STATIC ANALYSIS OF THE FREE-FREE TRUSSES BY USING A SELF-REGULARIZATION APPROACH

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ABSTRACT

Following the success of static analysis of free-free 2-D plane trusses by using a self-regularization approach uniquely, we further extend the technique to deal with 3-D problems of space trusses. The inherent singular stiffness of a free-free structure is expanded to a bordered matrix by adding r singular vectors corresponding to zero singular values, where r is the nullity of the singular stiffness matrix. Besides, r constraints are accompanied to result in a nonsingular matrix. Only the pure particular solution with nontrivial strain is then obtained but without the homogeneous solution of no deformation. To link with the Fredholm alternative theorem, the slack variables with zero values indicate the infinite solutions while those with nonzero values imply the case of no solutions. A simple space truss is used to demonstrate the validity of the proposed model. An alternative way of reasonable support system to result in a nonsingular stiffness matrix is also addressed. In addition, the finite-element commercial code ABAQUS is also implemented to check the results.

Keywords: Self-regularization approach, 3D free-free structure, Bordered matrix, Stiffness matrix, Space truss.

1. INTRODUCTION

It is well known that two kinds of rank-deficiency problems in the boundary element method (BEM) or finite element method (FEM) are present. For an airplane or a missile of free-free structures, inertia relief was employed to solve a static problem by considering the inertia force from the D'Alembert principle. To include this capability, a support card was provided in the NASTRAN implementation. Rigid body modes are found in a free-free structure for structural mechanics problems no matter which numerical method is employed. This indicates that the free-free stiffness matrix results in zero eigenvalues (singular values). Mathe-

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matically speaking, the nonunique solution appears for the Neumann problem of the Laplace equation or for the traction problem of the Navier equation, respectively [1-7].

In the BEM implementation, 2-D Dirichlet problems in potential theory or constrained structures for 2-D elasticity problems may result in non-uniqueness of solutions. It is physically unrealizable but mathematically realizable due to the single-layer integral representation for the solution. It is well known as a critical scale (degenerate scale). To avoid this ill-posed model, Chen *et al.* [8] proposed an idea to make it a well-posed model by employing the singular value decomposition (SVD) and the bordered matrix from Fichera's method [9]. One is to introduce a slack variable of an arbitrary constant field. The other is to enforce a constraint. Therefore, a singular system is transformed to a nonsingular bordered system. It is interesting to find that this idea can be applied to promote the rank of singular matrices inherent in free-free structures. An illustrative example corresponding to a free-free plane truss has been successfully demonstrated [8], although redundant (zero stress) members were found.

Following this successful experiences, we extend it to static analysis of a free-free space truss. Physical rigid-body modes for the displacement corresponding to zero singular values are found. The self-regularization technique is linked to the Fredholm alternative theorem. The solvability condition is examined by the zero values of the slack variables. On the other hand, generalized inverse of a matrix has been studied by Fredholm, Moore and Penrose [8]. It can be mathematically studied by using the bordered matrix [10]. However, its engineering applications in structural mechanics were not noticed in that book. The mathematical problem corresponding to the free-free structure analysis is how to inverse a singular matrix due to rigid body modes. Several mathematical and numerical methods [11-16] were proposed to deal with this problem. In the research done by Chen *et al.* [8], zero stress bars or so called redundant members were found. To make the example more attractive, a more general loading is considered. To extend to 3-D structures, four cases are considered with zero stress bars or without zero stress bars.

The rigid body mode exists for the Neumann problem. For the Dirichlet problem, the degenerate scale in the BEM also results in a singular matrix. To deal with these problems, the unified self-regularization method was applied in the analysis of complicated structures once the stiffness matrix is available. In this paper, a self-regularization technique is employed to solve the static responses of free-free space trusses. The particular solution with nontrivial stress in the two-force members will be obtained. Besides, reasonable support system is also one alternative to solve this problem in a similar way of the support card in the NASTRAN. Four examples are demonstrated for the proposed model. The finite-element results by the commercial code ABAQUS are also acquired for comparison and validation. We extend the applications of 2D to 3D cases from the viewpoint of structural analysis. In addition, we also use both the Fredholm alternative theorem and the SVD technique to address the non-uniqueness of the solution which was not mentioned in the paper by Chen *et al.* [8]. The validity and generality of the present approach are reconfirmed in this paper.

2. MOTIVATION FROM THE RANGE DEFICIENCY OF THE SINGLE-LAYER INTEGRAL REPRESENTATION FOR THE SOLUTION

In potential theory of BEM/BIEM, the single-layer representation model is employed to solve the boundary value problem (BVP) as given below:

$$u(\mathbf{x}) = \int_B U(\mathbf{x},\mathbf{s})\phi(\mathbf{s})dB(\mathbf{s}), \quad \mathbf{x} \in D, \tag{1}$$

where $u(\mathbf{x})$ is the potential field, $\phi(\mathbf{s})$ is the unknown boundary density, $U(\mathbf{x}, \mathbf{s})$ is the fundamental solution and B is the boundary of the domain D.

However, Eq. (1) may fail for the Dirichlet problem with a specific scale (degenerate scale). To overcome this problem due to the ill-posed model (rank-deficiency), Fichera proposed a regularized formulation by adding a constant, c, and a corresponding constraint as shown below:

$$u(\mathbf{x}) = \int_B U(\mathbf{x},\mathbf{s})\phi_R(\mathbf{s})dB(\mathbf{s}) + c, \quad \mathbf{x} \in D, \tag{2}$$

$$\int_B \phi_R(\mathbf{s})dB(\mathbf{s}) = 0, \quad \mathbf{s} \in B, \tag{3}$$

where $\phi_R(\mathbf{s})$ is the regularized boundary density. After discretizing the boundary by using constant elements, Eq. (1) reduces to

$$\mathbf{U}\underset{\sim}{\phi} = \underset{\sim}{b}, \tag{4}$$

where $\underset{\sim}{b}$ is the specified Dirichlet boundary condition (BC). By employing the boundary element implementation, Eq. (2) and Eq. (3) together yield

$$\begin{bmatrix} \mathbf{U}_{N\times N} & \{1\}_{N\times 1} \\ \{l\}^{\mathrm{T}}_{N\times 1} & 0 \end{bmatrix} \begin{Bmatrix} \underset{\sim}{\phi}_R \\ c \end{Bmatrix} = \begin{Bmatrix} \underset{\sim}{b} \\ 0 \end{Bmatrix}, \tag{5}$$

where $\mathbf{U}$ is the influence matrix and $\{l\}$ is the vector of length for boundary elements. It is noted that $\underset{\sim}{\phi}$ in Eq. (4) is the unregularized unknown vector in the singular system, while $\underset{\sim}{\phi}_R$ in Eq. (5) is the regularized unknown vector in the nonsingular system.

By analogy between the singular stiffness matrix for structural mechanics and the influence matrix for the indirect BEM as shown in Fig. 1, a regularized (bordered) matrix provides an alternative way to construct the free-free stiffness matrix.

The linear algebraic system is generally written as

$$\mathbf{A}\underset{\sim}{x} = \underset{\sim}{b}, \tag{6}$$

where $\mathbf{A}$ is obtained by using either the BEM or FEM.

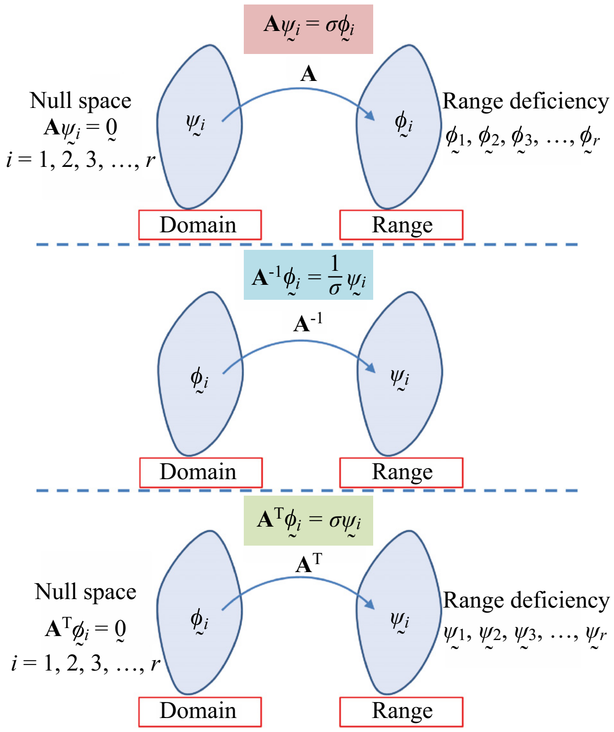


Fig. 1 Null space and range deficiency for the mapping matrix **A** or $\mathbf{A}^{\mathrm{T}}$ by using the SVD structure.

