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Revisit of the free terms of the dual boundary integral equations for elasticity

JENG-TZONG CHEN[*], WEI-CHIH CHEN[*], KUE-HONG CHEN[*] AND I-LIN CHEN[**]

[*]*Department of Harbor and River Engineering, National Taiwan Ocean University Keelung, Taiwan*
[**]*Department of Naval Architecture, National Kaohsiung Institute of Marine Technology, Kaohsiung, Taiwan.*

ABSTRACT

Dual boundary integral equations for elasticity problems with a smooth boundary are derived by using the contour approach surrounding the singularity. Both two and three-dimensional cases are considered. The potentials resulted from the four kernel functions in the dual formulation have different properities across the smooth boundary. The Hadamard principal value or the so called Hadamard finite part, is derived naturally and logically and is composed of two parts, the Cauchy principal value and the unbounded boundary term. After collecting the free terms, Cauchy principal value and unbounded terms, the dual boundary integral equations of the problems are obtained without infinity terms. A comparison between scalar (Laplace equation) and vector (Navier equation) potentials is also made.

Keywords: dual boundary integral equations; elasticity; free terms; a smooth boundary.

INTRODUCTION

Dual boundary integral equations (DBIEs) for crack problems were derived using a limiting approach by Chen (1986) and published two years later (Hong & Chen 1988). Also, the DBIEs for the Laplace equation with a degenerate boundary was developed (Chen & Hong 1993, Chen *et al.* 1992). A number of papers on the dual BEM were published by Aliabadi and his coworkers (1985). The dual formulation has been mainly applied to problems with a degenerate boundary (Chen & Hong 1999). e.g., a screen in an acoustic cavity (Chen & Chen 1998), a crack in an elastic body (Hong & Chen 1988) and cutoff wall in potential flow (Chen & Hong 1992, 1994). Later, the hypersingular formulation was found to play important roles in dealing with degenerate scale problems (Chen *et al.* 2001, 2002, 2002), spurious and fictitious solutions (Chen *et al.* 1998, 1999), adaptive BEM (Liang *et al.* 1999), and symmetric BEM (Bonnet *et* *al*. 1998, Chiu 1999). A detailed review for the DBIEs and the dual BEM can be found in Chen & Hong (1999).

Recently, the hypersingular equation has been utilized to provide a constraint at a corner in an analytical way (Elschner & Graham 1995, Gray 1989, 1990). Gray & Manne (1993) have applied the hypersingular equation as an additional constraint to ensure a unique solution by a limiting process from an interior point to a corner. The three-dimensional case was also extended by Gray & Lutz (1990). How to accurately determine the free term in hypersingular equations has received much attention in the dual BEM (Guiggiani 1995, Mantic 1985, 1993 and Mantic & Paris 1995). From the viewpoint of dual integral equations, the singular and hypersingular equations can provide sufficient constraints for a singular system with a corner. In the case of a nonsmooth boundary, e.g., a corner point, the jump terms of singular and hypersingular integral equations were found to be the same in the former derivations as reported by Lutz *et al*. (1991) and Chen & Hong (1992). Later, an additional free term in the hypersingular equation was obtained by Chen & Hong (1994) and Chen *et al*. (2001). Since the hypersingular integral equation can provide an additional constraint for problems with the Dirichlet boundary conditions, the free terms on a smooth boundary by approaching the interior domain to the exterior domain must be examined. Many researchers, for example, Guiggiani (1995). have derived the free terms in the boundary integral formulation for the Laplace and the Navier equations. Guiggiani also found an additional free term for the corner. In the recent papers (Mukherjee & Mukherjee 1998, Phan *et al*. 1998), the interpretation of finite parts has been shown to be consistent with those of the hypersingular boundary contour method formulation for 2-D and 3-D elasticity problems. The relationship between the Cauchy principal value (Mukherjee 2000 a) and the finite parts of an integral (Mukherjee 2000 b) was explored no matter whether the boundary source point is regular or irregular. However, they did not discuss the free terms using the bump - contour method.

To derive the free terms in a hypersingular equation, the bump-contour approach around the singularity can be considered by using the Taylor expansion for the boundary density. Therefore, the dual integral equations for a smooth boundary can be obtained. Following the same symbols of U_{ki}, L_{ki}, T_{ki} and M_{ki} kernel functions as in the book by Chen & Hong (1992) for a single-layer kernel and its traction derivative, double-layer kernel and its traction derivative, respectively, the bump contour method will be adopted to determine the free terms. Two alternatives for constraint equations can be chosen: (1) by using the singular *(UT)* equation; and (2) by using the hypersingular *(LM)* equation. Both the free terms of the two-dimensional and three-dimensional elasticity problems will be examined by using the bump contour approach in this paper. Their results will also be compared with the scalar potential case (Laplace equation) in Chen *et al*. (2000).

FREE TERMS OF DUAL BOUNDARY INTEGRAL FORMULATION FOR TWO-DIMENSIONAL ELASTICITY WITH A SMOOTH BOUNDARY

In solving the plane elasticity problem without a degenerate boundary, the standard integral representation for the i^{th} component of the displacement at the point x inside the body B is (Rizzo 1967).

$$u_i(x) = \int_{B'+B^-+B_\epsilon+B^+} \{U_{ki}(s,x)t_k(s) - T_{ki}(s,x)u_k(s)\}\, dB(s) \tag{1}$$

where $u_k(s)$ and $t_k(s)$ are the k^{th} components of the displacement and traction vectors on the boundary point s, respectively, B', B^-, $B_\in$ and B^+ are the contour integration paths including the singularity inside the domain, D, as shown in Fig.1, and U_{ki} and T_{ki} are the kernel functions. In many applications, it is not trivial to derive the second equation of DBIEs (Bui 1977, Brebbia 1978).

$$t_i(x) = \int_{B'+B^-+B_\epsilon+B^+} \{L_{ki}(s,x)t_k(s) - M_{ki}(s,x)u_k(s)\}\, dB(s) \tag{2}$$

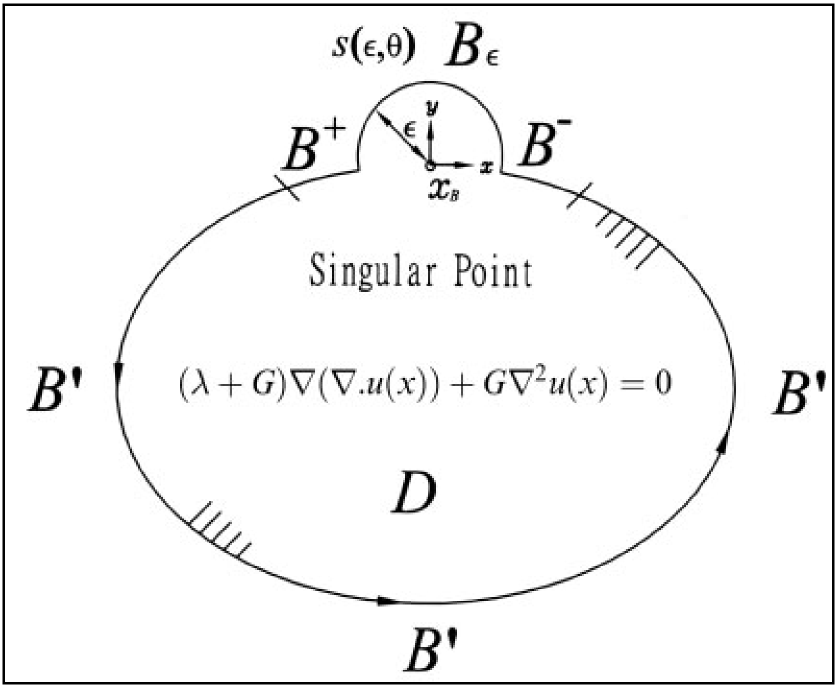

Fig.1: The considered boundary integration path for the two-dimensional elasticity problem.

