin m\theta \\ & + a_0 \ln r + \sum_{m=1}^{N} a_m r^{-m} \cos m\theta + \sum_{m=1}^{N} b_m r^{-m} \sin m\theta \\ & + c_0 r^2 \ln r + \sum_{m=1}^{N} c_m r^{2-m} \cos m\theta + \sum_{m=1}^{N} d_m r^{2-m} \sin m\theta,\end{aligned} \tag{31}$$

by using the Trefftz base. Since, it can be seen as an interior problem superimposing with an exterior case, both interior and exterior Trefftz bases are chosen. Based on the relation between stress and Airy stress function, we have

$$\begin{aligned}\sigma_{rr} = {} & \sum_{m=1}^{N} (m-m^2)\bar{a}_m r^{m-2} \cos m\theta + \sum_{m=1}^{N} (m-m^2)\bar{b}_m r^{m-2} \sin m\theta \\ & + 2\bar{c}_0 + \sum_{m=1}^{N} (m+2-m^2)\bar{c}_m r^m \cos m\theta \\ & + \sum_{m=1}^{N} (m+2-m^2)\bar{d}_m r^m \sin m\theta + a_0 \frac{1}{r^2} \\ & - \sum_{m=1}^{N} (m+m^2) a_m r^{-(m+2)} \cos m\theta \\ & - \sum_{m=1}^{N} (m+m^2) b_m r^{m-2} \sin m\theta + c_0(2\ln r + 1) \\ & + \sum_{m=1}^{N} (2-m-m^2) c_m r^{-m} \cos m\theta \\ & + \sum_{m=1}^{N} (2-m-m^2) d_m r^{-m} \sin m\theta.\end{aligned} \tag{32}$$

By matching the boundary condition $(\sigma_{rr}|_{r=b} = -P_o)$ for the outer boundary, we have

$$2\bar{c}_0 + a_0 \frac{1}{b^2} + c_0(2\ln b + 1) = -P_o. \tag{33}$$

By matching the inner boundary condition $(\sigma_{rr}|_{r=a} = -P_i)$, we have

$$2\bar{c}_0 + a_0 \frac{1}{a^2} + c_0(2\ln a + 1) = -P_i. \tag{34}$$

Since the solution is a single-valued function, c_0 must be zero. From Eqs. (33) and (34), we have

$$a_0 = \frac{a^2 b^2}{b^2 - a^2}(P_o - P_i), \tag{35}$$

$$\bar{c}_0 = \frac{1}{2}\frac{a^2 P_i - b^2 P_o}{b^2 - a^2}. \tag{36}$$

Therefore, the Airy stress function is

$$\phi(r,\theta) = \frac{1}{2}\frac{a^2 P_i - b^2 P_o}{b^2 - a^2} r^2 + \frac{a^2 b^2 (P_o - P_i)}{b^2 - a^2} \ln r + \bar{a}_0, \tag{37}$$

where $\bar{a}_0$ is a constant. Then, the stress components are

$$\sigma_{rr} = \frac{a^2 P_i - b^2 P_o}{b^2 - a^2} + \frac{a^2 b^2 (P_o - P_i)}{b^2 - a^2}\frac{1}{r^2}, \tag{38}$$

$$\sigma_{\theta\theta} = \frac{a^2 P_i - b^2 P_o}{b^2 - a^2} - \frac{a^2 b^2 (P_o - P_i)}{b^2 - a^2}\frac{1}{r^2}. \tag{39}$$

The results are the same with the Timoshenko and Goodier's solution.

2.3.2. Stress concentration factor problem

For the problem of an infinite plate with a circular hole subject to a uniform tension of magnitude S in the x direction, it can be decomposed into two parts by using the superposition technique, as shown in Fig. 3(a) and (b). One is an infinite plate subject to a uniform tension and the other is an infinite plate with a free-traction hole. On the boundary of the hole, it needs to satisfy the boundary conditions of traction free for the superposing total solution. For the problem of an infinite plate subject to a uniform tension, it can be seen as a circular plate with an infinite radius. The Airy stress function is represented as

$$\begin{aligned}\phi^{\infty}(r,\theta) = {} & \bar{a}_0 + \sum_{m=1}^{N} \bar{a}_m r^m \cos m\theta + \sum_{m=1}^{N} \bar{b}_m r^m \sin m\theta \\ & + \bar{c}_0 r^2 + \sum_{m=1}^{N} \bar{c}_m r^{m+2} \cos m\theta + \sum_{m=1}^{N} \bar{d}_m r^{m+2} \sin m\theta,\end{aligned} \tag{40}$$

by choosing the interior Trefftz bases in Table 1. We have the stress components as follows:

$$\begin{aligned}\sigma_{rr}^{\infty} = {} & \sum_{m=1}^{N} (m-m^2)\bar{a}_m r^{m-2} \cos m\theta + \sum_{m=1}^{N} (m-m^2)\bar{b}_m r^{m-2} \sin m\theta \\ & + 2\bar{c}_0 + \sum_{m=1}^{N} (m+2-m^2)\bar{c}_m r^m \cos m\theta \\ & + \sum_{m=1}^{N} (m+2-m^2)\bar{d}_m r^m \sin m\theta,\end{aligned} \tag{41}$$

$$\begin{aligned}\sigma_{r\theta}^{\infty} = {} & \sum_{m=1}^{N} m(m-1)\bar{a}_m r^{m-2} \sin m\theta - \sum_{m=1}^{N} m(m-1)\bar{b}_m r^{m-2} \cos m\theta \\ & + \sum_{m=1}^{N} m(m+1)\bar{c}_m r^m \sin m\theta - \sum_{m=1}^{N} m(m+1)\bar{d}_m r^m \cos m\theta,\end{aligned} \tag{42}$$

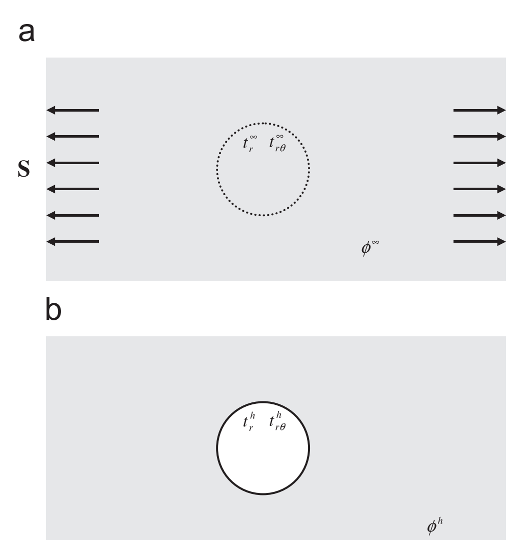

Fig. 3. (a) An infinite plate subject to a uniform tension (ϕ^{∞} field) and (b) an infinite plate with a hole (ϕ^h field).

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$$\sigma_{\theta\theta}^{\infty} = \sum_{m=1}^{N} m(m-1)\bar{a}_m r^{m-2} \cos m\theta + \sum_{m=1}^{N} m(m-1)\bar{b}_m r^{m-2} \sin m\theta + 2\bar{c}_0 + \sum_{m=1}^{N}(m+2)(m+1)\bar{c}_m r^m \cos m\theta + \sum_{m=1}^{N}(m+2)(m+1)\bar{d}_m r^m \sin m\theta. \quad (43)$$

when r approaches infinity, the boundary conditions, $\sigma_{rr}{}^{\infty} = (S/2)(1+\cos 2\theta)$, $\sigma_{r\theta}{}^{\infty} = -(1/2)S\sin 2\theta)$ and $\sigma_{\theta\theta}{}^{\infty} = (S/2)(1-\cos 2\theta)$, are needed to be satisfied. After comparing with the coefficient, we have

$$\bar{a}_2 = -\frac{1}{4}S, \quad \bar{c}_0 = \frac{1}{4}S, \quad \bar{a}_0 = \text{arbitrary} \quad (44)$$

and all other coefficients are equal to zero. The Airy stress function for the problem is obtained

$$\phi^{\infty}(r,\theta) = \frac{S}{4}r^2 - \frac{S}{4}r^2 \cos 2\theta + \bar{a}_0. \quad (45)$$

For the other part of the problem, an infinite plate with a hole, we choose the exterior Trefftz base for the Airy stress function as

$$\phi^h(r,\theta) = a_0 \ln r + \sum_{m=1}^{N} a_m r^{-m} \cos m\theta + \sum_{m=1}^{N} b_m r^{-m} \sin m\theta + c_0 r^2 \ln r + \sum_{m=1}^{N} c_m r^{2-m} \cos m\theta + \sum_{m=1}^{N} d_m r^{2-m} \sin m\theta. \quad (46)$$

The stress components are

$$\sigma_{rr}^h = \frac{a_0}{r^2} - \sum_{m=1}^{N}(m^2+m)a_m r^{-(m+2)} \cos m\theta - \sum_{m=1}^{N}(m^2+m)b_m r^{-(m+2)} \sin m\theta + c_0(2\ln r + 1) + \sum_{m=1}^{N}(2-m-m^2)c_m r^{-m} \cos m\theta + \sum_{m=1}^{N}(2-m-m^2)d_m r^{-m} \sin m\theta, \quad (47)$$

$$\sigma_{r\theta}^h = -\sum_{m=1}^{N} m(m+1)a_m r^{-(m+2)} \sin m\theta + \sum_{m=1}^{N} m(m+1)b_m r^{-(m+2)} \cos m\theta + \sum_{m=1}^{N} m(1-m)c_m r^{-m} \sin m\theta - \sum_{m=1}^{N} m(1-m)d_m r^{-m} \cos m\theta, \quad (48)$$

$$\sigma_{\theta\theta}^h = -\frac{a_0}{r^2} + \sum_{m=1}^{N} m(m+1)a_m r^{-(m+2)} \cos m\theta + \sum_{m=1}^{N} m(m+1)b_m r^{-(m+2)} \sin m\theta + c_0(2\ln r + 3) + \sum_{m=1}^{N}(2-m)(1-m)c_m r^{-m} \cos m\theta + \sum_{m=1}^{N} d_m(2-m)(1-m)r^{-m} \sin m\theta. \quad (49)$$