The matrix **A** may be a singular matrix which needs special care for the inversion. By employing the SVD technique, the matrix **A** is expressed as

$$\mathbf{A} = \Phi\Sigma\Psi^{\mathrm{T}} = \Phi\begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_N \end{bmatrix}\Psi^{\mathrm{T}}, \tag{7}$$

where the left singular matrix $\Phi = \{\underset{\sim}{\phi}_1, \underset{\sim}{\phi}_2, \ldots, \underset{\sim}{\phi}_N\}$ the right singular matrix $\Psi = \{\underset{\sim}{\psi}_1, \underset{\sim}{\psi}_2, \ldots, \underset{\sim}{\psi}_N\}$ and the singular values $\sigma_1 \le \sigma_2 \le \sigma_3 \le \ldots \le \sigma_N$. The unknown vector $\underset{\sim}{x}$ can be represented by using the right singular vector $\underset{\sim}{\psi}_i$ as

$$\underset{\sim}{x} = \sum_{i=1}^{N} \alpha_i \underset{\sim}{\psi}_i. \tag{8}$$

Similarly, the forcing vector $\underset{\sim}{b}_i$ is expanded by the superposition of the left singular vector $\underset{\sim}{\phi}_i$ as follows:

$$\underset{\sim}{b} = \sum_{i=1}^{N} \beta_i \underset{\sim}{\phi}_i. \tag{9}$$

If the singular value, σ_1, is zero, α_1 can not be determined. By suppressing α_1 to be zero in Eq. (8), it can be expressed as

$$\underset{\sim}{\psi}_1 \cdot \underset{\sim}{x}_R = 0, \tag{10}$$

where the regularized solution $\underset{\sim}{x}_R$ can be regarded as the pure particular solution without containing any complementary solution (rigid body mode) $\underset{\sim}{x}_c$ such that $\mathbf{A}\underset{\sim}{x}_c = \underset{\sim}{0}$. Since the range of **A** is deficient by $\underset{\sim}{\phi}_1$ as shown in Fig. 1, Eq. (6) can be regularized into

$$\mathbf{A}\underset{\sim}{x}_R + c_1\underset{\sim}{\phi}_1 = \underset{\sim}{b}, \tag{11}$$

where c_1 is an arbitrary constant to be determined. By combining Eq. (10) and Eq. (11), a regularized linear algebraic system is expressed as

$$\begin{bmatrix} \mathbf{A} & \underset{\sim}{\phi}_1 \\ \underset{\sim}{\psi}_1^{\mathrm{T}} & 0 \end{bmatrix}\begin{Bmatrix} \underset{\sim}{x}_R \\ c_1 \end{Bmatrix} = \begin{Bmatrix} \underset{\sim}{b} \\ 0 \end{Bmatrix}. \tag{12}$$

Then, the bordered matrix $\mathbf{A}_B$ is defined by

$$\mathbf{A}_B = \begin{bmatrix} \mathbf{A} & \underset{\sim}{\phi}_1 \\ \underset{\sim}{\psi}_1^{\mathrm{T}} & 0 \end{bmatrix}. \tag{13}$$

Since $\mathbf{A}_B$ in Eq. (13) is nonsingular, $\underset{\sim}{x}_R$ and c_1 in Eq. (12) can be easily solved [8].

3. LINKAGE OF THE SLACK VARIABLES IN THE BORDERED MATRIX TO THE FREDHOLM ALTERNATIVE THEOREM

In the literature, Fredholm alternative theorem plays an important role to ensure a unique solution for the ODE, the integral equation and the linear algebra. Now, the linkage of the slack variables corresponding to the bordered matrix to the Fredholm alternative theorem is constructed. If the determinant value of a matrix **A** is not equal to zero, the linear algebraic system has a unique solution. Otherwise, the singular matrix **A** has two possibilities for the nonuniqueness of solutions. One can use the zero or non-zero slack variables corresponding to the bordered matrix to check the infinite solutions or no solutions in the Fredholm alternative theorem, respectively.

The linear algebraic system in Eq. (6) has a unique solution if and only if the only solution to the homogeneous equation,

$$\mathbf{A}\underset{\sim}{x} = \underset{\sim}{0}, \tag{14}$$

is $\underset{\sim}{x} = \underset{\sim}{0}$. Alternatively, the homogeneous equation has at least one solution if the homogeneous adjoint equation

$$\mathbf{A}^{\mathrm{H}}\underset{\sim}{\phi} = \underset{\sim}{0}, \tag{15}$$

has a non-trivial solution $\underset{\sim}{\phi}$, where $\mathbf{A}^{\mathrm{H}}$ is the conjugate transpose matrix of **A** and the vector $\underset{\sim}{b}$ must satisfy the constraint ($\underset{\sim}{b}^{\mathrm{H}}\underset{\sim}{\phi} = 0$). If the matrix **A** is real-valued, the conjugate transpose of a matrix is simplified to its transpose only [17], i.e., $\mathbf{A}^{\mathrm{H}} = \mathbf{A}^{\mathrm{T}}$. In other words, Eq. (6) has at least one solution for $\underset{\sim}{x}$ if the homogeneous adjoint equation

$$\mathbf{A}^{\mathrm{T}} \underset{\sim}{\phi} = \underset{\sim}{0}, \tag{16}$$

has a non-trivial solution $\underset{\sim}{\phi}$, in which the constraint, $\underset{\sim}{b} \cdot \underset{\sim}{\phi} = 0$, must be satisfied.

By employing the singular value decomposition (SVD), the matrix $\mathbf{A}$ is represented in Eq. (7). If the matrix $\mathbf{A}$ is symmetric, the right and left singular matrices are the same due to the symmetric property. Therefore, $\mathbf{A} = \Phi\Sigma\Phi^{\mathrm{T}} = \Psi\Sigma\Psi^{\mathrm{T}}$. If the singular values, σ_1, σ_2, σ_3…, σ_r are zeroes, where r is the number of the rank deficiency. In other words, the matrix $\mathbf{A}$ has full rank N due to the non-zero determinant. If the matrix $\mathbf{A}$ is rank deficient by r, it means that the matrix $\mathbf{A}$ has rank $N - r$. It exists the nullity k ($k = N - (N - r) = r$). According to the SVD, we have

$$\mathbf{A} \underset{\sim}{\psi}_i = \sigma_i \underset{\sim}{\phi}_i, \quad i = 1,2,3,\ldots,N \tag{17}$$

$$\mathbf{A}^{\mathrm{T}} \underset{\sim}{\phi}_i = \sigma_i \underset{\sim}{\psi}_i, \quad i = 1,2,3,\ldots,N \tag{18}$$

$$\mathbf{A}^{-1} \underset{\sim}{\phi}_i = \frac{1}{\sigma_i} \underset{\sim}{\psi}_i, \quad i = 1,2,3,\ldots,N \tag{19}$$

If the matrix $\mathbf{A}$ is rank deficient by r, then Eq. (17) and Eq. (18) become

$$\mathbf{A} \underset{\sim}{\psi}_i = \underset{\sim}{0}, \quad i = 1,2,3,\ldots,r \tag{20}$$

$$\mathbf{A}^{\mathrm{T}} \underset{\sim}{\phi}_i = \underset{\sim}{0}, \quad i = 1,2,3,\ldots,r \tag{21}$$

It is interesting that the null space exists in the domain and the range deficiency of the singular system is shown in Fig. 1.