Table 1. Properties of potentials resulted from different kernels across the smooth boundary for 2-D elasticity problem

Kernel function $K(s,x)$ (direct method)	$U_{ki}(s,x)$	$T_{ki}(s,x)$	$L_{ki}(s,x)$	$M_{ki}(s,x)$
Kernel function $K(s,x)$ (indirect method)	$U_{ki}(s,x)$	$T_{ki}(s,x)$	$U^{*}{}_{ki}(s,x)$	$T^{*}{}_{ki}(s,x)$
Singularity 2D	$O(ln(r))$	$O(1/r)$	$O(1/r)$	$O(1/r^2)$
Density Function $\mu\ (s)$	$t_k(s)$	$-u_k(s)$	$t_k(s)$	$-u_k(s)$
Potential type $\int\ K(s,x)\ \mu\ (s)ds$	single-layer	double-layer	traction derivative of single-layer potential	traction derivative of double-layer potential
Continuity across boundary	continuous	discontinuous	discontinuous	pseudo continuous
Free term (Navier) (2D), $i=1,k=1$	no jump	$-\dfrac{u_1(x)}{2}$	$\dfrac{G(3-4\upsilon)}{16(1-v)}\cdot\left\{(\dfrac{\partial u_1}{\partial s_2}+\dfrac{\upsilon\partial u_2}{\partial s_1})\right\}\Big\|_{s=x}$	$\dfrac{-G}{8(1-v)}\cdot\dfrac{\partial u_1}{\partial s_2}\Big\|_{s=x}$
$i=2,k=1$		0	$\dfrac{G(-1+4\upsilon)}{8(1-v)(1-2v)}\cdot\left\{(1-v)\dfrac{\partial u_1}{\partial s_1}+\dfrac{\partial u_2}{\partial s_2})\right\}\Big\|_{s=x}$	$\dfrac{-G}{8(1-v)}\cdot\dfrac{\partial u_1}{\partial s_1}\Big\|_{s=x}$
$i=1,k=2$		0	$\dfrac{G(3-4\upsilon)}{16(1-v)}\cdot\left\{(\dfrac{\partial u_1}{\partial s_2}+\dfrac{\partial u_2}{\partial s_1})\right\}\Big\|_{s=x}$	$\dfrac{-G}{8(1-v)}\cdot\dfrac{\partial u_2}{\partial s_1}\Big\|_{s=x}$
$i=2,k=2$		$-\dfrac{u_2(x)}{2}$	$\dfrac{G(5-4\upsilon)}{8(1-v)(1-2v)}\cdot\left\{(1-v)\dfrac{\partial u_2}{\partial s_2}+\dfrac{\upsilon\partial u_1}{\partial s_1})\right\}\Big\|_{s=x}$	$\dfrac{-3G}{8(1-v)}\cdot\dfrac{\partial u_2}{\partial s_2}\Big\|_{s=x}$
Free term (Laplace) (2D)	no jump	πu	$-\dfrac{1}{2}\pi t$	$\dfrac{1}{2}\pi t$
Principal Value sense	R.P.V.	C.P.V.	C.P.V.	H.P.V.

where L_{ki} and M_{ki} as well as U_{ki} and T_{ki} are the four kernel functions in the dual integral equations with the properties shown in Table 1. The U_{ki} and M_{ki} kernels are weakly singular and hypersingular, respectively, while the T_{ki} and L_{ki} kernels are strongly singular. For the single and double-layer kernels, Aliabadi *et al.* (1985) have employed the Taylor expansion to reduce the order of singularity. Eq. (2) denotes the integral representation for the i^{th} component traction on the collocation point x. The B integration path in Fig.1 denotes the contour integration around the singularity with a radius ϵ, and $B' + B^{+} + B^{-}$ is just the definition of the integration region of the Cauchy principal value. The boundaries of B^{+} and B^{-} denote two elements in the B^{+} boundary near the singularity as shown in Fig.1. First of all, we will integrate the B_{ϵ} path integration to obtain the free terms for the four integrals of kernel functions. The four kernel functions in the dual integral formulation for the two-dimensional elasticity problems are shown below,

$$U_{ki}(s,x) = C_1(C_2\delta_{ki}ln(r) - \frac{y_i y_k}{r^2}) + A_{ki} \ , \tag{3}$$

$$T_{ki}(s,x) = \frac{-C_3}{r^2}[C_4(n_i y_k - n_k y_i) + (C_4\delta_{ki} + \frac{2y_i y_k}{r^2})\, y_j n_j], \tag{4}$$

$$L_{ki}(s,x) = \frac{C_3}{r^2}[C_4(\overline{n}_k y_i - \overline{n}_i y_k) + (C_4\delta_{ik} + \frac{2y_i y_k}{r^2})\, y_j \overline{n}_j], \tag{5}$$

$$\begin{aligned} L_{ki}(s,x) = \frac{-2C_3 G}{r^2}\Big\{ & \frac{y_l n_l}{r^2}[2a_2\overline{n}_i y_k + 2v(\delta_{ik} y_j \overline{n}_j + \overline{n}_k y_i) - \frac{8y_i y_j y_k \overline{n}_j}{r^2}] \\ & + n_i\Big(\frac{2v y_j y_k \overline{n}_j}{r^2} + a_2\overline{n}_k\Big) + n_j\overline{n}_j\ \Big(\frac{2v y_i y_k}{r^2} + a_2\delta_{ik}\Big) \\ & + n_k\Big(\frac{2a_2 y_i y_j \overline{n}_j}{r^2} - a_4\overline{n}_i\Big)\Big\}, \end{aligned} \tag{6}$$

where

$$C_1 = \frac{-1}{8\pi G(1-v)} \quad , \ C_2 = 3 - 4v \ ,$$

$$C_3 = \frac{-1}{4\pi(1-v)} \quad , \quad C_4 = 1 - 2v \ ,$$

$$a_2 = 1 - 2v \ , \ a_4 = 1 - 4v \ ,$$

$$r^2 = y_i y_i \ , \ y_i = x_i - s_i \ ,$$

in which, r is the distance between the source point s and the field point x, G and v are the shear modulus and Poisson's ratio, respectively, δ_{ki} are the components of the Kronecker delta, A_{ki} is the rigid body term, and n_i and $\overline{n}_i$ denote the i^{th} components of the normal vectors for the source point s and the field point x,

