SOME ADVANCED APPLICATIONS OF THE BOUNDARY ELEENT METHOD IN ENGINEERING MECHANICS

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Outline of Presentation

- A Brief Review of Research on Inverse Problems
- Computational Approach
- Two-Step Inverse Analysis
- Introduction of Recent Applications
- Concluding Remarks

Brief Review

- There are a tremendous number of inverse problems around the world. Even in the fields of engineering, we have to solve many inverse problems.

- The inverse problems can be classified into the following:
  1. Retrieval of the initial and/or boundary conditions
  2. Determination of domain or boundary shapes
  3. Estimation of sources
  4. Estimation of material properties
  5. Finding the differential equation of the field

Brief Review

- The methods of solution are available for inverse problems:
  1. Standard method of optimization with sensitivity analysis
  2. Iterative solution with filter theory
  3. Experimental design
  4. Cellular automata
  5. Genetic algorithm
  6. Neural network

  .

  .

Brief Review

The Simplest Method of Inverse Analysis:

1. The inverse problem is modeled as a parameters identification problem.

2. A suitable cost function is introduced.

3. The standard optimization technique is applied to find an optimal set of parameters.

Brief Review

The standard optimization technique uses sensitivity analysis with respect to parameters.

The above approach can get a successful solution, only if the initial guess of parameters is very near the exact solution. Otherwise, estimation would be less successful or in most cases fail.

A more robust technique should be innovated for the solution of the inverse problems.

Brief Review

The standard optimization technique has been so far applied successfully to optimal shape design of domain and/or boundary.

In most inverse problems, however, we should estimated lacking information of the direct problem, using some of the measured data as additional information. Such additional data, in general, includes measurement errors.

The inverse analysis with filter theory can be applied to the above cases rather successfully.

Brief Review

Main Difficulties of Inverse Analysis

◆Non-Uniqueness

There can be a number of solutions from the mathematical point of view. One useful solution should be selected from the engineering point of view.

◆Ill-Posedness

Solution process can be much influenced by errors encluded in the measured data. We have to make the problem well-posed, and to develop a robust method of inverse analysis. To this end, *a priori* information can be used as much as possible.

Brief Review

Two-Step Solution Procedure for Inverse Analysis

◆**1st Step**

A knowledge-engineering procedure should first be applied to obtain a rough, approximate solution of the inverse problem.

Neural network, Genetic Algorithm, Fuzzy, Experimental design, etc. would be successfully applied.

◆**2nd Step**

Starting from the above approximate solution, we can finally arrive at a refined, accurate solution through one of the "slope methods", which are based on sensitivity.

Standard optimization procedures and the methods of inverse analysis with filter theory can be applied successfully.

Brief Review

- Many excellent general-purpose computer codes based on FDM, FEM, BEM, etc. are available for analysis of *direct problems*. It has been attracting the attention of engineers to apply such computer codes to analysis of *inverse problems*.
- The main advantage of BEM is that only the boundary surface is discretized for most of linear problems, and that high accuracy of computation can be expected.
- Since in most inverse problems the measured data are available only at a selected number of points in domain or on boundary, BEM seems to be suitable for the inverse problems.

Brief Review

At the author's side, there are available BEMs for the direct problems governed by the following PDE:

- **Laplace and Poisson Equations (steady-state heat conduction, electric or magnetic fileds)**
- **Helmholtz Equation (steady-state acoustic field, etc.)**
- **Navier Equations (etastostatics, time-harmonic elastodynamics, unsteady elastodynamics, thermoelasticity)**
- **Diffusion Equation (Unsteady-state heat conduction, etc.)**
- **Biharmonic Equation (plate bending, etc.)**

Recent Investigation (1)

First, I will introduce a successful application of the two-step inverse analysis for identifying the material constants of an elastic plate subject to dynamic load.

Basic assumption

Strain components are measured at several points on the plate surface. The material constants of flexural rigidity and mass density are not known, and should be identified.

Recent Investigation (1)

Estimation of Material Constants and/or Applied Dynamic Load of an Elastic Plate Using Strain Measurements

(Refinement of a Rough Estimation via Extended Kalman Filter)

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Background

BEM for Dynamic Bending of Elastic Plates	Laplace-Transform BEM, Time-Stepping BEM, and Boundary-Domain-Element Method are available.
Extended Kalman Filter	Excellent methods of inverse analysis can be developed.

↓

Combination of two schemes can lead us to a successful method for the solution of inverse problems related to dynamic bending of plates:

Dynamic load distribution?

Position, magnitude of concentrated load?

Material properties?

Outline of Inverse Analysis

Inverse problem under consideration

↓ **Modeling**

Optimization problem to identify parameters minimizing a cost function

In our previous work:

Cost function → Square sum of residuals for deflection

Solver → Extended Kalman filter

Outline of Inverse Analysis (Continued)

In this work:

Cost function → Square sum of residuals for strains

Solver → Extended Kalman filter

Reasons:

1. **Strains on the plate surfaces can be easily measured.**
2. **We want to investigate the computational aspects of this inverse analysis.**

Outline of Inverse Analysis (Continued)

General feature of inverse analysis based on sensitivity:

Successful identification depends on the initial guess of parameter values at the beginning of analysis.

If the assumed parameter values are close to the exact ones, successful identification can be expected.

Otherwise, inverse analysis often fails, falling in a local minimum of the cost function.

To circumvent the above difficulty,
we propose a two-step inverse analysis.

Two-Step Inverse Analysis

First Step:

Rough estimation of parameter values using *a priori* information as much as possible

1. **through experimental design or**
2. **trial computations for an appropriate number of combinations of parameter values**

Second Step:

Precise inverse analysis
under the roughly-estimated parameter values
"Slope" method based on sensitivity analysis can be used for this purpose, which in many cases can lead to a successful estimation, if the initial guess is appropriate.

Present Inverse Analysis

First Step:

Rough estimation of parameter values through trial computations for an appropriate number of combinations of parameter values, based on *a priori* information

Second Step:

Precise inverse analysis via the extended Kalman filter and BEM under the roughly-estimated parameter values via the first step analysis

Dynamic Bending of Elastic Plates

◆Governing Partial Differential Equation (GPDE)

$$D\nabla^4 w + \rho h \frac{\partial^2 w}{\partial t^2} + c_b \frac{\partial w}{\partial t} = p$$

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

w	: Deflection
ρ	: Density of mass
h	: Plate thickness
c_b	: External damping
p	: Applied load
D	: Bending stiffness

◆Laplace Transform

$$\hat{f}(x,s) = \int_0^\infty f(x,t)e^{-st}dt$$

◆Laplace-Transformed GPDE

$$D\nabla^4 \hat{W}(x,s) + (\rho h s^2 + c_b s)\hat{W}(x,s) = \hat{P}(x,s)$$

Integral Equation Formulation

◆Integral Expression:

$$\int_{\Omega}\{D\nabla^4\hat{W}(x,s)+(\rho hs^2+c_b s)\hat{W}(x,s)-\hat{P}(x,s)\}W^*(x,y)d\Omega=0$$

◆Integral Equation for Deflection at Internal Point:

$$\begin{aligned}\hat{W}(y)=&\int_{\Gamma}\Big\{W^*(x,y)\hat{V}_n(x)-\Theta_n^*(x,y)\hat{M}_n^*(x)\\&+M_n^*(x,y)\hat{\Theta}_n(x)-V_n^*(x,y)\hat{W}(x)\Big\}d\Gamma\\&-(\rho hs^2+c_b s)\int_{\Omega}W^*(x,y)\hat{W}(x)d\Omega+\int_{\Omega}W^*(x,y)\hat{P}(x)d\Omega\\&-\sum_{k=1}^{K_c}\left[W^*(x,y)\hat{M}_{nt}(x)\right]_k+\sum_{k=1}^{K_c}\left[M_{nt}^*(x,y)\hat{W}(x)\right]_k\quad,y\in\Omega\end{aligned}$$

Fundamental Solution

◆**The fundamental solution for static bending problem:**

$$W^*(x,y) = \frac{1}{8\pi D} r^2 \ln r, \qquad r = |x-y|$$

$$D\nabla^4 W^*(x,y) = \delta(x-y)$$

is used for the integral equation formulation.