By extending the concept from [8], we have

$$\boldsymbol{\psi}_i^{\mathrm{T}} \underset{\sim}{x}_R = \underset{\sim}{x}_R \cdot \underset{\sim}{\psi}_i = 0, \quad i = 1,2,3,\ldots,r, \tag{22}$$

where the solution from the regularized system $\underset{\sim}{x}_R$ (pure particular solution) does not contain any component of rigid body modes (the complementary solution). Since the range of $\mathbf{A}$ is deficient by $\underset{\sim}{\phi}_1 \sim \underset{\sim}{\phi}_r$, Eq. (6) is regularized into

$$\mathbf{A} \underset{\sim}{x}_R + \sum_{i=1}^{r} c_i \underset{\sim}{\phi}_i = \underset{\sim}{b}, \quad i = 1,2,3,\ldots,r, \tag{23}$$

where c_i are slack variables.

By combining Eq. (22) and Eq. (23) for a matrix with r zero singular values, a regularized linear algebraic system is expressed as

$$\begin{bmatrix} \mathrm{A} & \underset{\sim}{\phi}_1 & \underset{\sim}{\phi}_2 & \underset{\sim}{\phi}_3 & \cdots & \underset{\sim}{\phi}_r \\ \underset{\sim}{\psi}_1^{\mathrm{T}} & 0 & 0 & 0 & \cdots & 0 \\ \underset{\sim}{\psi}_2^{\mathrm{T}} & 0 & 0 & 0 & \cdots & 0 \\ \underset{\sim}{\psi}_3^{\mathrm{T}} & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \underset{\sim}{\psi}_r^{\mathrm{T}} & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{Bmatrix} \underset{\sim}{x}_R \\ c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_r \end{Bmatrix} = \begin{Bmatrix} \underset{\sim}{b} \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{Bmatrix}. \tag{24}$$

By taking the inner product for Eq. (23) with respect to $\underset{\sim}{\phi}_1$ on both sides, it can be represented as

$$\begin{aligned} &\mathbf{A} \underset{\sim}{x}_R \cdot \underset{\sim}{\phi}_1 + c_1 \underset{\sim}{\phi}_1 \cdot \underset{\sim}{\phi}_1 + c_2 \underset{\sim}{\phi}_2 \cdot \underset{\sim}{\phi}_1 + c_3 \underset{\sim}{\phi}_3 \cdot \underset{\sim}{\phi}_1 \\ &+ \ldots + c_r \underset{\sim}{\phi}_r \cdot \underset{\sim}{\phi}_1 = \underset{\sim}{b} \cdot \underset{\sim}{\phi}_1. \end{aligned} \tag{25}$$

Since $\underset{\sim}{\phi}_i \cdot \underset{\sim}{\phi}_1 = 0$, $i = 2,3,4,\ldots,r$, due to the orthogonal property of each $\underset{\sim}{\phi}_i$, Eq. (25) reduces to

$$\mathbf{A} \underset{\sim}{x}_R \cdot \underset{\sim}{\phi}_1 + c_1 = \underset{\sim}{b} \cdot \underset{\sim}{\phi}_1. \tag{26}$$

Since $\mathbf{A} \underset{\sim}{x}_R \cdot \underset{\sim}{\phi}_1 = (\mathbf{A} \underset{\sim}{x}_R)^{\mathrm{T}} \underset{\sim}{\phi}_1 = \underset{\sim}{x}_R \mathbf{A}^{\mathrm{T}} \underset{\sim}{\phi}_1 = 0$ [8], the slack variable is obtained as shown below:

$$c_1 = \underset{\sim}{b} \cdot \underset{\sim}{\phi}_1. \tag{27}$$

Similarly, the slack variables is expressed as

$$c_i = \underset{\sim}{b} \cdot \underset{\sim}{\phi}_i, \quad i = 2,3,\ldots,r. \tag{28}$$

If the matrix $\mathbf{A}$ is rank-deficient by r order ($\sigma_i = 0$, $i = 1, 2, 3, \ldots, r$), the range of the operator $\mathbf{A}$ is range deficient by $\underset{\sim}{\phi}_i$, $i = 2,3,4,\ldots,r$. The slack variables,

$$c_i = \underset{\sim}{b} \cdot \underset{\sim}{\phi}_i = 0, \quad i = 1,2,3,\ldots,r, \tag{29}$$

imply that Eq. (6) has infinite solutions. Otherwise, there are no solutions for Eq. (6). Accordingly, the process for the judgement of the non-uniqueness of solutions from the Fredholm alternative theorem is summarized in Fig. 2.

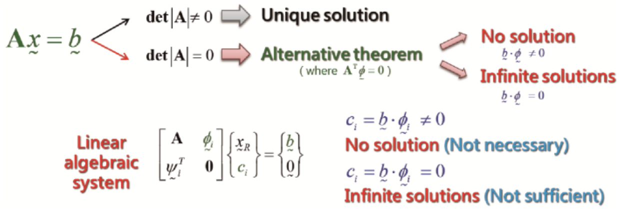
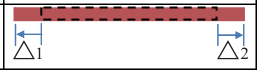
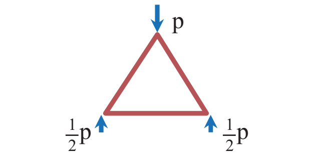
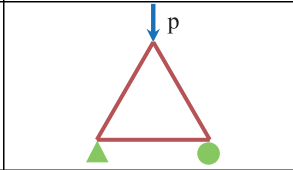
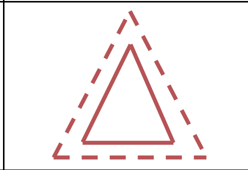
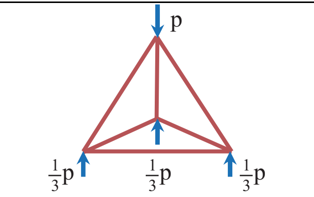
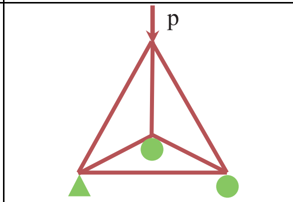
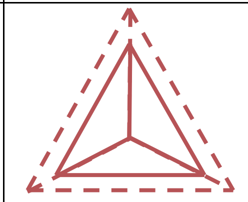


Fig. 2 Linking of the non-unique solution from the Fredholm alternative theorem and the bordered matrix.

4. NUMERICAL EXAMPLES

In the ODE, the PDE and the IE, three kinds of BCs, Dirichlet, Neumann and mixed types are used for different problems. For a structural system, similar corresponding BCs for the fixed end, the traction loading (nodal force of truss) and the mixed type BC are summarized in Table 1. For the structural engineering, it is easy and straightforward to understand the validity of the self-regularization method in 2D and 3D problems by using trusses. Four examples of free-free structures are demonstrated by using the self-regularized technique, the reasonable support system and the finite-element commercial code ABAQUS with automatic damping algorithm to deal with the non-unique solutions of the Neu-

Table 1 Analysis of 1-D, 2-D and 3-D structures subject to different BCs

Dimension \ BC Types	Neumann BC (free-free)	Mixed type BC (reasonable support)	Dirichlet BC (enforced displacement)
1-D			
2-D			
3-D			
Global stiffness (before condensation)	$[K_g]$	$[K_g]$	$[K_g]$
Stiffness matrix (after condensation or self-regularization)	$[K_b]$	$[K_m]$	$[K_D]$
Singularity	$\det \lvert[K_g]\rvert = 0$ $\det \lvert[K_b]\rvert \neq 0$	$\det \lvert[K_g]\rvert = 0$ $\det \lvert[K_m]\rvert \neq 0$	$\det \lvert[K_g]\rvert = 0$ $\det \lvert[K_D]\rvert \neq 0$

mann case. Four examples by using the reasonable support system are compared with these results. In addition, the rigid body mode would be discussed.

