



On the equivalence of method of fundamental solutions and Trefftz method for Laplace equation

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Outline

- ✿ Description of the Laplace problem
- ✿ Trefftz method
- ✿ Method of fundamental solutions (MFS)
- ✿ Connection between the Trefftz method
and the MFS for Laplace equation
- ✿ Concluding remarks
- ✿ Further research

- ✿ **Description of the Laplace problem**
- ✿ **Trefftz method**
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Description of the Laplace problem

Engineering applications:

1. Seepage problem
2. Heat conduction
3. Electrostatics
4. Torsion bar
5. Water wave

Two-dimensional Laplace problem with a circular domain

G.E. :

$$\nabla^2 u(x) = 0, \quad x \in D$$

B.C. :

$$u(x) = \bar{u}, \quad x \in B$$

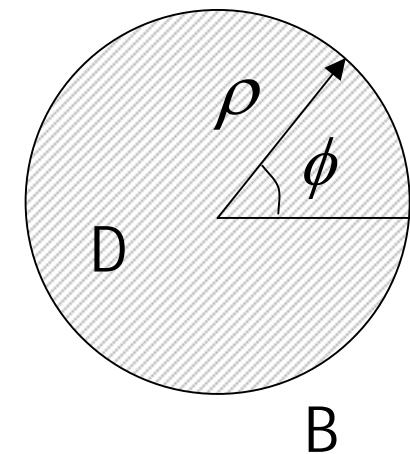
where

∇^2 denotes the Laplacian operator

$u(x)$ is the potential function

ρ is the radius of the circular domain

ϕ is the angle along the circular domain





Fourier series

Orthogonal basis function :

$$1, \cos(n\phi), \sin(n\phi)$$

Fourier series expansion

$$u(\rho, \phi) = \bar{a}_0 + \sum_{n=1}^N \bar{a}_n \cos(n\phi) + \sum_{n=1}^N \bar{b}_n \sin(n\phi)$$

where $\bar{a}_0, \bar{a}_n, \bar{b}_n$ are the Fourier coefficients

Field
Solution

$$u(r, \phi) = \bar{a}_0 + \sum_{n=1}^N \bar{a}_n \left(\frac{r}{\rho}\right)^n \cos(n\phi) + \sum_{n=1}^N \bar{b}_n \left(\frac{r}{\rho}\right)^n \sin(n\phi)$$

where $0 < r < \rho$

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Trefftz method

Field solution :

$$u(x) = \sum_{j=1}^{2N_T+1} w_j u_j(x)$$



where

$2N_T + 1$ is the number of complete functions

w_j is the unknown coefficient

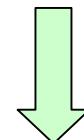
u_j is the T-complete function which satisfies the Laplace equation



T-complete set

T-complete set functions :

$$1, r^n \cos(n\phi), r^n \sin(n\phi)$$



$$u(r,\phi) = a_0 + \sum_{n=1}^{N_T} a_n r^n \cos(n\phi) + \sum_{n=1}^{N_T} b_n r^n \sin(n\phi), \quad 0 < r < \rho$$

$$w_j \rightarrow a_0, a_1, b_1, \dots a_n, b_n \quad n = 0, 1, 2, \dots$$



By matching the boundary condition at $r = \rho$

$$u(\rho, \phi) = a_0 + \sum_{n=1}^{N_T} a_n \rho^n \cos(n\phi) + \sum_{n=1}^{N_T} b_n \rho^n \sin(n\phi).$$

$$u(\rho, \phi) = \bar{a}_0 + \sum_{n=1}^N \bar{a}_n \cos(n\phi) + \sum_{n=1}^N \bar{b}_n \sin(n\phi)$$

→

$$a_0 = \bar{a}_0,$$
$$a_n = \frac{\bar{a}_n}{\rho^n}, \quad n = 1, 2, \dots, N_T$$
$$b_n = \frac{\bar{b}_n}{\rho^n} \quad n = 1, 2, \dots, N_T$$

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Method of Fundamental Solutions ($R > \rho$)