Integral Equation

◆**An integral equation which relates deflection at the source point to physical quantities on the boundary:**

$$\hat{W}(y) = \int_{\Gamma} \left\{ W^*(x,y)\hat{V}_n(x) - \Theta_n^*(x,y)\hat{M}_n^*(x) + M_n^*(x,y)\hat{\Theta}_n(x) - V_n^*(x,y)\left[\hat{W}(x) - \hat{W}(x_0)\right] \right\} d\Gamma$$
$$- (\rho h s^2 + c_b s)\int_{\Omega} W^*(x,y)\hat{V}_n(x)d\Omega + \int_{\Omega} W^*(x,y)\hat{P}(x)d\Omega$$
$$- \sum_{k=1}^{K_c} \left[W^*(x,y)\hat{M}_{nt}(x) + M_{nt}^*(x,y)\hat{W}(x,y) \right]_k, \quad y \in \Omega$$

x_0 : **The boundary point nearest to the inner source point**

K_c	: **Number of Edge nodes**
$\hat{\Theta}_n$	: **Normal slope**
$\hat{M}_n$	: **Bending moment**
$\hat{M}_{nt}$	: **Twisting moment**
$\hat{V}_n$	: **Equivalent shear force**

Boundary Integral Equations

◆**Regularized boundary integral equation for deflection:**

$$\int_\Gamma \left\{ W^*(x,y)\hat{V}_n(x) - \Theta_n^*(x,y)\hat{M}_n^*(x) + M_n^*(x,y)\hat{\Theta}_n(x) - V_n^*(x,y)\left[\hat{W}(x) - \hat{W}(y)\right] \right\} d\Gamma$$
$$-(\rho h s^2 + c_b s)\int_\Omega W^*(x,y)\hat{V}_n(x)d\Omega + \int_\Omega W^*(x,y)\hat{P}(x)d\Omega$$
$$-\sum_{k=1}^{K_c}\left[W^*(x,y)\hat{M}_{nt}(x) + M_{nt}^*(x,y)\hat{W}(x,y) \right]_k = 0$$

◆**Regularized boundary integral equation for normal slope:**

$$\int_\Gamma \left\{ \hat{W}^*(x,y)\hat{V}_n(x) - \hat{\Theta}_n^*(x,y)\hat{M}_n^*(x) + \hat{M}_n^*(x,y)\hat{\Theta}_n(x) - \hat{V}_n^*(x,y)\left[\hat{W}(x) - \hat{W}(y)\right] \right\} d\Gamma$$
$$-(\rho h s^2 + c_b s)\int_\Omega \hat{W}^*(x,y)\hat{V}_n(x)d\Omega + \int_\Omega \hat{W}^*(x,y)\hat{P}(x)d\Omega$$
$$-\sum_{k=1}^{K_c}\left[\hat{W}^*(x,y)\hat{M}_{nt}(x) + \hat{M}_{nt}^*(x,y)\hat{W}(x,y) \right]_k = 0$$

Boundary-Domain-Element Method

Discretization by boundary-domain-elements

↓

Three sets of algebraic equations resulting from

(1) Boundary integral equation for deflection
(2) Boundary integral equation for normal slope
(3) Integral equation for internal deflection

↓

Numerical solutions for various values of the Laplace-transform parameter *s*, and then numerical inversion can yield time-dependent physical solutions.

Approximation of Dynamic Load

- **2nd-order polynomial is applied to interpolation of dynamic load:**

$$P_i(t) = b_i + b_{i+1}(t - t_i) + b_{i+2}(t - t_i)(t - t_{i+1})$$

$$b_i = p_i$$

$$b_{i+1} = (p_{i+1} - b_i)/(t_{i+1} - t_i)$$

$$b_{i+2} = \{(p_{i+2} - b_i)/(t_{i+2} - t_i) - b_{i+1}\}/(t_{i+2} - t_{i+1})$$

p_i : Load at time t_i

Inverse Laplace Transform

◆Durbin's method for numerical inversion

$$f(t_n) = \frac{2e^{6n\Delta t}}{T}\left[-\frac{1}{2}\mathrm{Re}\left\{\hat{f}(\frac{6}{T})\right\} + \mathrm{Re}\left\{\sum_{K=0}^{N-1}(A(k)+\mathrm{i}B(k))W^{nk}\right\}\right]$$

$$W = \exp(\mathrm{i}\frac{2\pi}{N}), \quad s_n = \frac{6}{T} + \mathrm{i}(k+mN)$$

$$A(k) = \sum_{m=0}^{M}\mathrm{Re}\left[\hat{f}\left\{\frac{6}{T} + \mathrm{i}(k+mN)\frac{2n\pi}{T}\right\}\right]$$

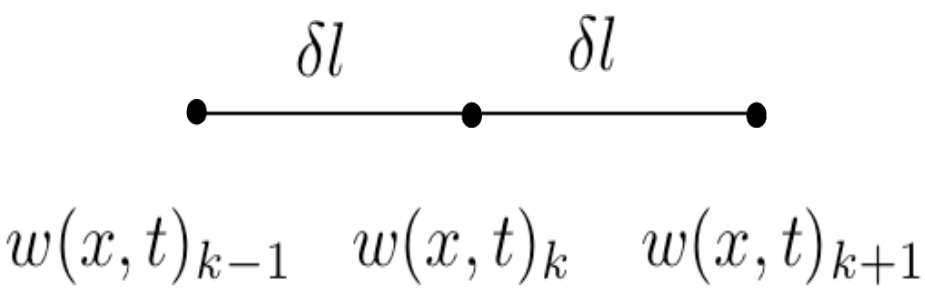
$$B(k) = \sum_{m=0}^{M}\mathrm{Im}\left[\hat{f}\left\{\frac{6}{T} + \mathrm{i}(k+mN)\frac{2n\pi}{T}\right\}\right]$$

T : **total time of analysis**
N : **sampling number of Laplace-transform parameter**
M : **total number of terms**
$\triangle t$: **time-step length**

$$\triangle t = \frac{T}{N}$$

Calculation of Strain Components

◆**Finite Different Scheme is used:**



$$\varepsilon_{ij}(x,t) = \frac{h}{2}\frac{w(x,t)_{k-1} - 2w(x,t)_k + w(x,t)_{k+1}}{(\delta l)^2}$$