4.1 Plane Truss (Zero Stress Bar to Non- Zero Stress Bar)

A two-dimensional, 3-node and 6-dof, triangular truss with reasonable support and the one without any support are shown in Fig. 3 and Fig. 4, respectively. By considering the direct stiffness method for the truss structure in Fig. 5, the free-free stiffness matrix $\mathbf{K}_{g\text{-}2D}$ of the truss is shown below:

$$\mathbf{K}_{g-2D} = k\begin{bmatrix} \frac{5}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{4} & \frac{\sqrt{3}}{4} & -1 & 0 \\ -\frac{\sqrt{3}}{4} & \frac{3}{4} & \frac{\sqrt{3}}{4} & -\frac{3}{4} & 0 & 0 \\ -\frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{1}{2} & 0 & -\frac{1}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & -\frac{3}{4} & 0 & \frac{3}{2} & -\frac{\sqrt{3}}{4} & -\frac{3}{4} \\ -1 & 0 & -\frac{1}{4} & -\frac{\sqrt{3}}{4} & \frac{5}{4} & \frac{\sqrt{3}}{4} \\ 0 & 0 & -\frac{\sqrt{3}}{4} & -\frac{3}{4} & \frac{\sqrt{3}}{4} & \frac{3}{4} \end{bmatrix}_{6\times 6}, \tag{30}$$

where $k = EA/L$, E denotes Young's modulus (N/m^2), A denotes the cross-sectional area (m^2) and L denotes the

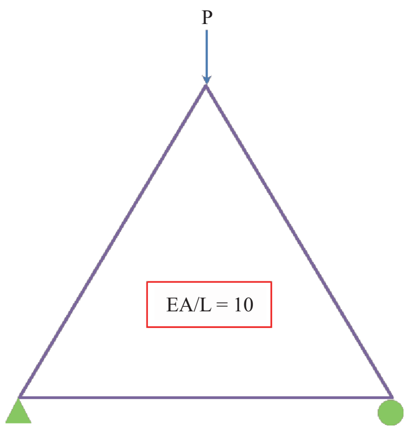


Fig. 3 The plane truss of the regular triangle with reasonable supports.

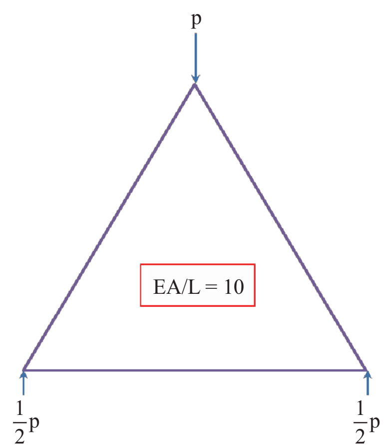


Fig. 4 The plane truss of the regular triangle with the external forces (free-free structure).

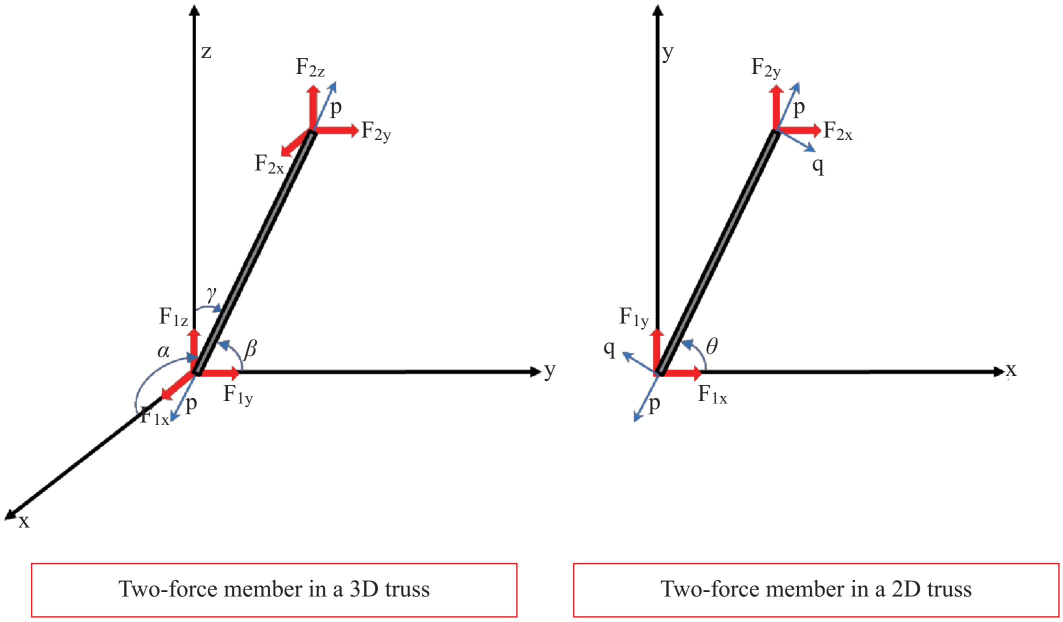


Fig. 5 Two-force members of the 2D truss and the 3D truss.

length (m) for each uniform member of trusses. By employing the SVD with respect to $\mathbf{K}_{g\text{-}2D}$ and setting $k = 10$ for simplicity, the matrices of the singular value and the singular vector are obtained as

$$\Sigma = k\begin{bmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & \frac{3}{2} & & \\ & & & & \frac{3}{2} & \\ & & & & & 3 \end{bmatrix}_{6\times 6}, \quad (31)$$

$$\Phi = \Psi = \begin{bmatrix} \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & -\frac{1}{2\sqrt{3}} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & -\frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & -\frac{1}{2\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & -\frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} \end{bmatrix}_{6\times 6}, \quad (32)$$

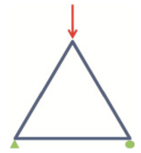
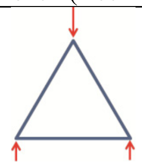
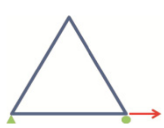
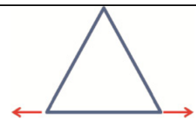
where Σ is the singular value matrix, the left singular matrix $\Phi = \{\underset{\sim}{\phi_1}, \underset{\sim}{\phi_2}, \ldots, \underset{\sim}{\phi_6}\}$, the right singular matrix $\Psi = \{\underset{\sim}{\psi_1}, \underset{\sim}{\psi_2}, \ldots, \underset{\sim}{\psi_6}\}$. Since there are three zero singular values, the matrix $\mathbf{K}_{g\text{-}2D}$ is rank deficient by 3. According to Eq. (24), the linear algebraic system $\mathbf{K}_{g\text{-}2D}\underset{\sim}{u} = \underset{\sim}{p}$ can be bordered to

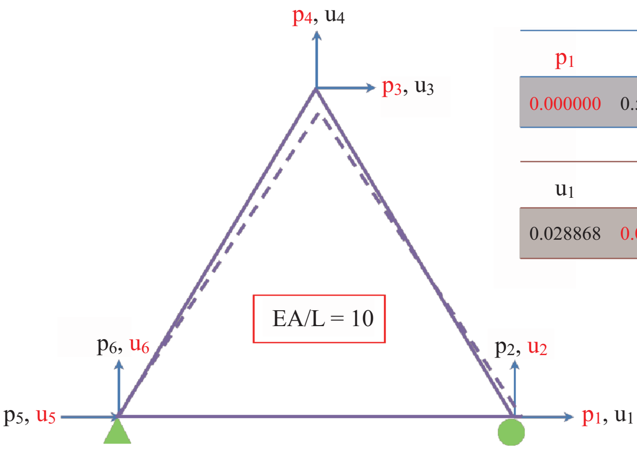
$$\mathbf{K}_{B-2D}\begin{Bmatrix} \underset{\sim}{u}_R \\ c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \begin{bmatrix} \mathbf{K}_{g-2D} & \underset{\sim}{\phi_1} & \underset{\sim}{\phi_2} & \underset{\sim}{\phi_3} \\ \underset{\sim}{\psi_1^T} & 0 & 0 & 0 \\ \underset{\sim}{\psi_2^T} & 0 & 0 & 0 \\ \underset{\sim}{\psi_3^T} & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \underset{\sim}{u}_R \\ c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \begin{Bmatrix} \underset{\sim}{p} \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \quad (33)$$

where $\underset{\sim}{\phi_1}$, $\underset{\sim}{\phi_2}$, $\underset{\sim}{\phi_3}$ and $\underset{\sim}{\psi_1}$, $\underset{\sim}{\psi_2}$, $\underset{\sim}{\psi_3}$ are the left singular vectors and the right singular vectors corresponding to three zero singular values of the free-free stiffness matrix $\mathbf{K}_{g\text{-}2D}$, respectively.

By considering the reasonable support or employing the self-regularization approach for the free-free structure corresponding to the zero stress bar case or the non-zero stress bar case, the 2-D cases are listed in Table 2.