Field solution :



$$u(x) = \sum_{j=1}^{N_M} c_j U(x, s_j), \quad s_j \in D^e$$

where

N_M is the number of source points in the MFS

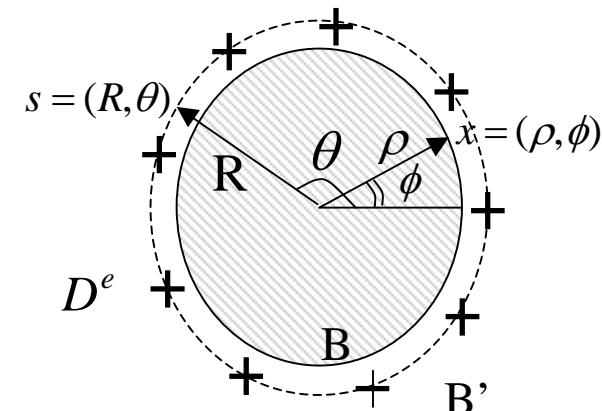
c_j is the unknown coefficient

$U(x, s_j)$ is the fundamental solution

D^e is the complementary domain

S is the source point

x is the collocation point





Degenerate kernel :

$$U(R, \theta, \rho, \phi) = \begin{cases} U^i(R, \theta, \rho, \phi) = \ln(R) - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\rho}{R}\right)^m \cos(m(\theta - \phi)), & R > \rho \\ U^e(R, \theta, \rho, \phi) = \ln(\rho) - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{R}{\rho}\right)^m \cos(m(\theta - \phi)), & R < \rho \end{cases}$$

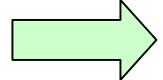
Symmetry property for kernel :

$$U(x, s_j) = U(s_j, x) \longrightarrow$$

$$u(x) = \sum_{j=1}^{N_M} c_j U(s_j, x), \quad s_j \in D^e.$$



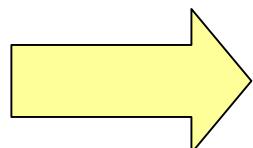
By matching the boundary condition



$$u(r, \phi) = \sum_{j=1}^{N_M} c_j [\ln(R) - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{r}{R}\right)^m \cos(m(\theta_j - \phi))]$$

$$u(\rho, \phi) = \bar{a}_0 + \sum_{n=1}^N \bar{a}_n \cos(n\phi) + \sum_{n=1}^N \bar{b}_n \sin(n\phi)$$

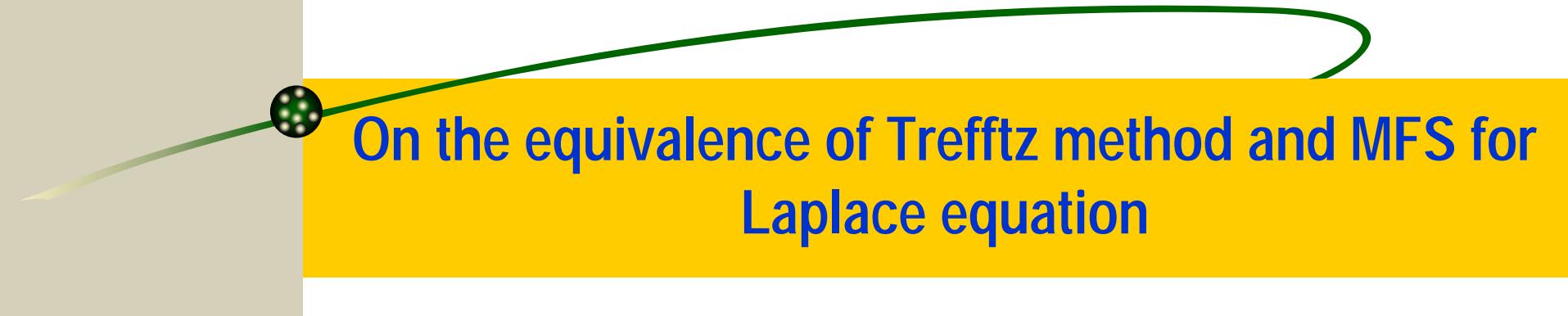
$$\bar{a}_0 = \sum_{j=1}^{N_M} c_j \ln(R)$$