Two-Step Inverse Analysis

1st Step

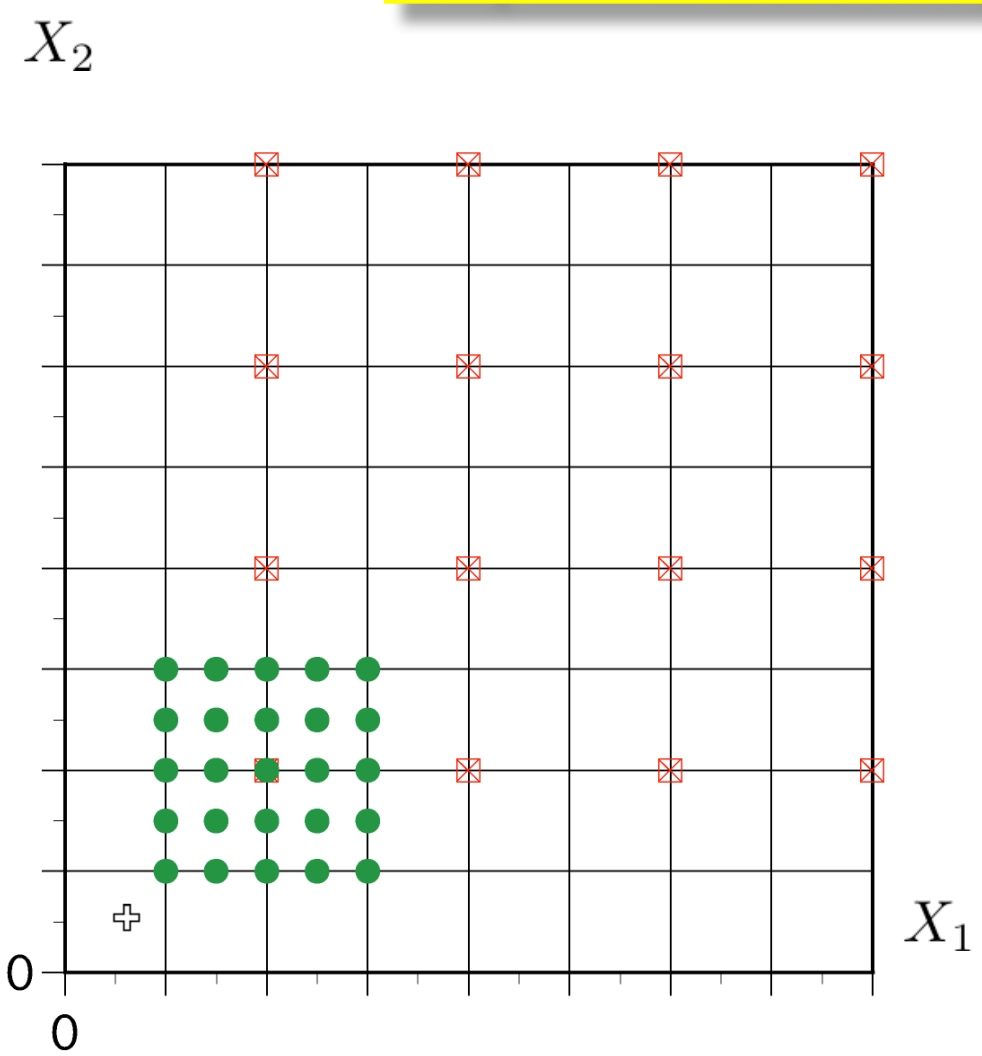
Assign an appropriate number of combinations of parameter values based on *a priori* information. Then, get a rough estimation of parameter values providing the minimum value of cost function among them.

2nd Step

Using the rough evaluation of parameters as the initial guess, carry out the inverse analysis based on the extended Kalman filter and BEM.

Rough Estimation - 1st Step Analysis

Explanation for two parameters:



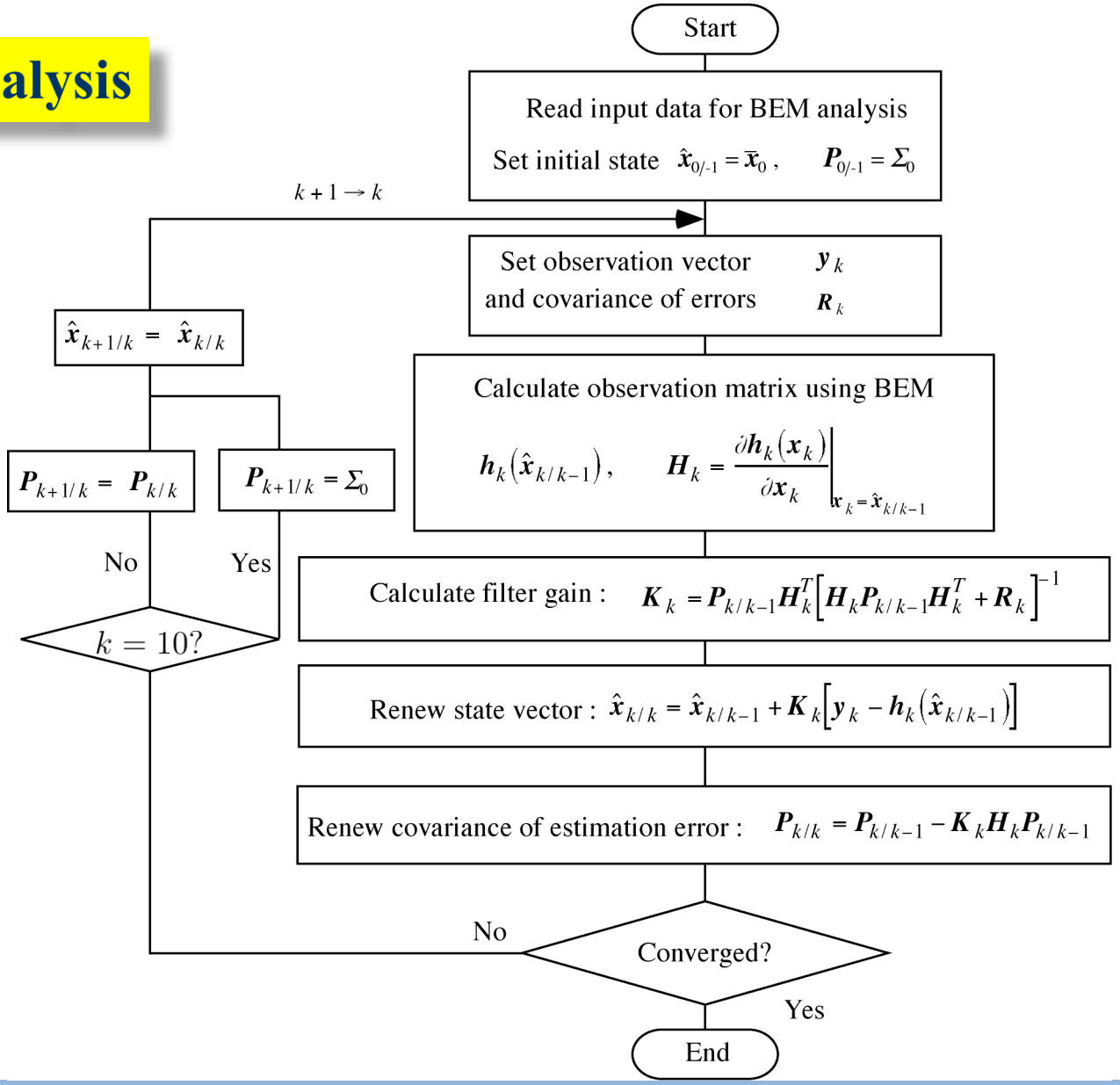
Initial
Initial
Exact

Parameters:

X_1 X_2

Inverse Analysis via BEM and Kalman Filter

Precise analysis



Inverse analysis via BEM and Kalman Filter

◆ Cost function:

$$W_k = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{v_i(X_k)-\bar{v}_i}{\bar{v}_i}\right)^2$$

v_i : Computed strain

$\bar{v}_i$: Measured strain

◆ Criterion of convergence:

$$\left|\frac{X_i(k+1)-X_i(k)}{X_i(k)}\right| < \epsilon$$

$X_i(k)$: Estimated parameter at kth iteration

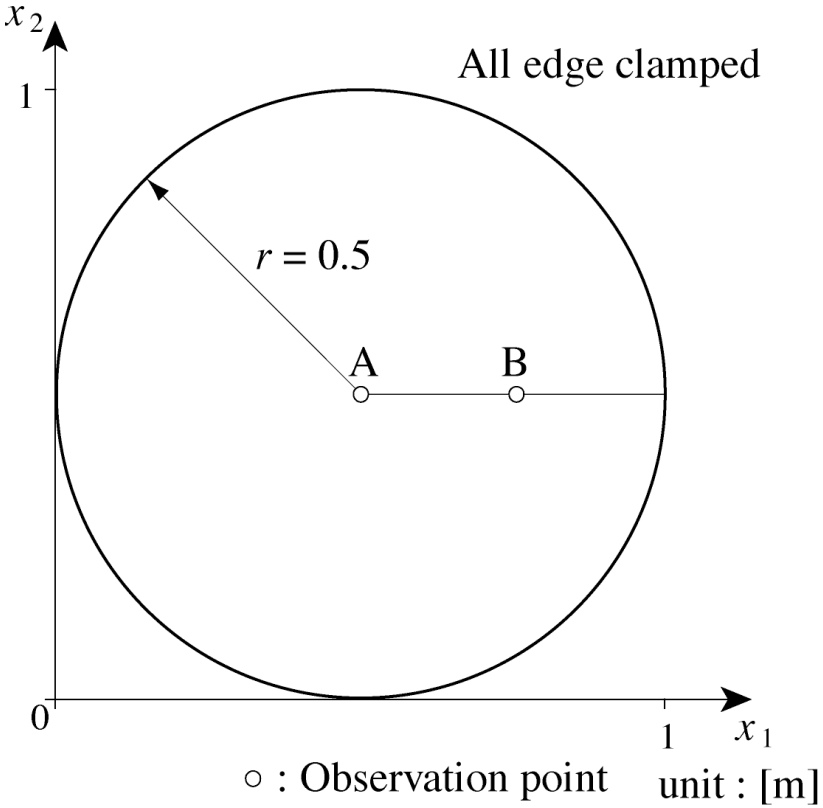
$\varepsilon = 10^{-3}$

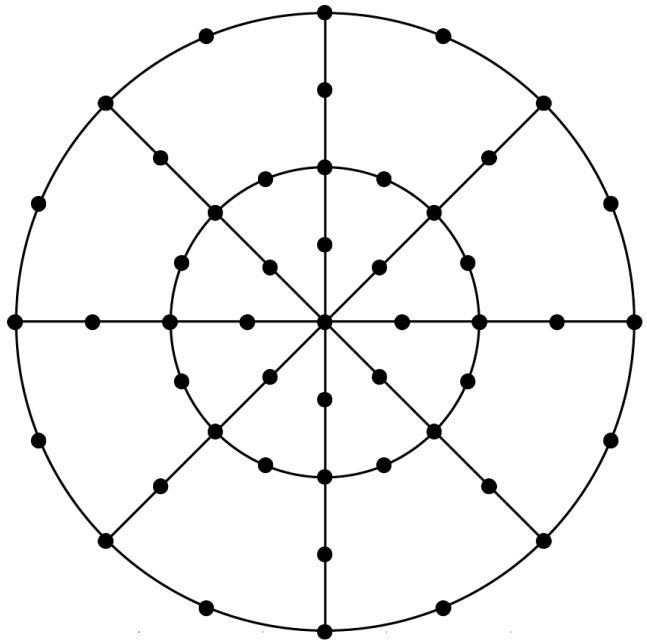
Example - Circular Plate

- **Analysis Model**
- **Discretization**

Material constants :

$$E_0 = 2.0 \times 10^{11} [Pa]$$
$$\rho_0 = 7.8 \times 10^{3} [kg/m^3]$$
$$\nu_0 = 0.3$$



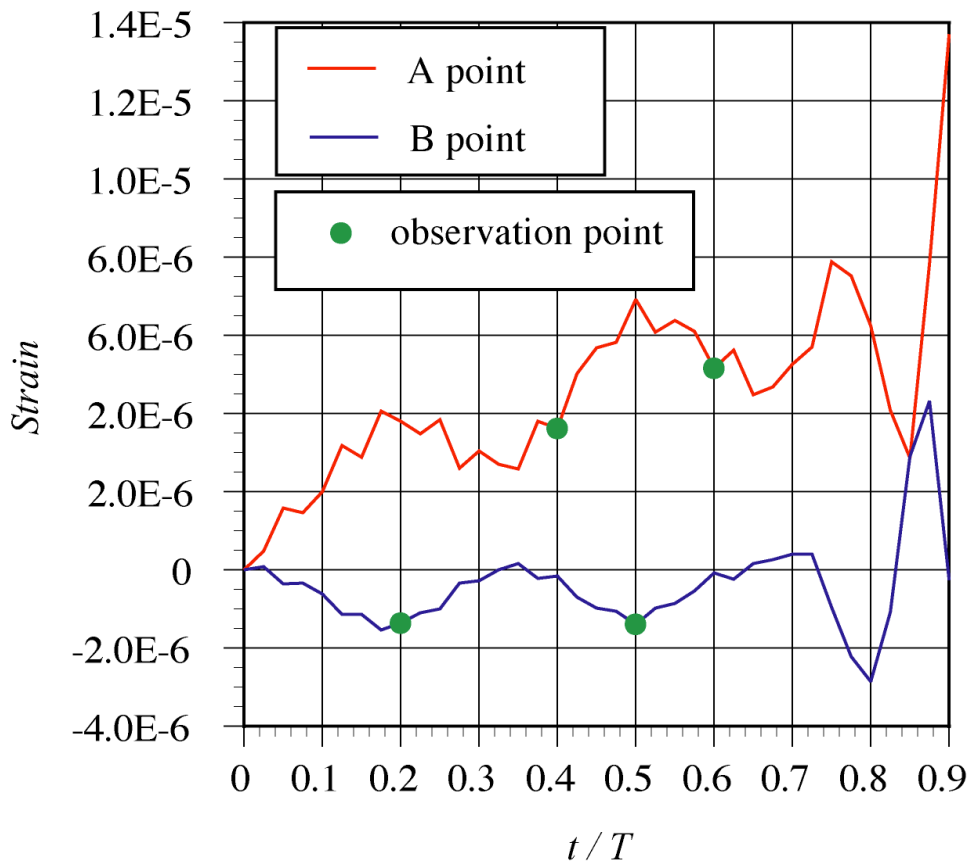


Numerical Results (1)

◆ Selection of Measured Data

Concentrated load of Heaviside-function type is applied at the center point of plate.

