方法一：相似轉換 (陳正宗教授參起)

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$\begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{matrix}$

e^{At} 解ODE

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix} \Rightarrow \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 2$$

$$(A - \lambda_1) V_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V_{11} \\ V_{12} \\ V_{13} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow V_{13} = 0$$

$$(A - \lambda_3) V_3 = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} V_{31} \\ V_{32} \\ V_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} V_{31} = 1 \\ V_{32} = 2 \\ V_{33} = 1 \end{cases}$$

重根 $\begin{cases} V_1 = (C_1, C_2, 0) = (1 \ 0 \ 0) \\ V_2 = (C_3, C_4, 0) \quad (0 \ 1 \ 0) \end{cases}$

單根 $\left(V_3 = (1, 2, 1) \quad (1 \ 2 \ 1) \right.$

$$\Rightarrow C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$AC = CD \Rightarrow A = CDC^{-1} \Rightarrow e^{A} = \sum_{n=0}^{\infty} \frac{1}{n!} A^n, \quad C^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$e^{At} \doteq \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e^t & 0 & e^{2t} \\ 0 & e^t & 2e^{2t} \\ 0 & 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^t & 0 & e^{2t} - e^t \\ 0 & e^t & 2e^{2t} - 2e^t \\ 0 & 0 & e^{2t} \end{pmatrix}$$

註：若重根
特徵向量不可求
則需使用到
Jordan form.
此題不需要

$$e^{At} = C \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{bmatrix} C^{-1}$$