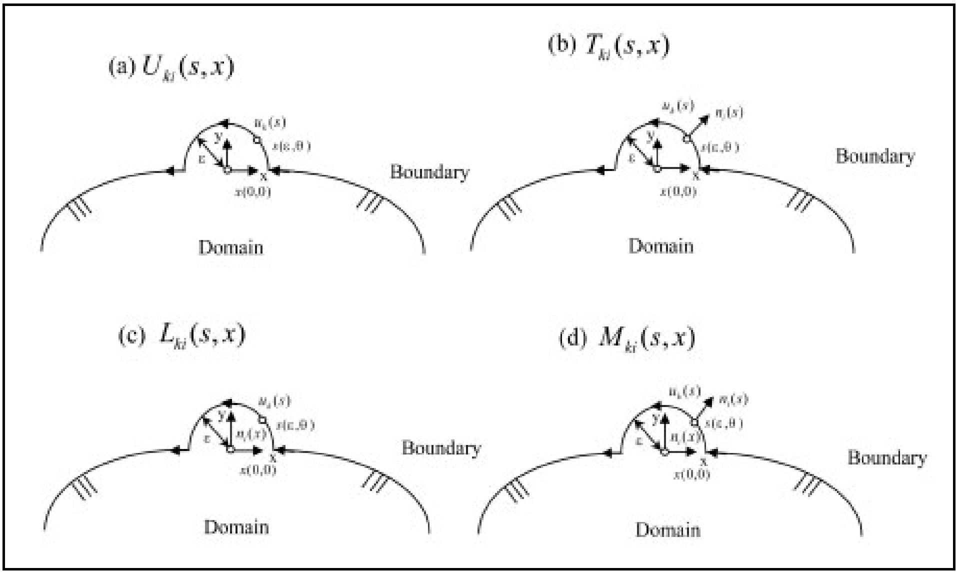

Fig.2: Related symbols of a smooth boundary for the two-dimensional elasticity problem.

respectively. Without loss of generality, we have the following notations for the cylindrical coordinate system in **Fig. 1** and **Fig. 2** :

$$x = (0,0), \tag{7}$$

$$s = (\epsilon \cos(\theta), -\epsilon \sin(\theta)), \tag{8}$$

$$r = |\, x - s \,|, \tag{9}$$

$$y_1 = -\epsilon \cos(\theta), \tag{10}$$

$$y_2 = \epsilon \sin(\theta), \tag{11}$$

$$n(s) = (n_1, n_2) = (\cos(\theta), -\sin(\theta)), \tag{12}$$

$$n(x) = (\overline{n}_1 \ , \ \overline{n}_2) = (0, 1) \qquad \text{for normal vector}, \tag{13}$$

$$u_1(s) = u_1(x) + \left.\frac{\partial u_1}{\partial s_1}\right|_{s=x} \epsilon\cos(\theta) - \left.\frac{\partial u_1}{\partial s_2}\right|_{s=x} \epsilon\sin(\theta) + \text{higher order terms}, \tag{14}$$

$$u_2(s) = u_2(x) + \left.\frac{\partial u_2}{\partial s_1}\right|_{s=x} \epsilon\cos(\theta) - \left.\frac{\partial u_2}{\partial s_2}\right|_{s=x} \epsilon\sin(\theta) + \text{higher order terms}, \tag{15}$$

where $0 < \theta < \pi$ for the surrounding contour on the smooth boundary. To obtain the strain field, ε_{ij}, we can differentiate the displacements, u_i, with respect to the spatial coordinates, i.e.,

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{16}$$

Stress, σ_{ij}, can be otained from Eq. (16) using Hook's law, which can be stated as

$$\sigma_{ij} = 2G(\varepsilon_{ij} + \frac{v}{1-2v}\delta_{ij}\epsilon_{kk}). \quad (17)$$

The traction vectors, t_i, can be determined according to the relationship of the stress tensor and its unit outward normal vector, i.e., $t_i = \sigma_{ij}n_j$ as shown below:

In the case of $n(s) = (\cos\theta, -\sin\theta)$, we have

$$t_1(s) = \frac{2G}{1-2v}\{(1-v)\frac{\partial u_1}{\partial s_1}\Big|_{s=x} + v\frac{\partial u_2}{\partial s_2}\Big|_{s=x}\}\cos(\theta) - G\{\frac{\partial u_1}{\partial s_2}\Big|_{s=x} + \frac{\partial u_2}{\partial s_1}\Big|_{s=x}\}\sin(\theta) \quad (18)$$

$$t_2(s) = G\{\frac{\partial u_1}{\partial s_2}\Big|_{s=x} + \frac{\partial u_2}{\partial s_1}\Big|_{s=x}\}\cos(\theta) - \frac{2G}{1-2v}\{(1-v)\frac{\partial u_2}{\partial s_2}\Big|_{s=x} + v\frac{\partial u_1}{\partial s_1}\Big|_{s=x}\}\sin(\theta) \quad (19)$$

In the case of $n(x) = (0,1)$, we have

$$t_1(x) = G\{\frac{\partial u_1}{\partial s_2}\Big|_{s=x} + \frac{\partial u_2}{\partial s_1}\Big|_{s=x}\}, \quad (20)$$

$$t_1(x) = \frac{2G}{1-2v}\{(1-v)\frac{\partial u_2}{\partial s_2}\Big|_{s=x} + v\frac{\partial u_1}{\partial s_1}\Big|_{s=x}\}. \quad (21)$$

According to the related symbols in **Fig.2**, the free terms of the boundary integration of the four kernel functions for the two-dimensional elasticity can be obtained as follows:

(1). Single-layer potential (U_{ki} kernel):

$$\lim_{\epsilon\to 0}\int_{B\epsilon} U_{ki}(s,x)\ t_k(s)\ dB(s) = \lim_{\epsilon\to 0}\{\epsilon\ (\text{finite value})\} = 0,\ (i=1,2; k=1,2) \quad (22)$$

where k is no sum.

(2). Double-layer potential (T_{ki} kernel):

$$\lim_{\epsilon\to 0}\int_{B\epsilon} T_{ki}(s,x)\ u_k(s)\ dB(s) = \lim_{\epsilon\to 0}\{-\frac{u_1(x)}{2} + \epsilon\ (\text{finite value})\} = -\frac{u_1(x)}{2}, (i=k=1) \quad (23)$$

$$\lim_{\epsilon\to 0}\int_{B\epsilon} T_{ki}(s,x)\ u_k(s)\ dB(s) = \lim_{\epsilon\to 0}\{-\frac{u_2(x)}{2} + \epsilon\ (\text{finite value})\} = -\frac{u_2(x)}{2}, (i=k=2) \quad (24)$$

$$\lim_{\epsilon\to 0}\int_{B\epsilon} T_{ki}(s,x)\ u_k(s)\ dB(s) = \lim_{\epsilon\to 0}\{\epsilon\ (\text{finite value})\} = 0, (i\neq k) \quad (25)$$

where k is no sum.