Two points in time as shown in the figure are selected for the two spatial points A and B.



Strain variations in time for normal strain in x-axis
T=0.0075[s]

Numerical Results (1)

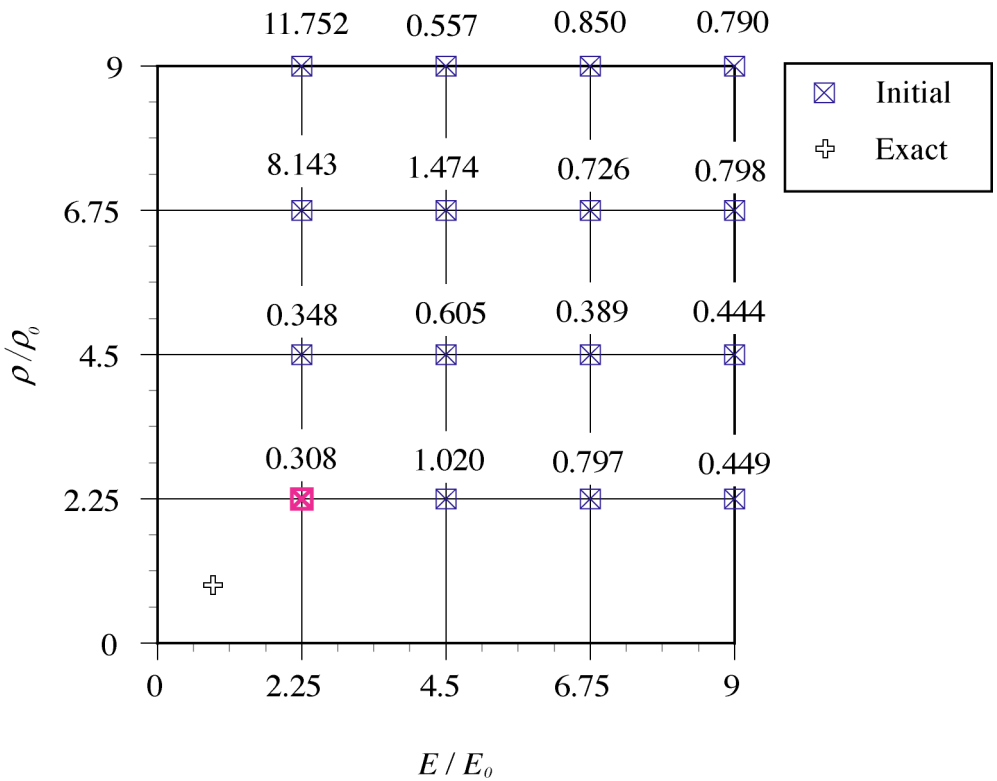
◆ 1st Step Analysis

Rough estimation:

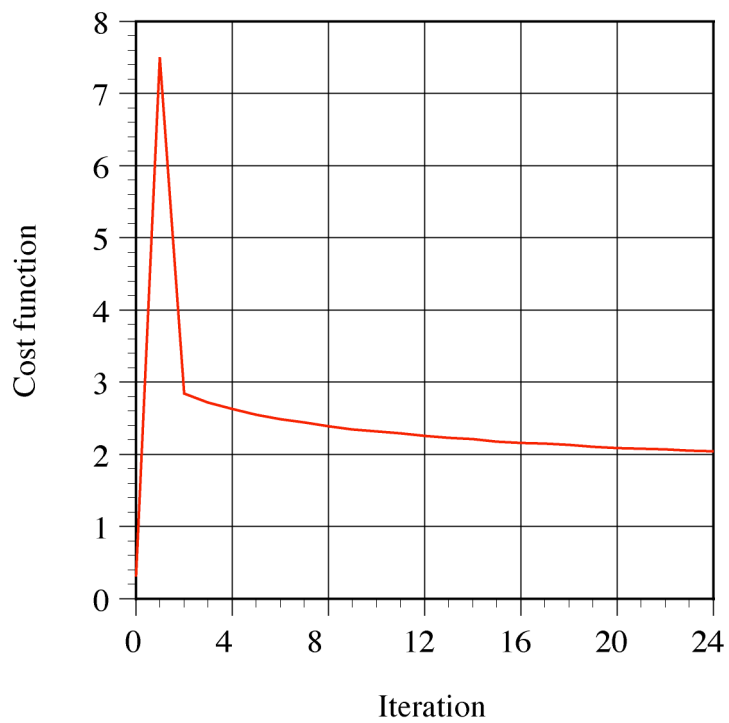
$$(E/E_0, \rho/\rho_0) = (2.25, 2.25)$$

Final estimation:

$$(E/E_0, \rho/\rho_0) = (-2.337, -2.330)$$



The Initial Parameter

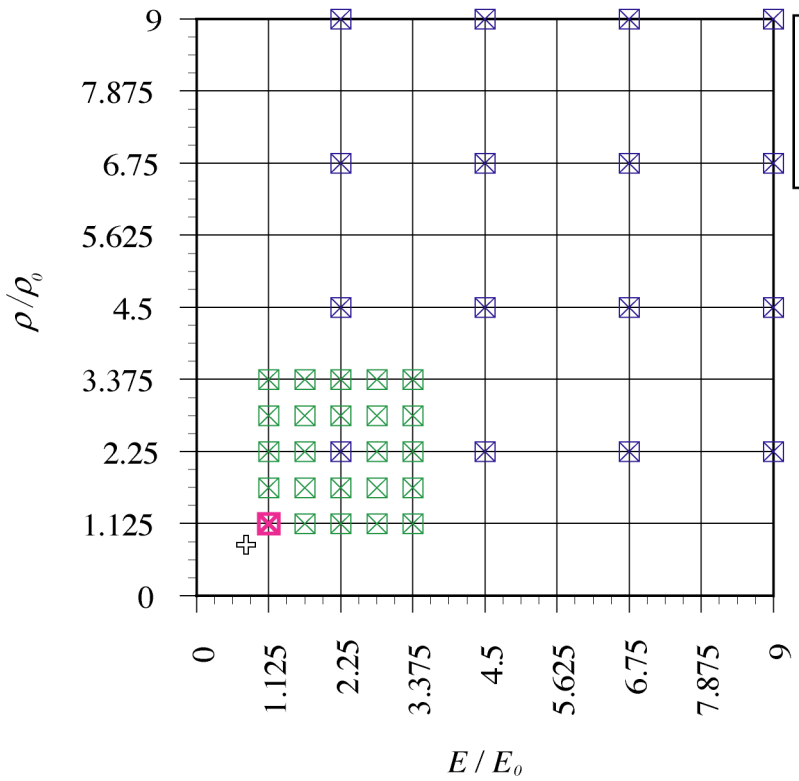


The Cost Function

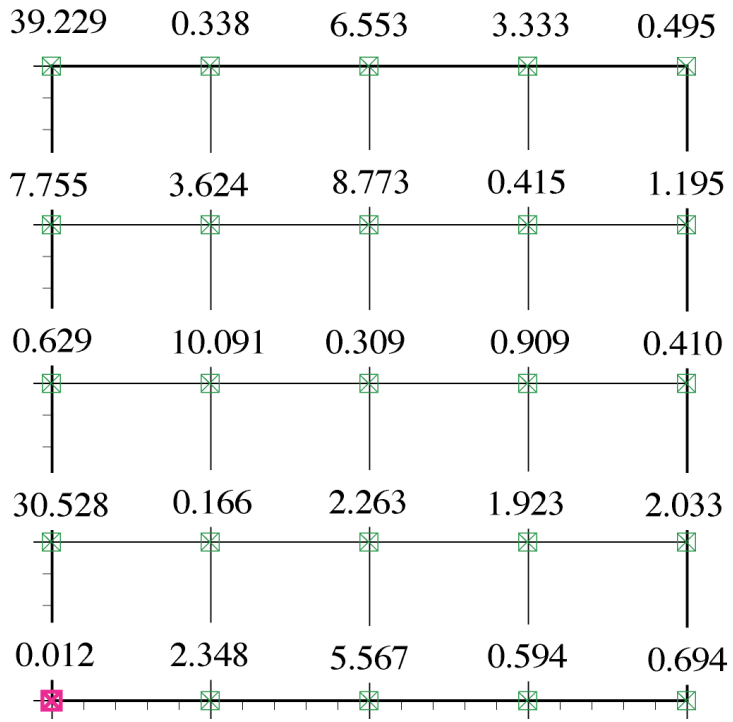
Numerical Results (1)

◆Second Trial of two-Step Inverse Analysis

Rough estimation: $(E/E_0, \rho/\rho_0) = (1.125, 1.125)$



The Initial Parameter



The Cost Function

Numerical Results (1)

◆Final Results Obtained:

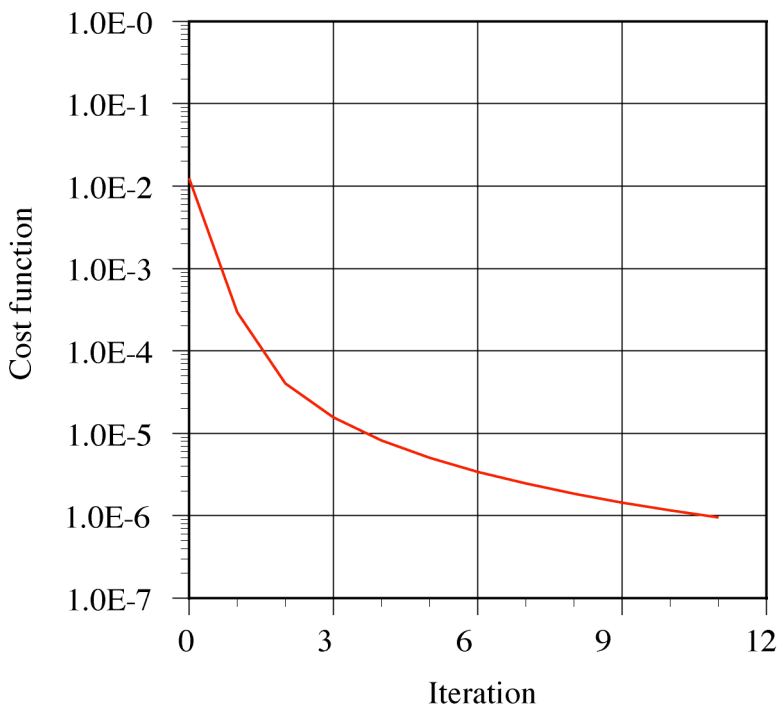
Rough estimation:

$(E/E_0, \rho/\rho_0) = (1.125, 1.125)$

→

Final estimation:

$(E/E_0, \rho/\rho_0) = (0.999, 0.999)$



The Cost Function

Resetting Variance of Estimated Values

◆**Drawback** of Kalman Filter:

First, we assume the co-variance of estimated values, and then its value is automatically renewed at each iteration.

The co-variance of estimated values rapidly becomes very small, and thereafter little modification of parameters is made, which results in a larger number of iterations to approach the target.

◆**Remedy** of the above dilemma:

Co-variance of estimated values is reset after an appropriate number of iterations, e.g., 5 or 10.

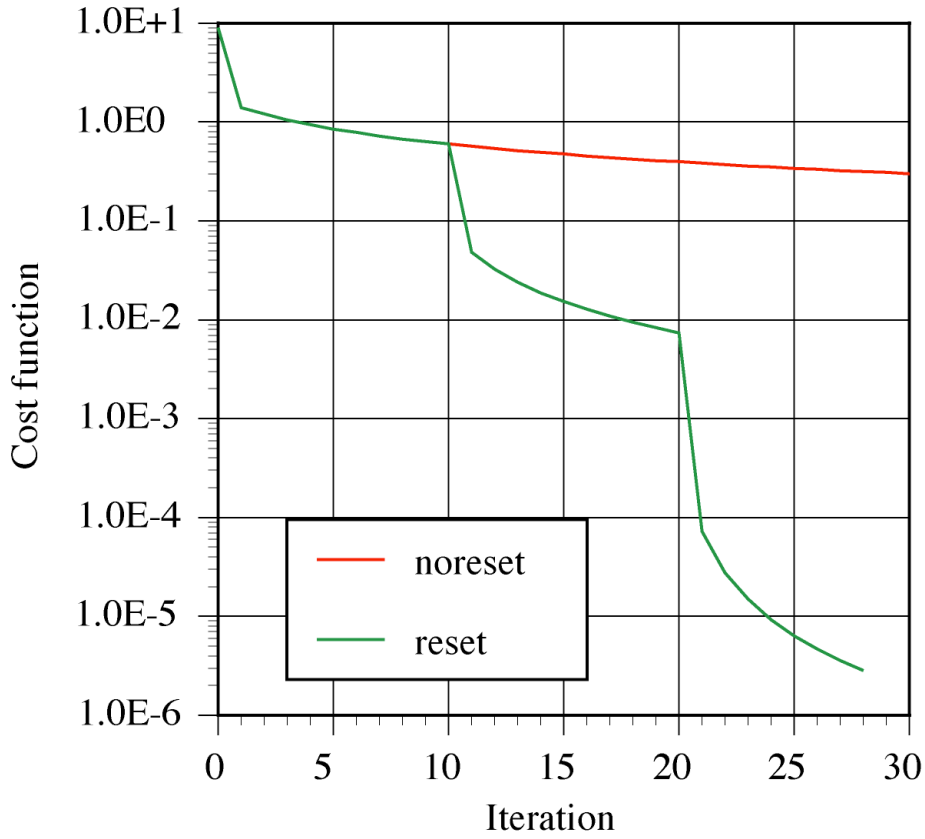
Numerical Results (2)