Table 2 2-D Cases for two loading cases (zero stress bar and non-zero stress bar) by using the reasonable support system or the self-regularization approach (free-free)

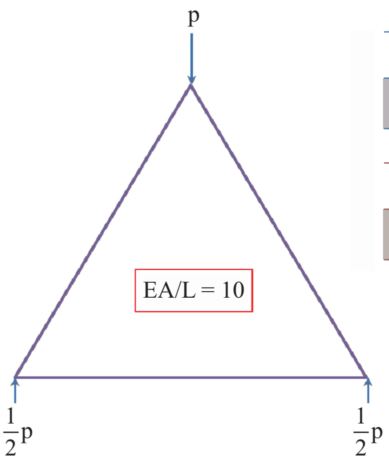
	Reasonable support	Self-regularization approach (free-free)
Non-zero stress bar	 Case 1-1	 Case 1-2
Zero stress bar	 Case 1-3	 Case 1-4 (Chen *et al.* 2014a)



p_1	p_2	p_3	p_4	p_5	p_6
0.000000	0.500000	0.000000	-1.000000	0.000000	0.500000

u_1	u_2	u_3	u_4	u_5	u_6
0.028868	0.000000	0.014434	-0.075000	0.000000	0.000000

Fig. 6 Case 1-1: The plane truss of the regular triangle with the mixed type BC by using the reasonable support (the solid line denotes the original plane truss and the dashed line denotes the deformed truss).



p_1	p_2	p_3	p_4	p_5	p_6
0.000000	0.500000	0.000000	-1.000000	0.000000	0.500000

u_1	u_2	u_3	u_4	u_5	u_6
0.014434	0.025000	0.000000	-0.050000	-0.014434	0.025000

Red: given nodal force
Black: unknown displacement

Fig. 7 Case 1-2: The plane truss of the regular triangle with the Neumann BC (free-free structure) by using the self-regularization approach (the solid line denotes the original plane truss and the dashed line denotes the deformed truss).

4.1.1 Reasonable Support System

Instead of solving the Neumann problem, the reasonable support system transforms the Neumann problem (singular) to the mixed BC problem (non-singular) which can be used to solve the linear algebraic system with a non-singular stiffness matrix. By adding the hinge and roller supports at nodes of left bottom and right bottom, respectively, the reasonable support system for the plane truss is constructed. The rigid body modes are constrained with reasonable supports by adding physical constraints, i.e. hinges or rollers. After the matrix condensation, the stiffness matrix of the reduced linear algebraic system becomes non-singular and yields to a unique solution.

4.1.2 Free-Free System Using the Present Approach (No Support)

For the complicated structure, it is not easy to add reasonable supports (suitable physical constraints) for the Neumann problems by engineering judgement. A self-regularization approach is systematically useful to solve the linear algebraic system for the singular stiffness matrix [8]. The advantage of the present approach is that only the self-information (singular vectors) from the free-free stiffness matrix is needed to construct the bordered matrix in this systematic method. The bordered stiffness matrix is non-singular and full-rank which is different from the original one, the unique solution can be ensured as a reference solution for the original system.

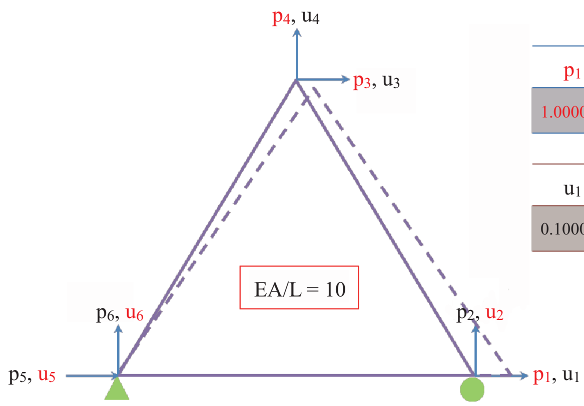
4.1.3 Results and Discussions

The deformations of the plane truss by using the reasonable support system (Case 1-1) and the self-regularization approach (Case 1-2) are plotted in Fig. 6 and Fig. 7, respectively. In contrast to the different displacements solved from the reasonable support system and the self-regularization approach, the stress states obtained by using two methods are the same. The difference between two solutions is the rigid body mode. Here, only translation rigid body modes are found in this case. The pure particular solution is obtained by using the self-regularization approach without containing any rigid body mode, translation and rotation.

It is interesting to find that the 2-D case in [8] has redundant (zero stress) members. The corresponding reasonable support system is established as shown in Fig. 8 (Case 1-3). If the given forces are specified symmetrically in Fig. 9 (Case 1-4) [8], redundant (zero stress) members are found. In this case, only a two-force member is subjected to stress. Similarly, reasonable support system in the Case 1-3 can yield the same stress state.

4.2 Space Truss (Zero Stress Bar to Non-Zero Stress Bar)

A 3-D truss of the regular tetrahedron with 4-node and 12-dof is shown in Fig. 10. The free-free stiffness matrix $\mathbf{K}_{g\text{-}3D}$ assembled from members as shown in Fig. 5, for the space truss is shown below

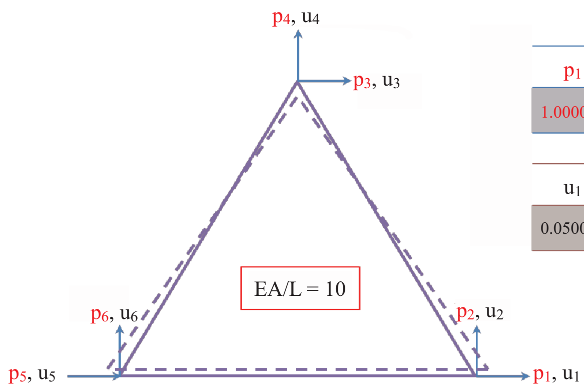


p_1	p_2	p_3	p_4	p_5	p_6
1.000000	0.000000	0.000000	0.000000	-1.000000	0.500000

u_1	u_2	u_3	u_4	u_5	u_6
0.100000	0.000000	0.050000	-0.028868	0.000000	0.000000

Red: given BC or nodal force
Black: unknown nodal force or displacement

Fig. 8 Case 1-3: The plane truss of the regular triangle with the mixed type BC by using the reasonable support - zero stress bar (the solid line denotes the original plane truss and the dashed line denotes the deformed truss).



p_1	p_2	p_3	p_4	p_5	p_6
1.000000	0.000000	0.000000	0.000000	-1.000000	0.000000

u_1	u_2	u_3	u_4	u_5	u_6
0.050000	0.009623	0.000000	-0.019245	-0.050000	0.009623

Red: given nodal force
Black: unknown displacement

Fig. 9 Case 1-4: The plane truss of the regular triangle with the Neumann BC (free-free structure) by using the self-regularization approach - zero stress bar (the solid line denotes the original plane truss and the dashed line denotes the deformed truss).

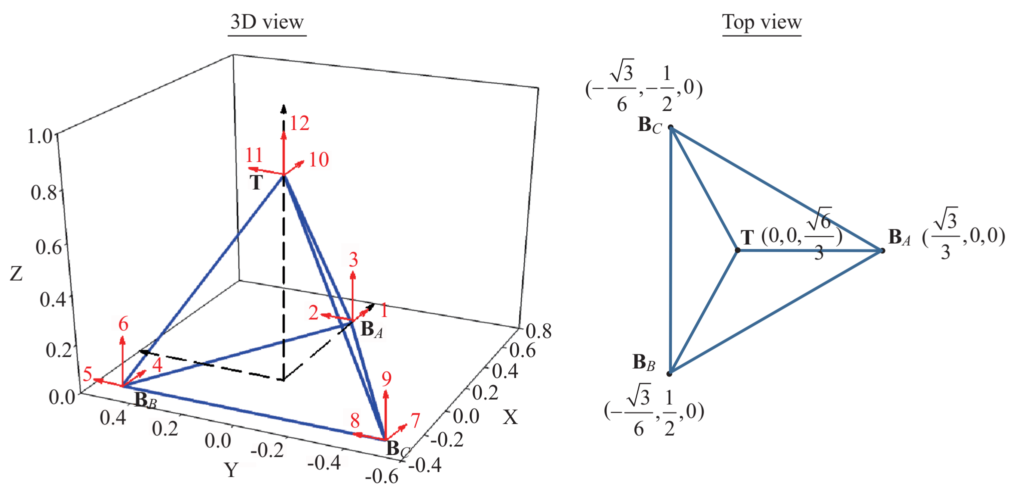


Fig. 10 A 3D space truss of the regular tetrahedron (Red arrows denote the direction of the force (p) and displacement (u) on the nodes).