$$\frac{\bar{a}_n}{\rho^n} = - \sum_{j=1}^{N_M} c_j \frac{1}{n} \left(\frac{1}{R}\right)^n \cos(n\theta_j), \quad n = 1, 2, \dots, N_M$$

$$\frac{\bar{b}_n}{\rho^n} = - \sum_{j=1}^{N_M} c_j \frac{1}{n} \left(\frac{1}{R}\right)^n \sin(n\theta_j), \quad n = 1, 2, \dots, N_M$$

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On the equivalence of Trefftz method and MFS for Laplace equation

We can find that the T-complete functions of Trefftz method are imbedded in the degenerate kernels of MFS :①

MFS:

$$U(x, s) = \begin{cases} U^i(x, s) = \ln \bar{\rho} - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\rho}{\bar{\rho}}\right)^m \cos(m(\bar{\theta} - \theta)), & \rho < \bar{\rho} \\ U^e(x, s) = \ln \rho - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\bar{\rho}}{\rho}\right)^m \cos(m(\bar{\theta} - \theta)), & \rho > \bar{\rho} \end{cases}$$

Trefftz:

$$\rho^m \cos m\theta, \rho^m \sin m\theta$$

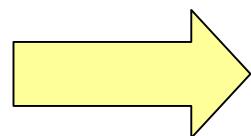
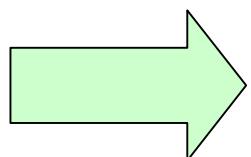


By setting $N_T = N_M = 2N + 1$

$$a_0 = \sum_{j=1}^{2N+1} c_j \ln(R)$$

$$a_n = - \sum_{j=1}^{2N+1} c_j \frac{1}{n} \left(\frac{1}{R}\right)^n \cos(n\theta_j), \quad n = 1, 2, \dots, 2N+1$$

$$b_n = - \sum_{j=1}^{2N+1} c_j \frac{1}{n} \left(\frac{1}{R}\right)^n \sin(n\theta_j), \quad n = 1, 2, \dots, 2N+1$$



$$\{\underline{u}\} = [K] \{\underline{v}\}$$

Trefftz

MFS



in which

$$\langle w_1 \rangle = \ln(R)[1, 1, \dots, 1],$$

$$\langle w_2 \rangle = \left(\frac{-1}{R} \right) [\cos(\theta_1), \cos(\theta_2), \dots, \cos(\theta_{2N+1})],$$

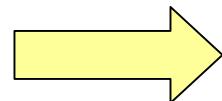
$$\langle w_3 \rangle = \left(\frac{-1}{R} \right) [\sin(\theta_1), \sin(\theta_2), \dots, \sin(\theta_{2N+1})],$$

$$\vdots$$

$$\langle w_{2N} \rangle = \frac{-1}{n} \left(\frac{1}{R} \right)^n [\cos(N\theta_1), \cos(N\theta_2), \dots, \cos(N\theta_{2N+1})],$$

$$\langle w_{2N+1} \rangle = \frac{-1}{n} \left(\frac{1}{R} \right)^n [\sin(N\theta_1), \sin(N\theta_2), \dots, \sin(N\theta_{2N+1})],$$

$$K = \begin{bmatrix} \langle w_1 \rangle \\ \langle w_2 \rangle \\ \vdots \\ \vdots \\ \langle w_{2N+1} \rangle \end{bmatrix}_{(2N+1) \times (2N+1)}$$