(3). Traction derivative of single-layer potential (L_{ki} kernel):

$$\lim_{\epsilon\to 0}\int_{B\epsilon} L_{ki}(s,x)\ t_k(s)\ dB(s) = \frac{G(3-4v)}{16(1-v)}\{ \left.\frac{\partial u_1}{\partial s_2}\right|_{s=x} + \left.\frac{\partial u_2}{\partial s_1}\right|_{s=x} \},\ (i=k=1) \tag{26}$$

$$\lim_{\epsilon\to 0}\int_{B\epsilon} L_{ki}(s,x)\ t_k(s)\ dB(s) = \frac{G(-1+4v)}{8(1-v)(1-2v)}\{ (1-v)\left.\frac{\partial u_1}{\partial s_1}\right|_{s=x} + v\left.\frac{\partial u_2}{\partial s_2}\right|_{s=x} \},\ (i=2,\ k=1) \tag{27}$$

$$\lim_{\epsilon\to 0}\int_{B\epsilon} L_{ki}(s,x)\ t_k(s)\ dB(s) = \frac{G(3-4v)}{16(1-v)}\{ \left.\frac{\partial u_1}{\partial s_2}\right|_{s=x} + \left.\frac{\partial u_2}{\partial s_1}\right|_{s=x} \},\ (i=1,\ k=2) \tag{28}$$

$$\lim_{\epsilon\to 0}\int_{B\epsilon} L_{ki}(s,x)\ t_k(s)\ dB(s) = \frac{G(5-4v)}{8(1-v)(1-2v)}\{ (1-v)\left.\frac{\partial u_2}{\partial s_2}\right|_{s=x} + v\left.\frac{\partial u_1}{\partial s_1}\right|_{s=x} \},\ (i=k=2), \tag{29}$$

where k is no sum.

(4). Traction derivative of double-layer potential (M_{ki} kernel):

$$\lim_{\epsilon\to 0}\int_{B\epsilon} M_{ki}(s,x)\ u_k(s)\ dB(s) = \frac{-G}{8(1-v)}\left.\frac{\partial u_1}{\partial s_2}\right|_{s=x} + \text{ Boundary term},\ (i=k=1), \tag{30}$$

where the boundary term $B(\epsilon)$ is

$$B(\epsilon) = \frac{-G}{\pi(1-v)}\ \frac{u_1(x)}{\epsilon}. \tag{31}$$

The free term is $\frac{-G}{\pi(1-v)}\left.\frac{\partial u_1}{\partial s_2}\right|_{s=x}$, and the boundary term, $B(\epsilon)$, in Eq. (31) is unbounded.

$$\lim_{\epsilon\to 0}\int_{B\epsilon} M_{ki}(s,x)\ u_k(s)\ dB(s) = \frac{-G}{8(1-v)}\left.\frac{\partial u_1}{\partial s_1}\right|_{s=x},\ (i=2,\ k=1) \tag{32}$$

$$\lim_{\epsilon\to 0}\int_{B\epsilon} M_{ki}(s,x)\ u_k(s)\ dB(s) = \frac{-G}{8(1-v)}\left.\frac{\partial u_2}{\partial s_1}\right|_{s=x},\ (i=1,\ k=2) \tag{33}$$

$$\lim_{\epsilon\to 0}\int_{B\epsilon} M_{ki}(s,x)\ u_k(s)\ dB(s) = \frac{-3G}{8(1-v)}\ \frac{\partial u_2}{\partial s_2}\bigg|_{s=x} + \text{ Boundary term, } (i=k=2) \quad (34)$$

where the boundary term $B(\epsilon)$ is

$$B(\epsilon) = \frac{G}{\pi(1-v)}\ \frac{u_2(x)}{\epsilon}. \quad (35)$$

The free term is $\frac{-3G}{8(1-v)}\ \frac{\partial u_2}{\partial s_2}\Big|_{s=x}$, and the boundary term, $B(\epsilon)$, in Eq. (35) is unbounded. In Eq. (30), Eq. (32) to Eq. (34), k is no sum.

The boundary term is infinite as ϵ approaches zero in Eq. (31) and Eq. (35). By combining the boundary term with the Cauchy principal value of the M_{ki} kernel integration over B' including B^+ and B^- as shown in **Fig. 1**, the finite part can be extracted and the infinity can be cancelled out. Therefore, the Hadamard principal value in the contour integration with a smooth boundary for the M_{ki} kernel can be defined by

$$H.P.V.\int_B M_{11}(s,x)\ u_1(s)\ dB(s) = C.P.V.\int_B M_{11}(s,x)\ u_1(s)\ dB(s) - \frac{G}{\pi(1-v)}\ \frac{u_1(x)}{\epsilon}, \quad (36)$$

$$H.P.V.\int_B M_{22}(s,x)\ u_2(s)\ dB(s) = C.P.V.\int_B M_{22}(s,x)\ u_2(s)\ dB(s) - \frac{G}{\pi(1-v)}\ \frac{u_2(x)}{\epsilon}, \quad (37)$$

where $C.P.V$ and $H.P.V.$ denote the Cauchy and Hadamard principal values, respectively. It is found that both displacement and traction on the boundary can be solved by using either UT or LM equations, respectively. After collecting the free terms and unbounded values, we can derive the dual boundary integral equations on the smooth boundary point for the two-dimensional elasticity problems as follows.

$$\frac{1}{2}u_i(x) = R.P.V.\int_B\ U_{ki}(s,x)\ t_k(s)\ dB(s) - C.P.V.\int_B\ T_{ki}(s,x)\ u_k(s)\ dB(s), \quad (38)$$

$$\frac{1}{2}t_i(x) = C.P.V.\int_B\ L_{ki}(s,x)\ t_k(s)\ dB(s) - H.P.V.\int_B\ M_{ki}(s,x)\ u_k(s)\ dB(s), \quad (39)$$

by using

$$\int_{B'+B^-B^+} U_{ki}(s,x)\ t_k(s)\ dB(s) = R.P.V.\int_B\ U_{ki}(s,x)\ t_k(s)\ dB(s), \quad (40)$$

$$\int_{B'+B^- B^+} T_{ki}(s,x)\ u_k(s)\ dB(s) = C.P.V. \int_B T_{ki}(s,x)\ u_k(s)\ dB(s), \qquad (41)$$

$$\int_{B'+B^- B^+} L_{ki}(s,x)\ t_k(s)\ dB(s) = C.P.V. \int_B L_{ki}(s,x)\ t_k(s)\ dB(s), \qquad (42)$$

$$\int_{B'+B^- B^+} M_{ki}(s,x)\ u_k(s)\ dB(s) = H.P.V. \int_B M_{ki}(s,x)\ u_k(s)\ dB(s) + \frac{G}{\pi(1-v)}\ \frac{u_i(x)}{\epsilon}. \qquad (43)$$

FREE TERMS OF DUAL BOUNDARY INTEGRAL FORMULATION FOR THREE-DIMENSIONAL ELASTICITY WITH A SMOOTH BOUNDARY

Similarly, we can extend the two-dimensional elasticity to the three-dimensional case by changing the four kernel functions of Eq. (3) $\sim$ Eq. (6) into the three-dimensional case as shown below:

$$U_{ki}(s,x) = \frac{1}{16\pi(1-v)G}\ \frac{1}{r}\,[(3-4v)\delta_{ki} + \frac{y_i y_k}{r^2}], \qquad (44)$$

$$T_{ki}(s,x) = \frac{1}{8\pi(1-v)r^2}\ \{(1-2v)(\frac{n_i y_k}{r} - \frac{n_k y_i}{r}) + [\frac{3 y_i y_k}{r^2} + (1-2v)\delta_{ik}]\ \frac{y_l y_l}{r}\}, \qquad (45)$$

$$L_{ki}(s,x) = \frac{-1}{8\pi(1-v)r^2}\ \{(1-2v)(\frac{\overline{n}_k y_i}{r} - \frac{\overline{n}_i y_k}{r}) + [\frac{3 y_i y_k}{r^2} + (1-2v)\delta_{ik}]\ \frac{y_l \overline{n}_l}{r}\}, \qquad (46)$$