◆ **Co-variance is reset after 10 iterations**

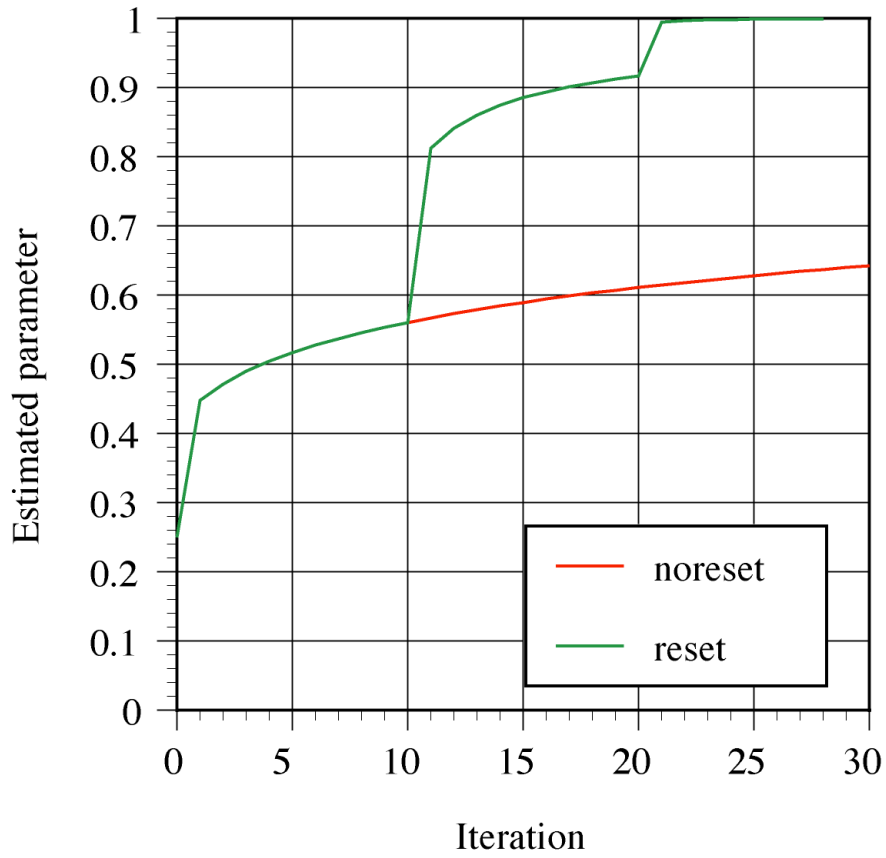
Initial guess : $(E/E_0, \rho/\rho_0) = (0.25, 0.25)$

Initial values of variances:

$$E/E_0 : 10^{-5}$$
$$\rho/\rho_0 : 10^{-5}$$



Convergence of cost function



Convergence of parameter

Numerical Results (2)

◆ **Estimated results with and without resetting co-variance of estimated values**

Measured data at two points in time for each of two spatial points A and B are used.

Initial values of co-variances
$E/E_0 : 10^{-5}$ $\rho/\rho_0 : 10^{-5}$

		Not reset			Reset		
Initial		Estimated			Estimated		
E/E_0	ρ/ρ_0	E/E_0	ρ/ρ_0	K	E/E_0	ρ/ρ_0	K
0.25	0.25	0.771189	0.770885	150	0.999041	0.999004	28
0.5	0.5	0.976582	0.976470	150	1.000023	1.000020	21
0.75	0.75	0.998778	0.998765	150	0.999973	0.999951	11
1.25	1.25	0.999033	0.999031	32	0.999026	1.000026	11
1.5	1.5	0.998313	0.998321	150	0.999831	0.999809	11
1.75	1.75	0.950864	0.951557	150	1.000036	1.000028	11

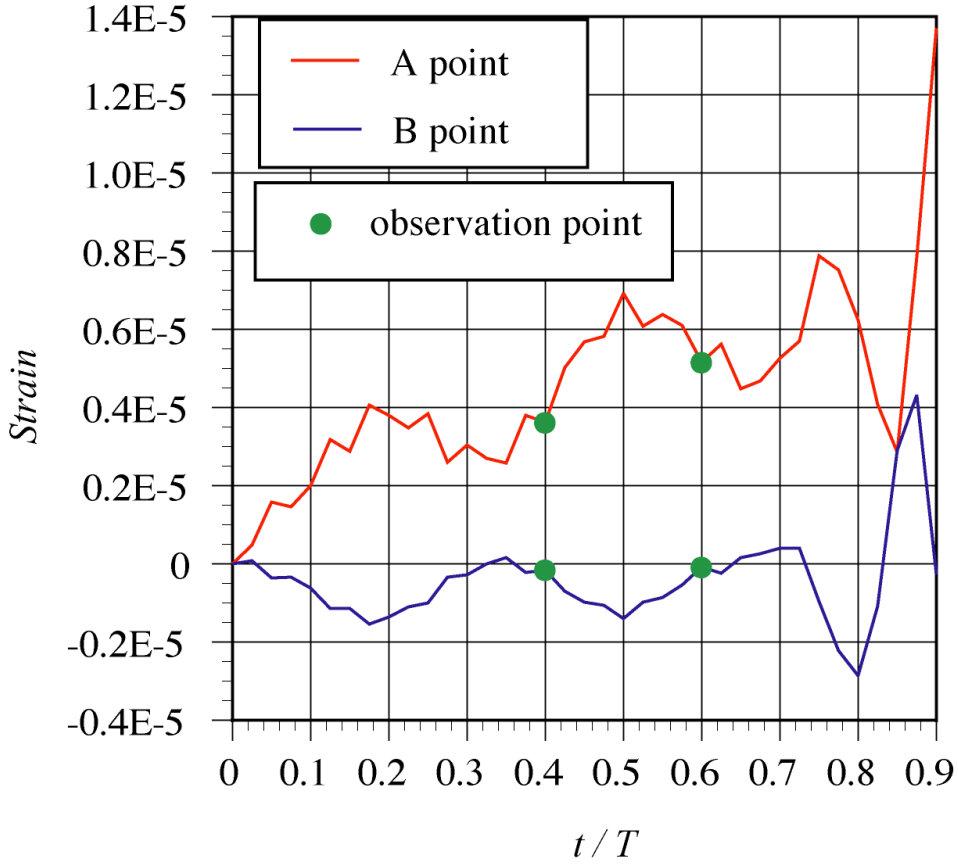
Numerical Results (3)

◆ **Simulation for measured data with measurement errors**

Results calculated by BEM are assumed to be averaged data, and random noise is superposed onto these data:

$$y_i = y_m + \sigma\lambda$$

y_i	: **Measured value**
y_m	: **Computed value**
σ	: **Variance of errors**
λ	: **Normalised random numbers**



Numerical Results (3)