$$
\mathbf{K}_{g-3D} = k\begin{bmatrix}
\frac{11}{6} & 0 & -\frac{\sqrt{2}}{3} & -\frac{3}{4} & \frac{\sqrt{3}}{4} & 0 & -\frac{3}{4} & -\frac{\sqrt{3}}{4} & 0 & -\frac{1}{3} & 0 & \frac{\sqrt{2}}{3} \\
0 & \frac{1}{2} & 0 & \frac{\sqrt{3}}{4} & -\frac{1}{4} & 0 & -\frac{\sqrt{3}}{4} & -\frac{1}{4} & 0 & 0 & 0 & 0 \\
-\frac{\sqrt{2}}{3} & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{3} & 0 & -\frac{2}{3} \\
-\frac{3}{4} & \frac{\sqrt{3}}{4} & 0 & \frac{5}{6} & -\frac{1}{\sqrt{3}} & \frac{1}{3\sqrt{2}} & 0 & 0 & 0 & -\frac{1}{12} & \frac{1}{4\sqrt{3}} & -\frac{1}{3\sqrt{2}} \\
\frac{\sqrt{3}}{4} & -\frac{1}{4} & 0 & -\frac{1}{\sqrt{3}} & \frac{3}{2} & -\frac{1}{\sqrt{6}} & 0 & -1 & 0 & \frac{1}{4\sqrt{3}} & -\frac{1}{4} & \frac{1}{\sqrt{6}} \\
0 & 0 & 0 & \frac{1}{3\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{2}{3} & 0 & 0 & 0 & -\frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{2}{3} \\
-\frac{3}{4} & -\frac{\sqrt{3}}{4} & 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{\sqrt{3}} & \frac{1}{3\sqrt{2}} & -\frac{1}{12} & -\frac{1}{4\sqrt{3}} & -\frac{1}{3\sqrt{2}} \\
-\frac{\sqrt{3}}{4} & -\frac{1}{4} & 0 & 0 & -1 & 0 & \frac{1}{\sqrt{3}} & \frac{3}{2} & \frac{1}{\sqrt{6}} & -\frac{1}{4\sqrt{3}} & -\frac{1}{4} & -\frac{1}{\sqrt{6}} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{2}{3} & -\frac{1}{3\sqrt{2}} & -\frac{1}{\sqrt{6}} & -\frac{2}{3} \\
-\frac{1}{3} & 0 & \frac{\sqrt{2}}{3} & -\frac{1}{12} & \frac{1}{4\sqrt{3}} & -\frac{1}{3\sqrt{2}} & -\frac{1}{12} & -\frac{1}{4\sqrt{3}} & -\frac{1}{3\sqrt{2}} & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{4\sqrt{3}} & -\frac{1}{4} & \frac{1}{\sqrt{6}} & -\frac{1}{4\sqrt{3}} & -\frac{1}{4} & -\frac{1}{\sqrt{6}} & 0 & \frac{1}{2} & 0 \\
\frac{\sqrt{2}}{3} & 0 & -\frac{2}{3} & -\frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{2}{3} & -\frac{1}{3\sqrt{2}} & -\frac{1}{\sqrt{6}} & -\frac{2}{3} & 0 & 0 & 2
\end{bmatrix}_{12\times 12}, \tag{34}
$$

By employing the SVD with respect to $\mathbf{K}_{g\text{-}3D}$ and setting $k = 10$, the singular value and the singular vector are obtained as

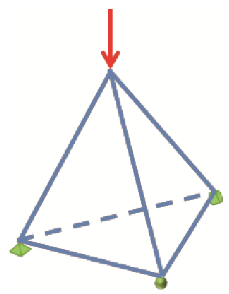
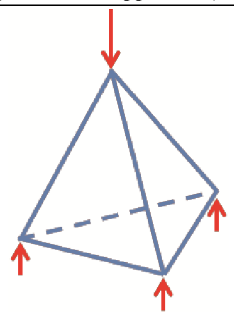
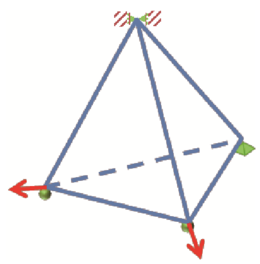
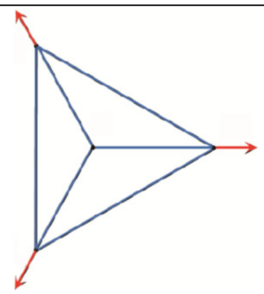
$$
\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \\ \sigma_7 \\ \sigma_8 \\ \sigma_9 \\ \sigma_{10} \\ \sigma_{11} \\ \sigma_{12} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 2 \\ 2 \\ 2 \\ 4 \end{Bmatrix}_{12\times 1} . \tag{35}
$$

and

$$\underset{\sim}{\phi}_1=\underset{\sim}{\psi}_1=\begin{Bmatrix}0\\0\\\frac{1}{2}\\0\\0\\\frac{1}{2}\\0\\0\\\frac{1}{2}\\0\\0\\\frac{1}{2}\end{Bmatrix}_{12\times1},\ \underset{\sim}{\phi}_2=\underset{\sim}{\psi}_2=\begin{Bmatrix}0\\\frac{1}{2}\\0\\0\\\frac{1}{2}\\0\\0\\\frac{1}{2}\\0\\0\\\frac{1}{2}\\0\end{Bmatrix}_{12\times1},\ \underset{\sim}{\phi}_3=\underset{\sim}{\psi}_3=\begin{Bmatrix}\frac{1}{2}\\0\\0\\\frac{1}{2}\\0\\0\\\frac{1}{2}\\0\\0\\\frac{1}{2}\\0\\0\end{Bmatrix}_{12\times1},\ \underset{\sim}{\phi}_4=\underset{\sim}{\psi}_4=\begin{Bmatrix}-\frac{1}{12}\\-\frac{\sqrt{3}}{4}\\-\frac{1}{3\sqrt{2}}\\\frac{1}{4}\\\frac{1}{4\sqrt{3}}\\0\\-\frac{5}{12}\\\frac{1}{4\sqrt{3}}\\\frac{1}{3\sqrt{2}}\\\frac{1}{4}\\\frac{1}{4\sqrt{3}}\\0\end{Bmatrix}_{12\times1},\ \underset{\sim}{\phi}_5=\underset{\sim}{\psi}_5=\begin{Bmatrix}0\\-\sqrt{\frac{2}{5}}\\0\\\sqrt{\frac{3}{10}}\\\frac{1}{\sqrt{10}}\\0\\-\sqrt{\frac{3}{10}}\\\frac{1}{\sqrt{10}}\\0\\0\\0\\0\end{Bmatrix}_{12\times1},\ \underset{\sim}{\phi}_6=\underset{\sim}{\psi}_6=\begin{Bmatrix}-\frac{1}{12}\\\frac{\sqrt{3}}{4}\\-\frac{1}{3\sqrt{2}}\\-\frac{5}{12}\\-\frac{1}{4\sqrt{3}}\\\frac{1}{3\sqrt{2}}\\\frac{1}{4}\\-\frac{1}{4\sqrt{3}}\\0\\\frac{1}{4}\\-\frac{1}{4\sqrt{3}}\\0\end{Bmatrix}_{12\times1}, \tag{36}$$

where $\underset{\sim}{\phi}_1$, $\underset{\sim}{\phi}_2$, $\underset{\sim}{\phi}_3$, $\underset{\sim}{\phi}_4$, $\underset{\sim}{\phi}_5$, $\underset{\sim}{\phi}_6$ and $\underset{\sim}{\psi}_1$, $\underset{\sim}{\psi}_2$, $\underset{\sim}{\psi}_3$, $\underset{\sim}{\psi}_4$, $\underset{\sim}{\psi}_5$, $\underset{\sim}{\psi}_6$ are the left singular vectors and the right singular vectors corresponding to six zero singular values of the free-free stiffness matrix $\mathbf{K}_{g\text{-}3D}$, respectively. Therefore, the matrix $\mathbf{K}_{g\text{-}3D}$ is rank deficient by 6. Similarly, the method by adding the reasonable support and the self-regularization approach to the free-free structure corresponding to the zero stress bar case or the non-zero stress bar case of the 3-D cases are listed in Table 3. A space truss of the regular tetrahedron subject to the mixed type BC (Case 2-1) is shown in Fig. 11, and the deformation of this case is shown in Fig. 12.