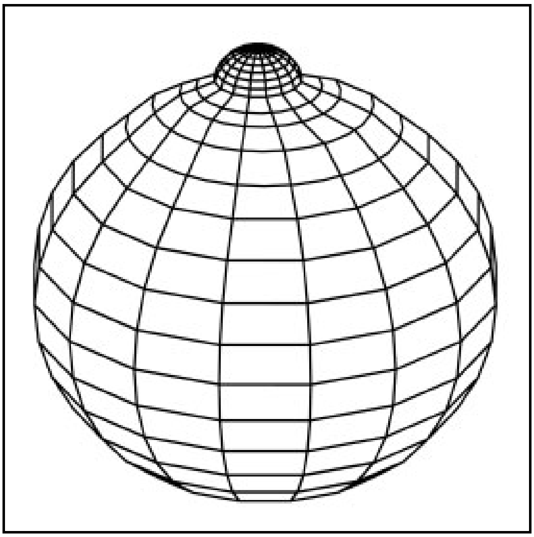

Fig.3: The considered boundary integration path for the three-dimensional elasticity problem.

$$M_{ki}(s,x) = \frac{G}{4\pi(1-v)r^3}\{(\frac{3_y n_e}{r^2})[(1-2v)\bar{n}_i y_k + v(\delta_{ik} y_j \bar{n}_j + \bar{n}_k y_i) - \frac{5 y_i y_j y_k \bar{n}_j}{r^2}]$$

$$+n_i[\frac{3v y_j y_k \bar{n}_j}{r^2} + (1-2v)\bar{n}_k] + n_j \bar{n}_j[\frac{3v y_i y_k}{r^2} + (1-2v)\delta_{ik}]$$

$$+n_k[\frac{3(1-2v) y_i y_j \bar{n}_j}{r^2} - (1-4v)\bar{n}_i]\}. \quad (47)$$

Without loss of generality, we have the following notations for the spherical coordinate system in Fig. 4:

$$x = (0,0,0), \quad (48)$$

$$s = (\epsilon \sin\theta \sin\phi, \ \epsilon \sin\theta \cos\phi, \ \epsilon \cos\theta) \quad (49)$$

$$r = |x - s|. \quad (50)$$

$$y_1 = -\epsilon \sin\theta \sin\phi, \quad (51)$$

$$y_2 = -\epsilon \sin\theta \cos\phi, \quad (52)$$

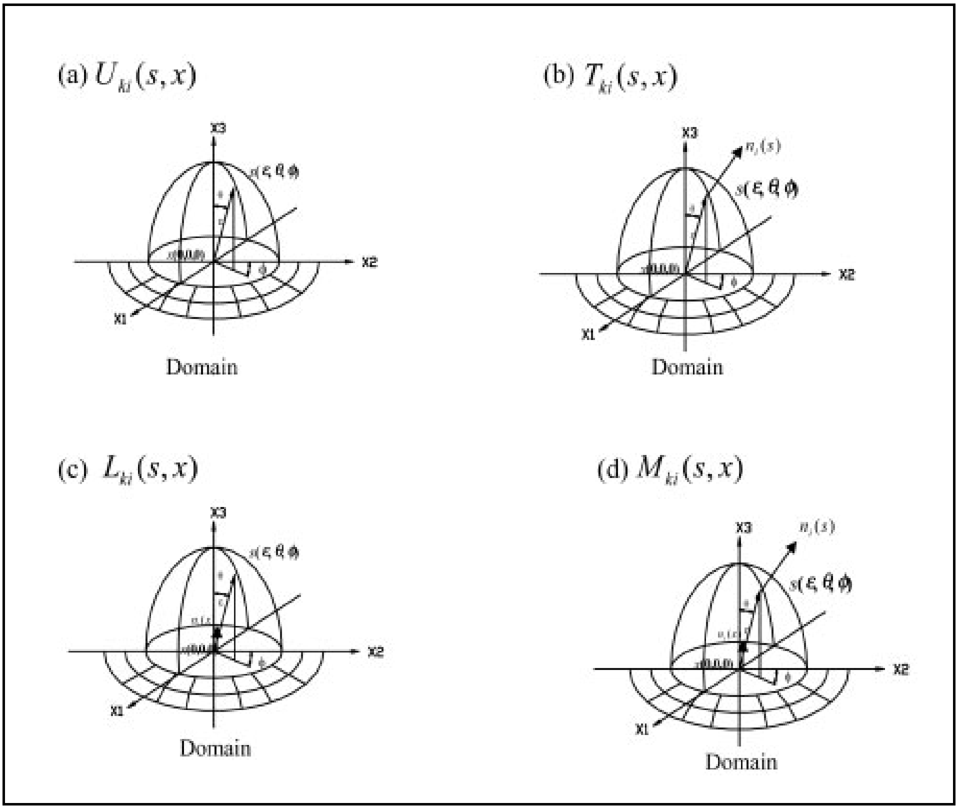

Fig.4: Related symbols of a smooth boundary for the three-dimensional elasticity problem.

$$y_3 = -\epsilon \cos\theta, \quad (53)$$

$$n(s) = (n_1,\ n_2,\ n_3) = (\sin(\theta)\sin(\phi), \sin(\theta)\cos(\phi),\ \cos(\theta)), \quad (54)$$

$$n(x) = (\overline{n}_1,\ \overline{n}_2,\ \overline{n}_3) = (0,\ 0,\ 1), \qquad \text{for normal vector.} \quad (55)$$

$$u_1(s) = u_1(x) + \left.\frac{\partial u_1}{\partial s_1}\right|_{s=x} \epsilon\sin\theta\sin\phi + \left.\frac{\partial u_1}{\partial s_2}\right|_{s=x} \epsilon\sin\theta\cos\phi + \left.\frac{\partial u_1}{\partial s_3}\right|_{s=x} \epsilon\cos\theta + \text{higher order terms}, \quad (56)$$

$$u_2(s) = u_2(x) + \left.\frac{\partial u_2}{\partial s_1}\right|_{s=x} \epsilon\sin\theta\sin\phi + \left.\frac{\partial u_2}{\partial s_2}\right|_{s=x} \epsilon\sin\theta\cos\phi + \left.\frac{\partial u_2}{\partial s_3}\right|_{s=x} \epsilon\cos\theta + \text{higher order terms}, \quad (57)$$

$$u_3(s) = u_3(x) + \left.\frac{\partial u_3}{\partial s_1}\right|_{s=x} \epsilon\sin\theta\sin\phi + \left.\frac{\partial u_3}{\partial s_2}\right|_{s=x} \epsilon\sin\theta\cos\phi + \left.\frac{\partial u_3}{\partial s_3}\right|_{s=x} \epsilon\cos\theta + \text{higher order terms}, \quad (58)$$

where $0 < \theta < \frac{\pi}{2}$, and $0 < \phi < 2\pi$. Similarly, the traction for the normal vector can be determined as shown below:

In the case of $n(s) = (\sin(\theta)\sin(\phi), \sin(\theta)\cos(\phi),\ \cos(\theta))$, we have

$$t_1(s) = G\{(\left.\frac{\partial u_1(s)}{\partial s_2}\right|_{s=x} + \left.\frac{\partial u_2(s)}{\partial s_1}\right|_{s=x})\ \sin\theta\cos\phi + (\left.\frac{\partial u_1(s)}{\partial s_3}\right|_{s=x} + \left.\frac{\partial u_3(s)}{\partial s_1}\right|_{s=x})\cos\theta + \frac{2}{1-2v}[(1-v)\ \left.\frac{\partial u_1(s)}{\partial s_1}\right|_{s=x} + v(\left.\frac{\partial u_2(s)}{\partial s_2}\right|_{s=x} + \left.\frac{\partial u_3(s)}{\partial s_3}\right|_{s=x})]\sin\theta\sin\phi\}. \quad (59)$$

$$t_2(s) = G\{(\left.\frac{\partial u_1(s)}{\partial s_2}\right|_{s=x} + \left.\frac{\partial u_2(s)}{\partial s_1}\right|_{s=x})\ \sin\theta\cos\phi + (\left.\frac{\partial u_2(s)}{\partial s_3}\right|_{s=x} + \left.\frac{\partial u_3(s)}{\partial s_2}\right|_{s=x})\cos\theta + \frac{2}{1-2v}[(1-v)\ \left.\frac{\partial u_2(s)}{\partial s_2}\right|_{s=x} + v(\left.\frac{\partial u_1(s)}{\partial s_1}\right|_{s=x} + \left.\frac{\partial u_3(s)}{\partial s_3}\right|_{s=x})]\sin\theta\sin\phi\}. \quad (60)$$

$$t_3(s) = G\{(\left.\frac{\partial u_1(s)}{\partial s_3}\right|_{s=x} + \left.\frac{\partial u_3(s)}{\partial s_1}\right|_{s=x})\ \sin\theta\cos\phi + (\left.\frac{\partial u_2(s)}{\partial s_3}\right|_{s=x} + \left.\frac{\partial u_3(s)}{\partial s_2}\right|_{s=x})\sin\theta\cos\phi$$

$$+\frac{2}{1-2v}[(1-v)\left.\frac{\partial u_3(s)}{\partial s_3}\right|_{s=x}+v(\left.\frac{\partial u_1(s)}{\partial s_1}\right|_{s=x}+\left.\frac{\partial u_2(s)}{\partial s_2}\right|_{s=x})]\cos\theta\}. \tag{61}$$

In the case of $n(x)=(0,0,1)$, we have

$$t_1(x)=G\{\left.\frac{\partial u_1(s)}{\partial s_3}\right|_{s=x}+\left.\frac{\partial u_3(s)}{\partial s_1}\right|_{s=x}\}, \tag{62}$$

$$t_2(x)=G\{\left.\frac{\partial u_2(s)}{\partial s_3}\right|_{s=x}+\left.\frac{\partial u_3(s)}{\partial s_2}\right|_{s=x}\}, \tag{63}$$

$$t_3(x)=\frac{2G}{1-2v}[(1-v)\left.\frac{\partial u_3(s)}{\partial s_3}\right|_{s=x}+v(\left.\frac{\partial u_1(s)}{\partial s_1}\right|_{s=x}+\left.\frac{\partial u_2(s)}{\partial s_2}\right|_{s=x})\}. \tag{64}$$

The free terms of the boundary integration of the four kernel functions for the three-dimensional elasticity can be obtained as follows:

(1). Single-layer potential (U_{ki} kernel):

$$\lim_{\epsilon\to 0}\int_{B\epsilon} U_{ki}(s,x)\ t_k(s)\ dB(s)=\lim_{\epsilon\to 0}\{\epsilon\ (\text{finite value})\}=0,(\mathrm{i}=1,2,3;\ \mathrm{k}=1,2,3) \tag{65}$$

where k is no sum.

(2). Double-layer potential (T_{ki} kernel):

$$\lim_{\epsilon\to 0}\int_{B\epsilon} T_{ki}(s,x)\ u_k(s)\ dB(s)=\lim_{\epsilon\to 0}\{-\frac{u_1(x)}{2}+\epsilon\ (\text{finite value})\}=-\frac{\mathrm{u}_1(\mathrm{x})}{2},(\mathrm{i}=\mathrm{k}=1) \tag{66}$$

$$\lim_{\epsilon\to 0}\int_{B\epsilon} T_{ki}(s,x)\ u_k(s)\ dB(s)=\lim_{\epsilon\to 0}\{-\frac{u_2(x)}{2}+\epsilon\ (\text{finite value})\}=-\frac{\mathrm{u}_2(\mathrm{x})}{2},(\mathrm{i}=\mathrm{k}=2) \tag{67}$$

$$\lim_{\epsilon\to 0}\int_{B\epsilon} T_{ki}(s,x)\ u_k(s)\ dB(s)=\lim_{\epsilon\to 0}\{-\frac{u_3(x)}{2}+\epsilon\ (\text{finite value})\}=-\frac{\mathrm{u}_3(\mathrm{x})}{2},(\mathrm{i}=\mathrm{k}=3) \tag{68}$$

$$\lim_{\epsilon\to 0}\int_{B\epsilon} T_{ki}(s,x)\ u_k(s)\ dB(s)=\lim_{\epsilon\to 0}\{\epsilon(\text{finite value})\}=0,(i\neq\ k) \tag{69}$$

where k is no sum.

(3). Traction derivative of single-layer potential (L_{ki} kernel):

$$\lim_{\epsilon\to 0}\int_{B\epsilon} L_{ki}(s,x)\ t_k(s)\ dB(s) = \frac{G(4-5v)}{30(1-v)}\{ \left.\frac{\partial u_1}{\partial s_3}\right|_{s=x} + \left.\frac{\partial u_3}{\partial s_1}\right|_{s=x} \},\ (i=k=1) \quad (70)$$

$$\lim_{\epsilon\to 0}\int_{B\epsilon} L_{ki}(s,x)\ t_k(s)\ dB(s) = 0,\ (i=2,\ k=1) \quad (71)$$

$$\lim_{\epsilon\to 0}\int_{B\epsilon} L_{ki}(s,x)\ t_k(s)\ dB(s) = \frac{G(-1+5v)}{15(1-v)(1-2v)}\{(1-v)\left.\frac{\partial u_1}{\partial s_1}\right|_{s=x} + v(\left.\frac{\partial u_2}{\partial s_2}\right|_{s=x} + \left.\frac{\partial u_3}{\partial s_3}\right|_{s=x})\},\ (i=3,\ k=1) \quad (72)$$

$$\lim_{\epsilon\to 0}\int_{B\epsilon} L_{ki}(s,x)\ t_k(s)\ dB(s) = 0,\ (i=1,\ k=2) \quad (73)$$