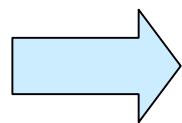


$$[K] = [T_R][T_\theta]$$

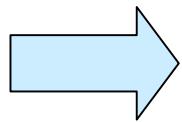
$$[T_\theta] = \begin{bmatrix} 1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\ \cos\theta_1) & \cos\theta_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \cos\theta_{2N+1}) \\ \sin\theta_1) & \sin\theta_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \sin\theta_{2N+1}) \\ \cos(2\theta_1) & \cos(2\theta_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \cos(2\theta_{2N+1}) \\ \sin(2\theta_1) & \sin(2\theta_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \sin(2\theta_{2N+1}) \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \cos(N\theta_1) & \cos(N\theta_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \cos(N\theta_{2N+1}) \\ \sin(N\theta_1) & \sin(N\theta_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \sin(N\theta_{2N+1}) \end{bmatrix}_{(2N+1) \times (2N+1)}$$

Matrix T_θ

$$[T_\theta] = \begin{bmatrix} 1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\ \cos\theta_1) & \cos\theta_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \cos\theta_{2N+1}) \\ \sin\theta_1) & \sin\theta_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \sin\theta_{2N+1}) \\ \cos(2\theta_1) & \cos(2\theta_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \cos(2\theta_{2N+1}) \\ \sin(2\theta_1) & \sin(2\theta_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \sin(2\theta_{2N+1}) \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \cos(N\theta_1) & \cos(N\theta_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \cos(N\theta_{2N+1}) \\ \sin(N\theta_1) & \sin(N\theta_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \sin(N\theta_{2N+1}) \end{bmatrix}_{(2N+1) \times (2N+1)}$$



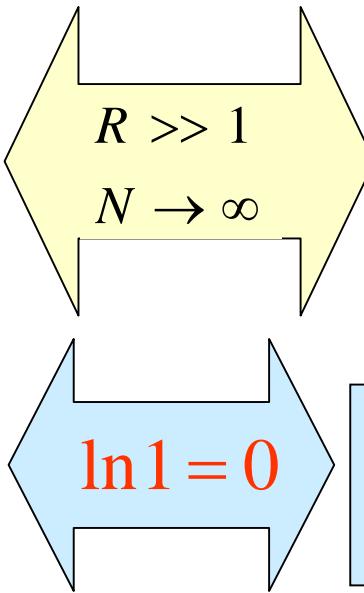
$$[T_\theta][T_\theta]^T = \begin{bmatrix} 2N+1 & 0 & \cdots & \cdots & 0 \\ 0 & \frac{2N+1}{2} & \cdots & \cdots & 0 \\ 0 & 0 & \frac{2N+1}{2} & \cdots & \vdots \\ \vdots & \vdots & \cdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \frac{2N+1}{2} \end{bmatrix}_{(2N+1) \times (2N+1)}$$



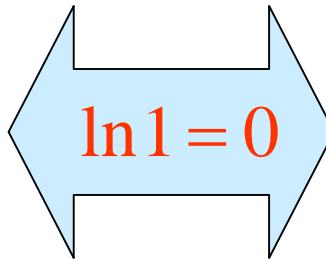
$$\det[T_\theta] = \frac{(2N+1)^{\frac{1}{2}}}{2^N} \neq 0, \quad N \in \text{natural number}$$

Matrix T_R

$$[T_R] = \begin{bmatrix} \ln(R) & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & \frac{-1}{R} & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \frac{-1}{R} & 0 & \cdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \frac{-1}{2}(\frac{1}{R})^2 & \cdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \frac{-1}{2}(\frac{1}{R})^2 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & \frac{-1}{N}(\frac{1}{R})^N & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & \frac{-1}{N}(\frac{1}{R})^N \end{bmatrix}_{(2N+1) \times (2N+1)}$$