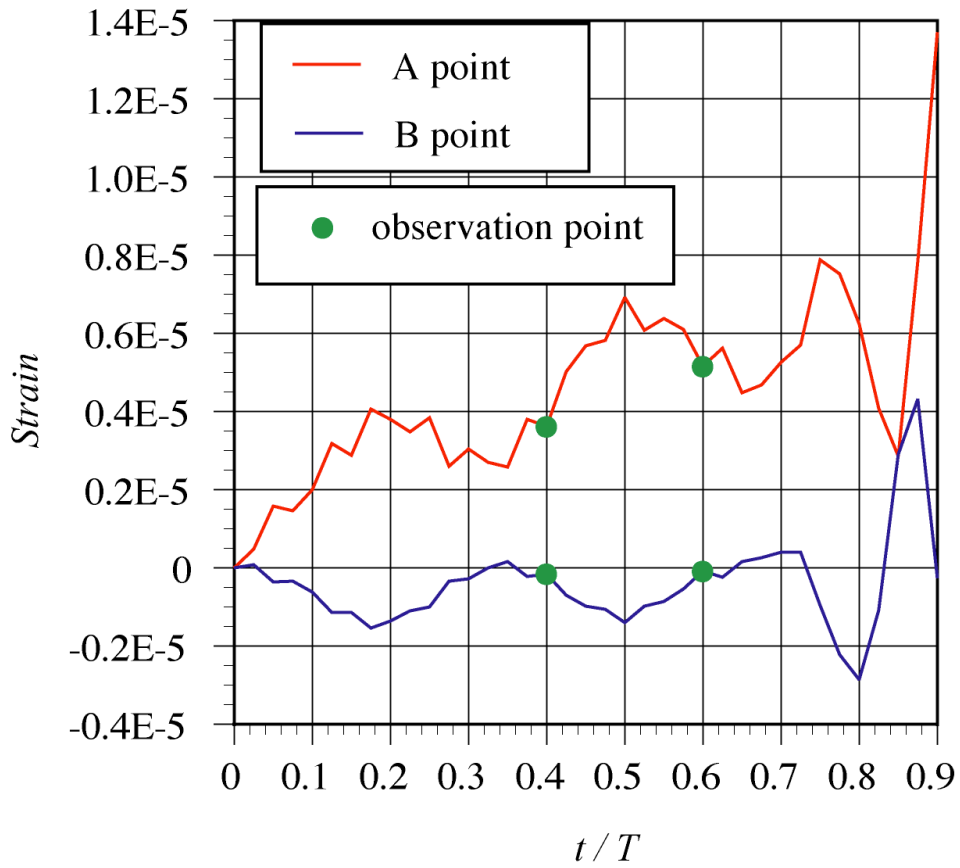
◆ **Estimated results for three kinds of variance for measurement errors**

		$\sigma^2 = 10^{-40}$			$\sigma^2 = 10^{-20}$			$\sigma^2 = 10^{-15}$		
Initial		Estimated			Estimated			Estimated		
E/E_0	ρ/ρ_0	E/E_0	ρ/ρ_0	K	E/E_0	ρ/ρ_0	K	E/E_0	ρ/ρ_0	K
0.25	0.25	0.9990	0.9990	28	0.9990	0.9990	28	0.9991	0.9990	68
0.5	0.5	1.0000	1.0000	21	1.0000	1.0000	21	0.9992	0.9991	61
0.75	0.75	1.0000	1.0000	11	1.0000	1.0000	11	0.9992	0.9991	53
1.25	1.25	0.9990	1.0000	11	1.0000	1.0000	11	1.0009	1.0009	68
1.5	1.5	0.9998	0.9998	11	0.9998	0.9998	11	1.1129	1.2531	100
1.75	1.75	1.0000	1.0000	11	1.0000	1.0000	21	1.1132	1.2534	100

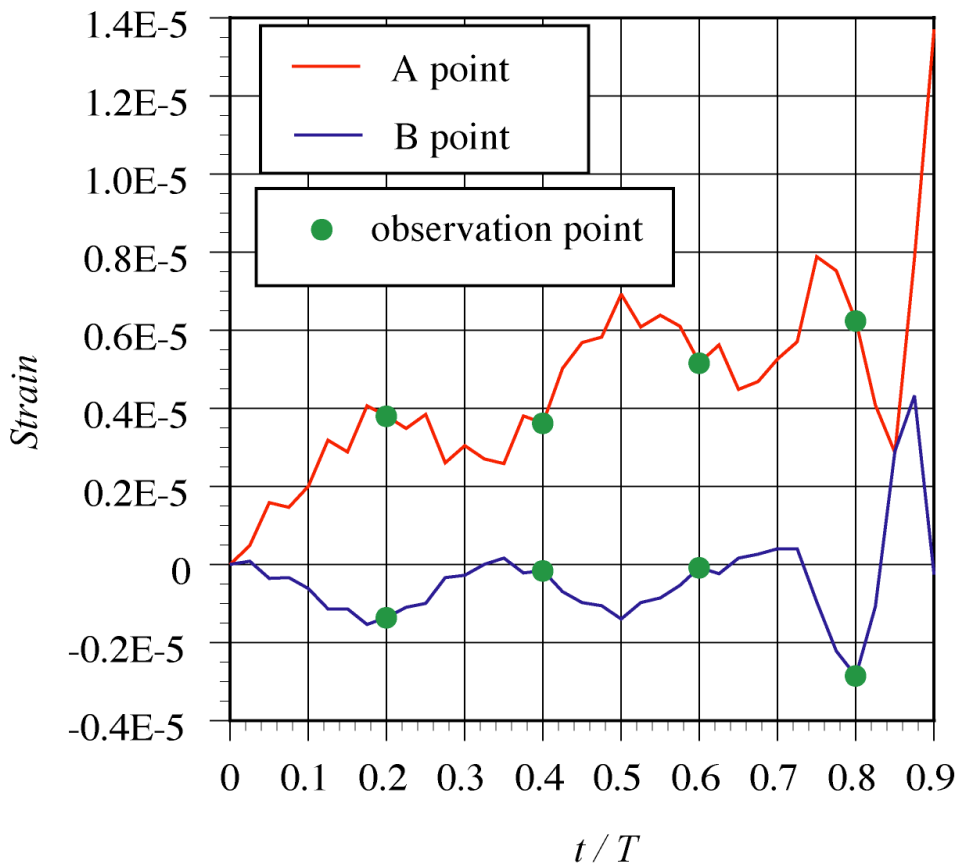
Numerical Results (4)

◆ Increase of measuring points in time

Normal strain in axis x at spatial points A and B



Two pints in time



Four points in time

Numerical Results (4)

Variance of errors:

$$\sigma^2 = 10^{-15}$$

Increase of measuring points in time can improve the solutions.

		2 time point			4 time point		
Initial		Estimated			Estimated		
E/E_0	ρ/ρ_0	E/E_0	ρ/ρ_0	K	E/E_0	ρ/ρ_0	K
0.25	0.25	0.999116	0.999011	68	0.999145	0.999057	45
0.5	0.5	0.999227	0.999136	61	0.999154	0.999066	37
0.75	0.75	0.999178	0.999081	53	0.999243	0.999165	31
1.25	1.25	1.000845	1.000948	68	1.000815	1.000899	31
1.5	1.5	1.112924	1.253080	100	1.000900	1.000993	35
1.75	1.75	1.113206	1.253370	100	1.000760	1.000838	41

Concluding Remarks

- A two-step inverse analysis has been applied to the parameter identification problem of elastic plates subjected to dynamic loading.
- For rough estimation of parameter values, trial computations are carried out for a number of their combinations to find the best one producing the lowest value of cost function.
- It is demonstrated through numerical experiment that precise inverse analysis based on the extended Kalman and BEM can provide a successful estimation if the initial guess of the rough estimation is used.

Concluding Remarks (Continued)

- In the inverse analysis using the extended Kalman filter, resetting the co-variance of estimated parameters after several iterations is very effective for acceleration of convergence.
- Increase of measured data in time could lead to a better estimation of parameters.
- Experimental design can also be applied to rough estimation of parameters.
- Further computation is required for identification of other parameters, e.g. locations of the applied load, its variations in time, etc.