$$\lim_{\epsilon\to 0}\int_{B\epsilon} L_{ki}(s,x)\ t_k(s)\ dB(s) = \frac{G(4-5v)}{30(1-v)}\{ \left.\frac{\partial u_2}{\partial s_3}\right|_{s=x} + \left.\frac{\partial u_3}{\partial s_2}\right|_{s=x} \},\ (i=k=2) \quad (74)$$

$$\lim_{\epsilon\to 0}\int_{B\epsilon} L_{ki}(s,x)\ t_k(s)\ dB(s) = \frac{G(-1+5v)}{15(1-v)(1-2v)}\{(1-v)\left.\frac{\partial u_2}{\partial s_2}\right|_{s=x} + v(\left.\frac{\partial u_1}{\partial s_1}\right|_{s=x} + \left.\frac{\partial u_3}{\partial s_3}\right|_{s=x})\},\ (i=3,\ k=2) \quad (75)$$

$$\lim_{\epsilon\to 0}\int_{B\epsilon} L_{ki}(s,x)\ t_k(s)\ dB(s) = \frac{G(4-5v)}{30(1-v)}\{ \left.\frac{\partial u_1}{\partial s_3}\right|_{s=x} + \left.\frac{\partial u_3}{\partial s_1}\right|_{s=x} \},\ (i=1,\ k=3) \quad (76)$$

$$\lim_{\epsilon\to 0}\int_{B\epsilon} L_{ki}(s,x)\ t_k(s)\ dB(s) = \frac{G(4-5v)}{30(1-v)}\{ \left.\frac{\partial u_2}{\partial s_3}\right|_{s=x} + \left.\frac{\partial u_3}{\partial s_2}\right|_{s=x} \},\ (i=2,\ k=3) \quad (77)$$

$$\lim_{\epsilon\to 0}\int_{B\epsilon} L_{ki}(s,x)\ t_k(s)\ dB(s) = \frac{G(7-5v)}{15(1-v)(1-2v)}\{(1-v)\left.\frac{\partial u_3}{\partial s_3}\right|_{s=x} + v(\left.\frac{\partial u_1}{\partial s_1}\right|_{s=x} + \left.\frac{\partial u_2}{\partial s_2}\right|_{s=x})\},\ (i=k=3) \qquad (78)$$

where k is no sum.

(4). Traction derivative of double-layer potential (M_{ki} kernel):

$$\lim_{\epsilon\to 0}\int_{B\epsilon} M_{ki}(s,x)\ u_k(s)\ dB(s) = \frac{-7+5v}{30(1-v)}\left.\frac{\partial u_1}{\partial s_3}\right|_{s=x} + \text{Boundary term},\ (i=k=1), \qquad (79)$$

where the boundary term $B(\epsilon)$ is shown as follows:

$$B(\epsilon) = \frac{G(-2+v)u_1(x)}{4(1-v)\ \epsilon}. \qquad (80)$$

The free term is $\frac{-7+5v}{30(1-v)}\left.\frac{\partial u_1}{\partial s_3}\right|_{s=x}$, and the boundary term, $B(\epsilon)$, in Eq. (80) is unbounded.

$$\lim_{\epsilon\to 0}\int_{B\epsilon} M_{ki}(s,x)\ u_k(s)\ dB(s) = 0,\ (i=2,\ k=1) \qquad (81)$$

$$\lim_{\epsilon\to 0}\int_{B\epsilon} M_{ki}(s,x)\ u_k(s)\ dB(s) = \frac{-G(1+5v)}{15(1-v)}\left.\frac{\partial u_1}{\partial s_1}\right|_{s=x},\ (i=3,\ k=1) \qquad (82)$$

$$\lim_{\epsilon\to 0}\int_{B\epsilon} M_{ki}(s,x)\ u_k(s)\ dB(s) = 0,\ (i=1,\ k=2) \qquad (83)$$

$$\lim_{\epsilon\to 0}\int_{B\epsilon} M_{ki}(s,x)\ u_k(s)\ dB(s) = \frac{G(-7+5v)}{30(1-v)}\left.\frac{\partial u_2}{\partial s_3}\right|_{s=x} + \text{Boundary term},\ (i=k=2) \qquad (84)$$

where the boundary term $B(\epsilon)$ is shown as follows:

$$B(\epsilon) = \frac{G(-2+v)u_2(x)}{4(1-v)\ \epsilon}. \qquad (85)$$

Table 2. Properties of potentials resulted from different kernels across the smooth boundary for 3-D elasticity problem

Kernel function $K(s,x)$ (direct method)		$U_{ki}(s,x)$	$T_{ki}(s,x)$	$L_{ki}(s,x)$	$M_{ki}(s,x)$
Kernel function $K(s,x)$ (indirect method)		$U_{ki}(s,x)$	$T_{ki}(s,x)$	$U^{*}{}_{ki}(s,x)$	$T^{*}{}_{ki}(s,x)$
Singularity 3D		$O(1/r)$	$O(1/r^2)$	$O(1/r^2)$	$O(1/r^3)$
Density Function $\mu\ (s)$		$t_k(s)$	$-u_k(s)$	$t_k(s)$	$-u_k(s)$
Potential type $\int\ K(s,x)\ \mu\ (s)ds$		single-layer	double-layer	traction derivative of single-layer potential	traction derivative of double-layer potential
Continuity across boundary		continuous	discontinuous	discontinuous	pseudo continuous
Free term (Navier) (3D)	$i=1,k=1$	no jump	$-\frac{u_1(x)}{2}$	$\frac{G(4-5\upsilon)}{30(1-v)}\cdot\left\{\left(\frac{\partial u_1}{\partial s_3}+\frac{\partial u_3}{\partial s_1}\right)\right\}\Big\vert_{s=x}$	$\frac{G(-7+5v)}{30(1-v)}\cdot\frac{\partial u_1}{\partial s_3}\Big\vert_{s=x}$
	$i=2,k=1$		0	0	0
	$i=3,k=1$		0	$\frac{G(-1+5\upsilon)}{15(1-v)(1-2v)}\cdot\left\{(1-v)\frac{\partial u_1}{\partial s_1}+v\left(\frac{\partial u_2}{\partial s_2}+\frac{\partial u_3}{\partial s_3}\right)\right\}\Big\vert_{s=x}$	$\frac{G(1+5v)}{15(1-v)}\cdot\frac{\partial u_1}{\partial s_1}\Big\vert_{s=x}$
	$i=1,k=2$		0	0	0
	$i=2,k=2$		$-\frac{u_2(x)}{2}$	$\frac{G(4-5\upsilon)}{30(1-v)}\cdot\left\{\left(\frac{\partial u_2}{\partial s_3}+\frac{\partial u_3}{\partial s_2}\right)\right\}\Big\vert_{s=x}$	$\frac{G(-7+5v)}{30(1-v)}\cdot\frac{\partial u_2}{\partial s_3}\Big\vert_{s=x}$
	$i=3,k=2$		0	$\frac{G(-1+5\upsilon)}{15(1-v)(1-2v)}\cdot\left\{(1-v)\frac{\partial u_2}{\partial s_2}+v\left(\frac{\partial u_1}{\partial s_1}+\frac{\partial u_3}{\partial s_3}\right)\right\}\Big\vert_{s=x}$	$\frac{G(1+5v)}{15(1-v)}\cdot\frac{\partial u_2}{\partial s_2}\Big\vert_{s=x}$
	$i=1,k=3$		0	$\frac{G(4-5\upsilon)}{30(1-v)}\cdot\left\{\left(\frac{\partial u_1}{\partial s_3}+\frac{\partial u_3}{\partial s_1}\right)\right\}\Big\vert_{s=x}$	$\frac{G(-7+5v)}{30(1-v)}\cdot\frac{\partial u_3}{\partial s_1}\Big\vert_{s=x}$
	$i=2,k=3$		0	$\frac{G(4-5\upsilon)}{30(1-v)}\cdot\left\{\left(\frac{\partial u_2}{\partial s_3}+\frac{\partial u_3}{\partial s_2}\right)\right\}\Big\vert_{s=x}$	$\frac{G(-7+5v)}{30(1-v)}\cdot\frac{\partial u_3}{\partial s_2}\Big\vert_{s=x}$
	$i=3,k=3$		$-\frac{u_3(x)}{2}$	$\frac{G(7-5\upsilon)}{15(1-v)(1-2v)}\cdot\left\{(1-v)\frac{\partial u_3}{\partial s_3}+v\left(\frac{\partial u_1}{\partial s_1}+\frac{\partial u_2}{\partial s_2}\right)\right\}\Big\vert_{s=x}$	$-\frac{8G}{15(1-v)}\cdot\frac{\partial u_3}{\partial s_3}\Big\vert_{s=x}$
Free term (Laplace) (3D)		no jump	$2\pi u$	$-\frac{2}{3}\pi\, t$	$\frac{4}{3}\pi\, t$
Principal value sense		R.P.V.	C.P.V.	C.P.V.	H.P.V.