**ill-posed
problem**

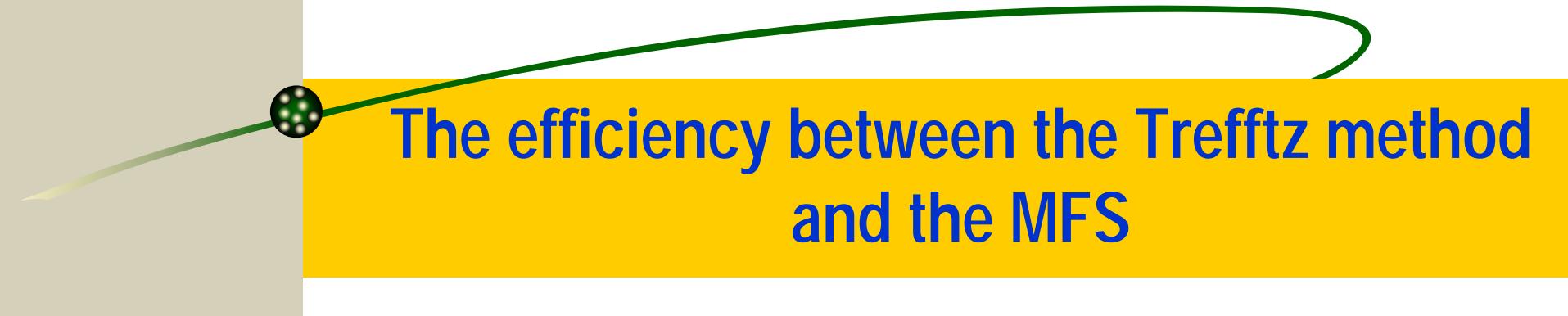


**Degenerate
scale problem**



Degenerate scale problem

1. J. T. Chen, S. R. Kuo and J. H. Lin, 2002, Analytical study and numerical experiments for degenerate scale problems in the boundary element method for two-dimensional elasticity, *Int. J. Numer. Meth. Engng.*, Vol.54, No.12, pp.1669-1681. (SCI and EI)
2. J. T. Chen, C. F. Lee, I. L. Chen and J. H. Lin, 2002 An alternative method for degenerate scale problem in boundary element methods for the two-dimensional Laplace equation, *Engineering Analysis with Boundary Elements*, Vol.26, No.7, pp.559-569. (SCI and EI)
3. J. T. Chen, J. H. Lin, S. R. Kuo and Y. P. Chiu, 2001, Analytical study and numerical experiments for degenerate scale problems in boundary element method using degenerate kernels and circulants, *Engineering Analysis with Boundary Elements*, Vol.25, No.9, pp.819-828. (SCI and EI)
4. 胡海昌, 平面調和函數的充要的邊界積分方程, 1992, 中國科學學報, Vol.4, pp.398-404
5. 胡海昌, 調和函數邊界積分方程的充要條件, 1989, 固體力學學報, Vol.2, No.2, pp.99-104



The efficiency between the Trefftz method and the MFS

We propose an example for exact solution:

$$u(r, \theta) = r^{50} \cos(50\theta),$$

Trefftz method :

$$N_T = 50$$



N=101 terms

MFS :