We have

$$a_0 = -\frac{S}{2}a^2, \quad a_2 = -\frac{S}{4}a^4, \quad c_2 = \frac{S}{2}a^2, \quad c_0 = \text{arbitrary}. \quad (50)$$

Since the solution is a single-valued function in physics, c_0 must be zero. The Airy stress function of this part is

$$\phi^h(r,\theta) = -\frac{S}{2}a^2 \ln r - \frac{S}{4}\frac{a^4}{r^2} \cos 2\theta + \frac{S}{2}a^2 \cos 2\theta. \quad (51)$$

Then, the total Airy stress function is

$$\phi = \phi^{\infty} + \phi^h = \frac{S}{4}(r^2 - 2a^2 \ln r) - \frac{S}{4}\left(r^2 + \frac{a^4}{r^2} - 2a^2\right)\cos 2\theta + \bar{a}_0. \quad (52)$$

After obtaining the Airy stress function, we can obtain the corresponding stress components

$$\sigma_{rr} = \frac{S}{2}\left(1 - \frac{a^2}{r^2}\right) + \frac{S}{2}\left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right)\cos 2\theta, \quad (53)$$

$$\sigma_{\theta\theta} = \frac{S}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{S}{2}\left(1 + \frac{3a^4}{r^4}\right)\cos 2\theta, \quad (54)$$

$$\sigma_{r\theta} = -\frac{S}{2}\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\sin 2\theta. \quad (55)$$

On the circular boundary ($r = a$) of the hole, the hoop stress is

$$\sigma_{\theta\theta} = S - 2S\cos 2\theta. \quad (56)$$

when θ approaches $\pi/2$ or $3\pi/2$, $\sigma_{\theta\theta}$ reaches the maximum $3S$. The result is the same with the Timoshenko and Goodier's solution. Fig. 4 shows the contour of hoop stress along the hole boundary in the infinite plate. Good agreement is made. The comparison between the present method and the Timoshenko and Goodier's approach is summarized in Table 2.

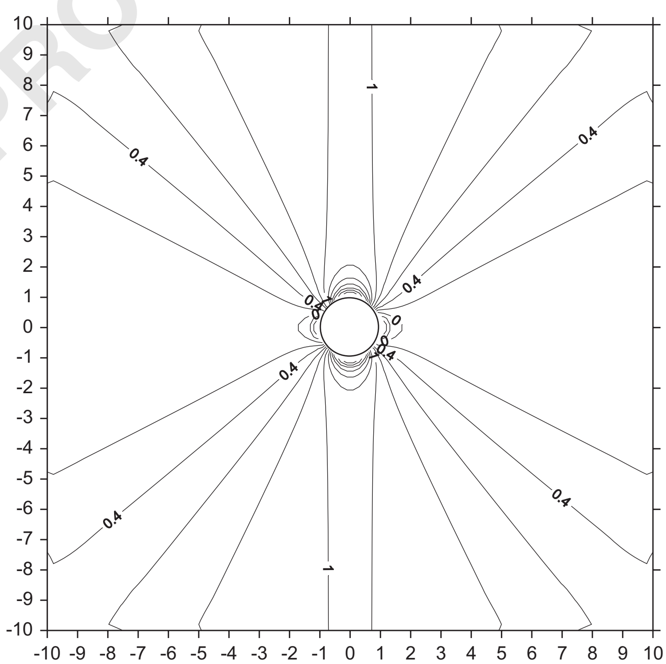

Fig. 4. The contour of hoop stress of the infinity plate.

Table 2
Comparison of the present method and Timoshenko and Goodier's approach for the SCF problem.

	Present method	Timoshenko and Goodier
Coordinate	Polar coordinate	Polar coordinate
Superposition	Stress (on boundary)	Stress (at infinity)
Geometry	Interior case+exterior case	Two annular domains
Base function	*T*-complete function	Assumption
Method	Direct	Semi-inverse

3. Concluding remarks

For the two classical elasticity problems, we have revisited the analytical solution by using the Trefftz method. Instead of using inverse or semi-inverse approach in the textbook, this paper has derived the solution in a direct, logical and natural way. The stress concentration factor problem and the Lamé problem were demonstrated to see the validity of the Trefftz formulation. Good agreements were made after comparing the exact solutions with those of Timoshenko and Goodier's textbook.

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