The free term is $G\frac{(-7+5v)}{30(1-v)}\left.\frac{\partial u_2}{\partial s_3}\right|_{s=x}$, and the boundary term, $B(\epsilon)$, in Eq. (85) is unbounded.

$$\lim_{\epsilon\to 0}\int_{B\epsilon} M_{ki}(s,x)\ u_k(s)\ dB(s) = -\frac{G(1+5v)}{15(1-v)}\left.\frac{\partial u_2}{\partial s_2}\right|_{s=x},\ (i=3;\ k=2) \tag{86}$$

$$\lim_{\epsilon\to 0}\int_{B\epsilon} M_{ki}(s,x)\ u_k(s)\ dB(s) = \frac{G(-7+5v)}{30(1-v)}\left.\frac{\partial u_3}{\partial s_1}\right|_{s=x},\ (i=1;\ k=3) \tag{87}$$

$$\lim_{\epsilon\to 0}\int_{B\epsilon} M_{ki}(s,x)\ u_k(s)\ dB(s) = \frac{G(-7+5v)}{30(1-v)}\left.\frac{\partial u_3}{\partial s_2}\right|_{s=x},\ (i=2;\ k=3) \tag{88}$$

$$\lim_{\epsilon\to 0}\int_{B\epsilon} M_{ki}(s,x)\ u_k(s)\ dB(s) = \frac{-8G}{15(1-v)}\left.\frac{\partial u_3}{\partial s_3}\right|_{s=x} + \text{Boundary term}, (i=k=3) \tag{89}$$

where the boundary term $B(\epsilon)$ is

$$B(\epsilon) = \frac{-G}{2(1-v)}\frac{u_3(x)}{\epsilon} \tag{90}$$

The free term is $\frac{-8G}{15(1-v)}\left.\frac{\partial u_3}{\partial s_3}\right|_{s=x}$, and the boundary term, $B(\epsilon)$, in Eq. (90) is unbounded. In Eq. (79), Eq. (81) to Eq. (84) and Eq. (86) to Eq. (89), k is no sum.

The dual boundary integral equations on the smooth boundary point for the three dimensional elasticity problems are derived as the same forms of Eq. (38) and Eq. (39). All the above results are summarized in **Table 1** and **Table 2** for 2-D and 3-D cases, respectively.

CONCLUSIONS

The free terms of the dual boundary integral equations for the two and three-dimenstional elasticity problems were derived by using the bump-contour approach. The behavior of potential across the boundary for the four kernel functions: single layer, double layer, traction derivatives of single layer and double layer, were also examined. The finite part for the hypersingular integral was found to be composed of two parts, which are the cauchy principal value and unbounded boundary term. The classical definition of one-dimensional

Hadamard principal value can be seen as a special case of the present formulation. After collecting the free terms and unbounded boundary term, the dual boundary integral equations on the smooth boundary point for two and three-dimensional elasticity problems can be implemented with the Navier case as shown in **Table 1** and **Table 2** for 2-D and 3-D cases, respectively.

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NOTATIONS

DBIEs	dual boundary integral equations
BEM	boundary element method
B', B^+, B_ϵ, B^-	contour integration path including the singularity
R.P.V.	Riemann principal value
C.P.V.	Cauchy principal value
H.P.V.	Hadamard principal value
$U_{ki}(s,x)$	kernel function of the first dual integral equation (vector field)
$T_{ki}(s,x)$	kernel function of the first dual integral equation (vector field)
$L_{ki}(s,x)$	kernel function of the second dual integral equation (vector field)
$M_{ki}(s,x)$	Kernel function of the second dual integral equation (vector field)
x	position vector of the field point
s	position vector of the source point
$u_k(x)$	the k^{th} potential on the boundary point x
$u_k(s)$	the k^{th} potential on the boundary point s
$n_i(s)$	the i^{th} component of the normal vector on the point s
$\overline{n}_i(x)$	the i^{th} component of the normal vector on the point x
$t_i(s)$	traction of the souce point s
$t_i(x)$	traction of the field point x
ϵ	radius of the contour integration around the singularity
(ϵ, θ)	polar coordinate
(ϵ, θ, ϕ)	spherical coordinate

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قسم هندسة الموانئ والأنهار - جامعة تايوان الوطنية للمحيطات
كيلونج 20224 - تايوان

خلاصة

في هذا البحث تم استنتاج ثنائي المعادلات التكاملية الحديثة لمسألة المرونة في وجود إطار ملس باستخدام طريقة الإحاطة بالنقاط (المتفردة) وذلك في نظام البعدين والثلاثة أبعاد. وقد لوحظ أن الجهود الناتجة عن الدوال الأربع وذلك عند تكوين، المعادلات الثنائية، كانت ذات خصائص متباينة خلال الحد الملس. ولقد تم استنتاج القاعدة الحدية لهادامارد أو ما يطلق عليها الجزء المحدود لهادامارد بطريقة طبيعية ومنطقية حيث تألفت من جزئين هما خاصية النتيجة لكوشي بالإضافة إلى حد حدي لا محدود.

وبتجميع الحدود اللامقيدة فإن خاصية النتيجة كوشي وكذلك الحدود الحدية اللامحدودة فإن ثنائي المعادلات التكاملية الحدية للمسألة قد تم الحصول عليها بدون الحاجة إلى الحدود اللامحدودة.

وقد اشتمل البحث على عمل مقارنة بين الجهود باستخدام (معادلة لابلاس) اللامتجهة (ومعادلة نافير) المتجهة.