Table 3 3-D Cases two loading cases (zero stress bar and non-zero stress bar) by using the reasonable support system or the self-regularization approach (free-free)

	Reasonable support	Self-regularization approach (free-free)
Non-zero stress bar	 Case 2-1	 Case 2-2
Zero stress bar	 Case 2-3	 Case 2-4

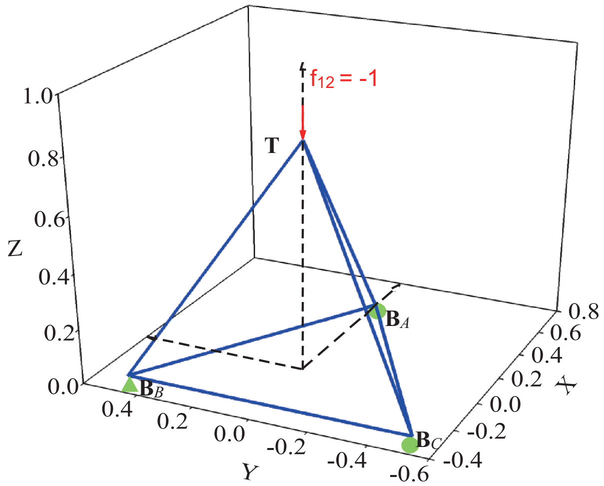


Fig. 11 Case 2-1: A space truss subject to the mixed type BC (reasonable support).

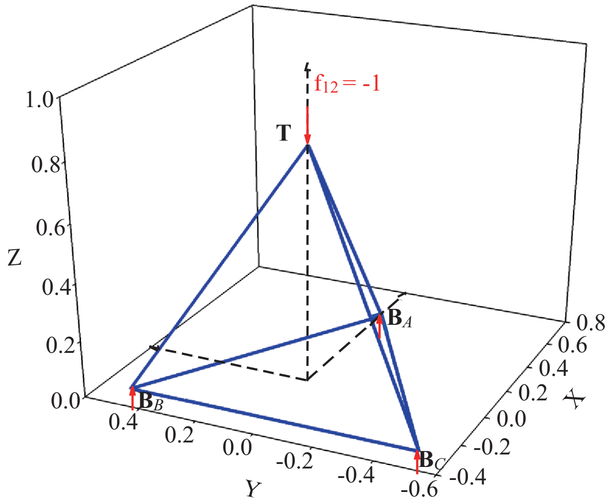


Fig. 13 Case 2-2: The space truss subject to the Neumann BC (free-free structure).

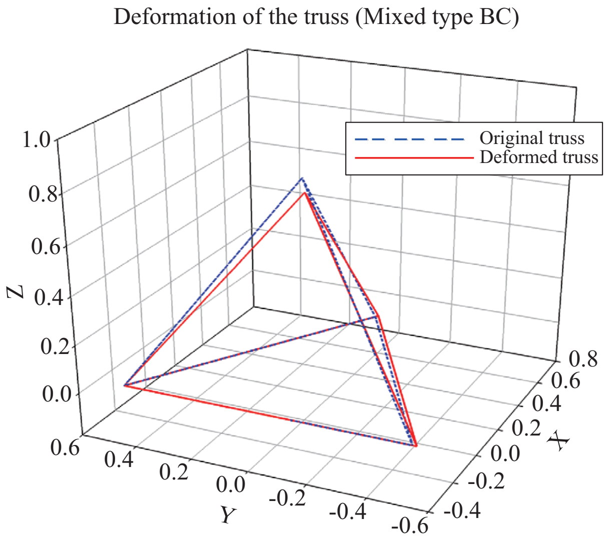


Fig. 12 Deformation of the space truss subject to the mixed type BC (Case 2-1).

Instead of considering the reasonable support system, the self-regularization approach is proposed to deal with the Neumann problem (Case 2-2) in Fig. 13. According to Eq. (24), the linear algebraic system $\mathbf{K}_{g\text{-}3D}\underset{\sim}{u} = \underset{\sim}{p}$ can be bordered to

$$\mathbf{K}_{B-3D}\begin{Bmatrix} u_r \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{Bmatrix} = \begin{bmatrix} \mathbf{K}_{g-3D} & \underset{\sim}{\phi_1} & \underset{\sim}{\phi_2} & \underset{\sim}{\phi_3} & \underset{\sim}{\phi_4} & \underset{\sim}{\phi_5} & \underset{\sim}{\phi_6} \\ \underset{\sim}{\psi}_1^T & 0 & 0 & 0 & 0 & 0 & 0 \\ \underset{\sim}{\psi}_2^T & 0 & 0 & 0 & 0 & 0 & 0 \\ \underset{\sim}{\psi}_3^T & 0 & 0 & 0 & 0 & 0 & 0 \\ \underset{\sim}{\psi}_4^T & 0 & 0 & 0 & 0 & 0 & 0 \\ \underset{\sim}{\psi}_5^T & 0 & 0 & 0 & 0 & 0 & 0 \\ \underset{\sim}{\psi}_6^T & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_r \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{Bmatrix} = \begin{Bmatrix} \underset{\sim}{p} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}. \quad (37)$$

Table 4 Internal force of members for the space truss of Case 2-1 and Case 2-2

Member ID	Mixed type problem (after enforcing reasonable support)	Neumann problem	ABAQUS (automatic damping algorithm)
$\overline{B_A B_B}$	0.136083 (Tension)	0.136083 (Tension)	0.135774 (Tension)
$\overline{B_B B_C}$	0.136083 (Tension)	0.136083 (Tension)	0.136367 (Tension)
$\overline{B_C B_A}$	0.136083 (Tension)	0.136083 (Tension)	0.135774 (Tension)
$\overline{B_A T}$	0.408248 (Compression)	0.408248 (Compression)	0.407473 (Compression)
$\overline{B_B T}$	0.408248 (Compression)	0.408248 (Compression)	0.408384 (Compression)
$\overline{B_C T}$	0.408248 (Compression)	0.408248 (Compression)	0.408384 (Compression)

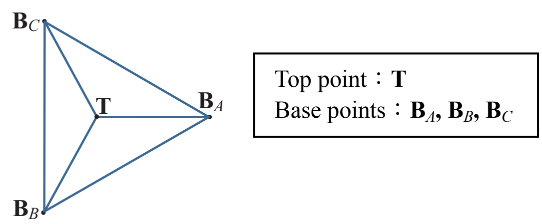


Table 4 shows member forces for the space truss of the regular tetrahedron by using the method of the reasonable support system (Case 2-1), the self-regularization approach (Case 2-2) and the finite-element commercial code ABAQUS using the automatic damping algorithm. In the ABAQUS program, the singular stiffness matrix is considered as a kind of instability which may occur because unconstrained rigid body motions exist. The ABAQUS provides the automatic damping algorithm

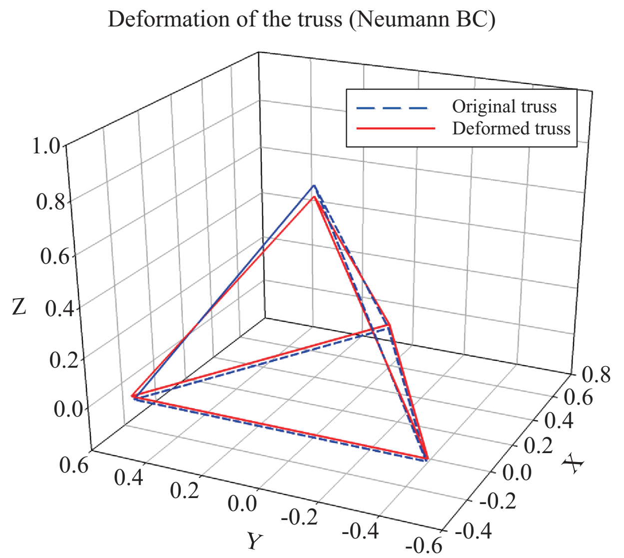


Fig. 14 Deformation of the space truss subject to the Neumann BC by using the self-regularization approach (Case 2-2).