$$N_M < 50$$



N < 101 terms

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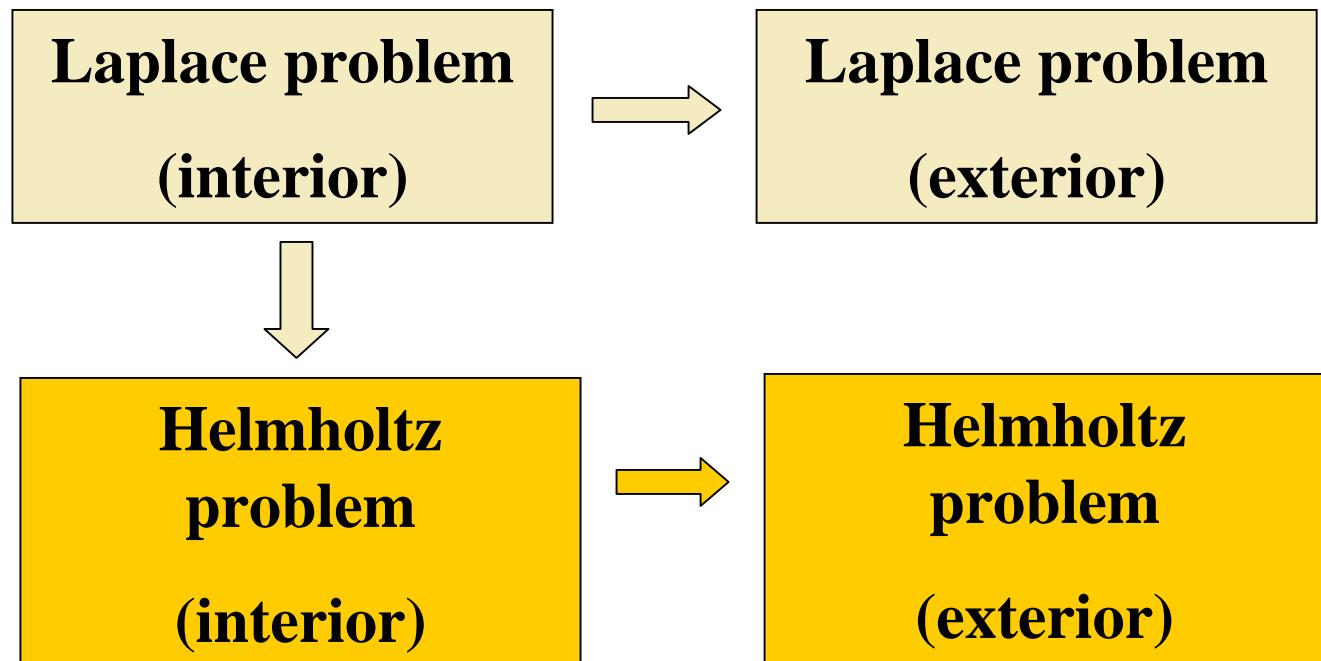
Concluding Remarks

1. The proof of the mathematical equivalence between the Trefftz method and MFS for Laplace equation was derived successfully.
2. The T-complete set functions in the Trefftz method for interior and exterior problems are imbedded in the degenerate kernels of the fundamental solutions as shown in Table 1 for 1-D, 2-D and 3-D Laplace problems.
3. The sources of degenerate scale and ill-posed behavior in the MFS are easily found in the present formulation.
4. It is found that MFS can approach the exact solution more efficiently than the Trefftz method under the same number of degrees of freedom.

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Further research



Simply-connected → Multiply-connected ?

Numerical examples ?



The End

Thanks for your
attention



1

$$\nabla^2 1 = 0$$

$$r^n \cos(n\phi)$$

$$\nabla^2 r^n \cos(n\phi) = 0$$

$$r^n \sin(n\phi)$$

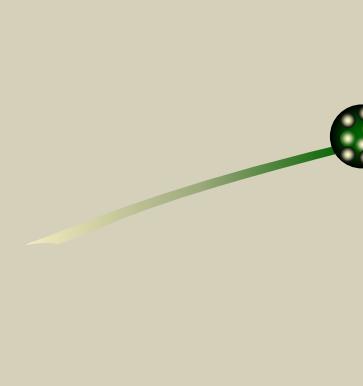
Laplace
equation

$$\nabla^2 r^n \sin(n\phi) = 0$$

where

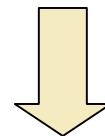
$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2}$$