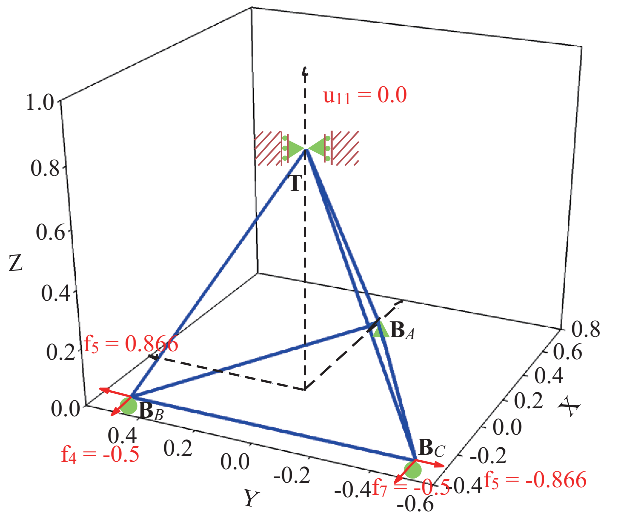


Fig. 16 Case 2-3: The space truss subject to the different mixed type BC - zero stress bar (reasonable support).

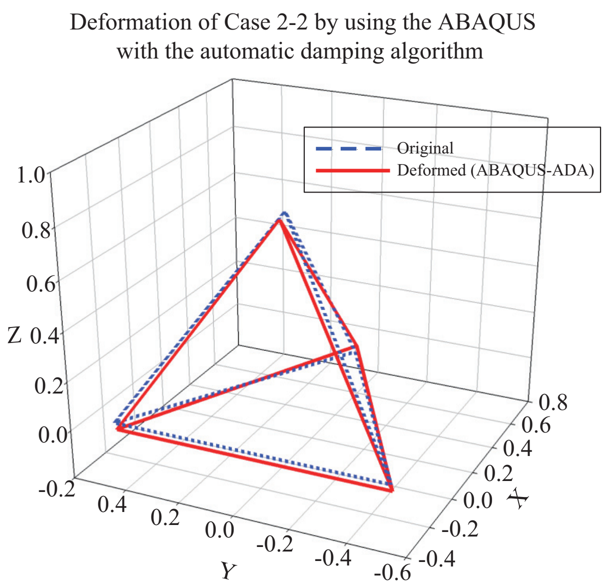


Fig. 15 Deformation of the space truss subject to the Neumann BC by using the ABAQUS with the automatic damping algorithm (Case 2-2, deformation scale factor = 1.0e^{-2})

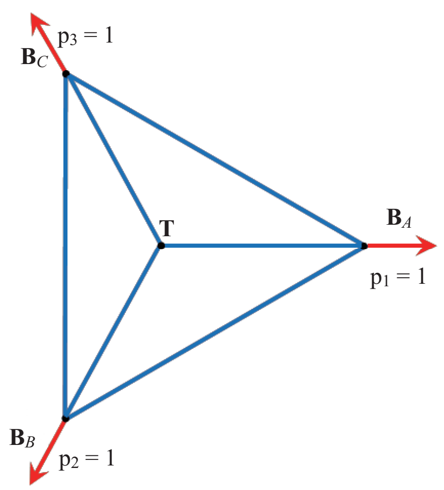


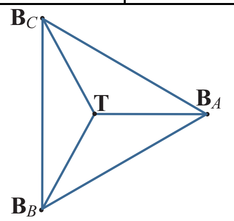
Fig. 17 Case 2-4: The space truss subject to the Neumann BC corresponding to the Case 2-3.

to stabilize the unstable problem. By the automatic stabilization, viscous force is added to overcome instabilities and to eliminate rigid body modes without considerably distorting the solutions. The plots of the deformation by using the self-regularization approach and ABAQUS with automatic damping algorithm are shown in Fig. 14 and Fig. 15, respectively. Although the displacements solved from the method of the reasonable support system and the self-regularization approach are different, the stress states obtained from these two methods are the same. The reason is that the solution obtained by using the self-regularization approach is a pure particular solution. In other words, the obtained displacement field does not contain any rigid body mode.

Reasonable support system for three zero stress bars (Case 2-3) is also constructed in Fig. 16 with a similar treatment from the 2-D truss cases. For this simple case, it is not easy to find the reasonable support. Similar to the 2-D case for the loading in Fig. 17 (Case 2-4), there are zero stress bars which can be considered as a redundant. The results of the space truss for the zero stress bar and non-zero stress bar are shown in Table 5. The deformation of Case 2-3 and Case 2-4 is shown in Fig. 18. All rigid body modes can be obtained straightforwardly by using the SVD in our approach. The procedure to obtain the rigid body mode is implemented in this paper. Since it is difficult to find the reasonable support for unsymmetrical and complicated structures, our approach can work straightforwardly once the stiffness matrix is determined free of adding the reasonable support.

Table 5 Internal force of members for the space truss of Case 2-3 and Case 2-4

Member ID	Mixed type BC	Neumann problem
$\overline{B_A B_B}$	0.57735 (Tension)	0.57735 (Tension)
$\overline{B_B B_C}$	0.57735 (Tension)	0.57735 (Tension)
$\overline{B_C B_A}$	0.57735 (Tension)	0.57735 (Tension)
$\overline{B_A T}$	0 (Redundant)	0 (Redundant)
$\overline{B_B T}$	0 (Redundant)	0 (Redundant)
$\overline{B_C T}$	0 (Redundant)	0 (Redundant)



Top point : **T**
Base points : **B**$_A$, **B**$_B$, **B**$_C$

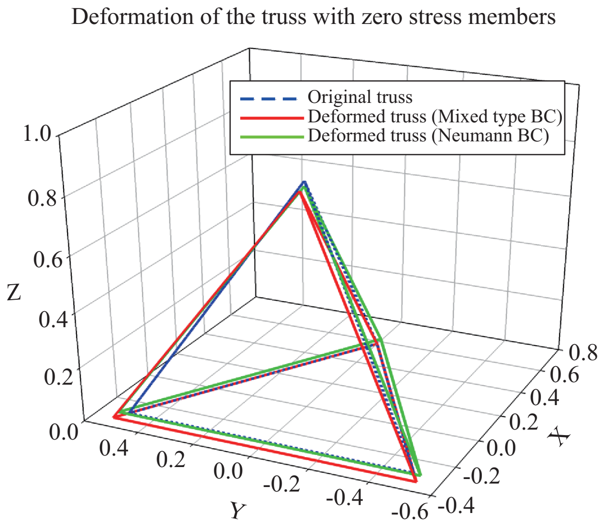


Fig. 18 Deformation of the space truss subject to different types of BCs (Case 2-3 (the mixed type BC) and Case 2-4 (the Neumann BC)).

5. CONCLUSIONS

In this paper, the indeterminacy can be removed in the rigid body mode inherent in the free-free plane and space trusses by adding slack variables and then enforcing the corresponding constraints. In this way, the corresponding solution space is added since the range of the mapping is deficient. Therefore, a singular stiffness matrix can be bordered to a nonsingular matrix. According to the Fredholm alternative theorem, the slack variables, c_i, is used to judge whether the algebraic system has the infinite solutions ($c_i = 0$) or no solutions ($c_i \neq 0$). By inversing the nonsingular matrix, the reference solution which is a pure particular solution without containing any rigid body mode can be obtained. A method of the reasonable support system to constrain the rigid body mode is also an alternative. Four examples of plane and space trusses were demonstrated to see the validity of the present formulation.

Besides, zero stress bars were also discussed for the special loading system. The finite-element commercial code ABAQUS was also applied for comparison and validation. ABAQUS provides the automatic damping algorithm to deal with the inherent instabilities and rigid body modes in free-free structures. It is found that the present method yields the agreeable stress of all bars although displacement fields from rigid body modes are different after comparing with those of using FEM and the method of reasonable support system. Furthermore, we emphasize the accurate stress field rather than the displacement field which can be superimposed by any rigid body mode reasonably. We obtain the reasonable reference displacement field which can be different from others by a rigid body mode.

ACKNOWLEDGEMENTS

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