Fundamental solution

$$\nabla_x^2 U(s, x) = \delta(x - s)$$



$$U(s, x) = \ln(r)$$

$$r = |\underline{s} - \underline{x}|$$

where

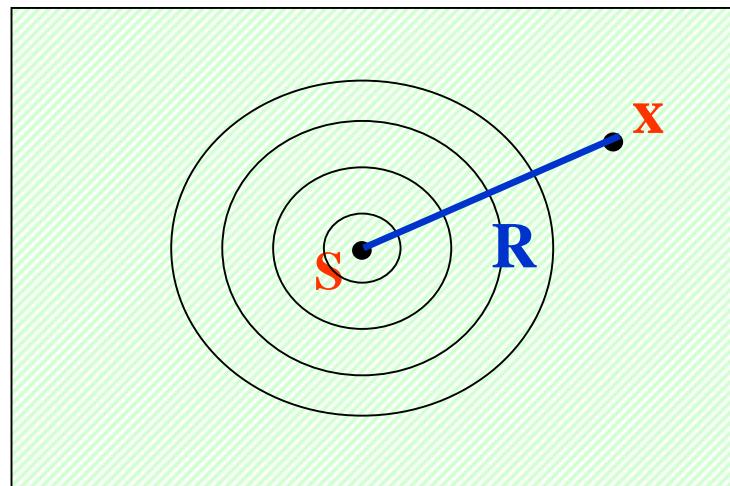
$$\nabla^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2}$$



Degenerate kernel (step1)

Step 1

$$U(s, x) = \ln(R) = \ln|s - x|$$

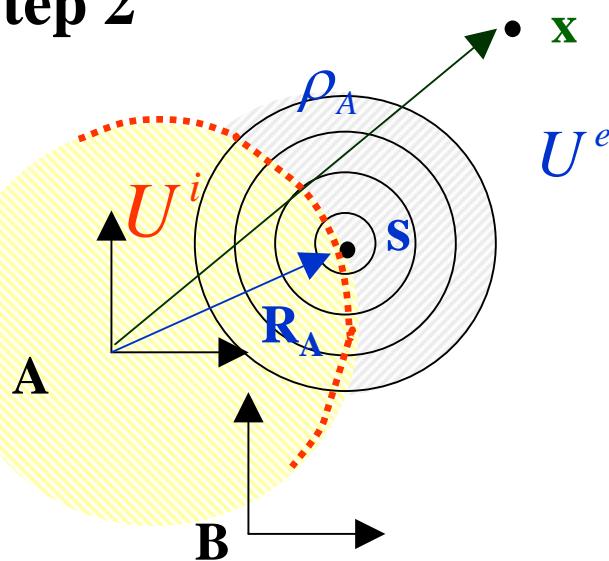


x: variable

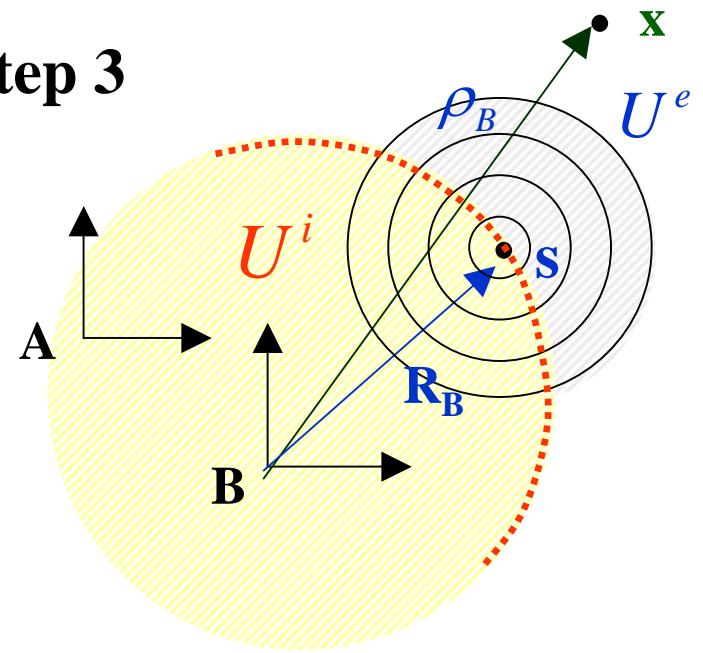
s: fixed

Degenerate kernel (step2, step3)

Step 2



Step 3



$$U^e(R, \theta, \rho, \phi) = \ln(\rho) - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{R}{\rho} \right)^m \cos(m(\theta - \phi)), \quad R > \rho$$

$$U^i(R, \theta, \rho, \phi) = \ln(R) - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\rho}{R} \right)^m \cos(m(\theta - \phi)), \quad R < \rho$$

