Study on the degenerate problems using BEM

Contents

Table captions

Figure captions

Notations

a	radius of a circular with keyway
B	boundary
B_1	boundary of domain
B_2	boundary of domain
b	radius of a circular with keyway
C^+	degenerate boundary
C^-	degenerate boundary
c	rigid body term
D	domain
D^e	complementary domain
G	shear modulus
$H_n^{(1)}(\cdot)$	the n-th order Hankel function of the first kind
$J_n(\cdot)$	the n-th order Bessel function of the first kind
J_n^i	the i-th zero for the n-th order Bessel function $J_n(\cdot)$
$J_n'(\cdot)$	the derivative of $J_n(\cdot)$
$J_n'^i$	the i-th zero for the dervative of n-th order Bessel function $J_n'(\cdot)$
k	wave number
k_i	the i-th true eigenvalue
k_f	fictitious frequency
k_s	spurious eigenvalue
k_t	true eigenvalue
$L(s,x)$	kernel function and matrix of the second dual integral eguation
$L^e(s,x)$	degenerate kernel function of $L(s,x)$ for $R < \rho$
$L^i(s,x)$	degenerate kernel function of $L(s,x)$ for $R > \rho$
l_i	length of boundary element
$M(s,x)$	kernel function and matrix of the second dual integral eguation
$M^e(s,x)$	degenerate kernel function of $M(s,x)$ for $R < \rho$

$M^i(s,x)$	degenerate kernel function of $M(s,x)$ for $R > \rho$
n_i	the i-th component of the outer normal vector at the source point s
r	distance between the source point s and the field point x, $r = \lvert x - s \rvert$
s	position vector of source point
$T(s,x)$	kernel function and matrix of the first dual integral eguation
$T^e(s,x)$	degenerate kernel function of $T(s,x)$ for $R < \rho$
$T^i(s,x)$	degenerate kernel function of $T(s,x)$ for $R > \rho$
T_r	torsional rigidity
$t(s)$	directional derivative of $u(s)$ along the outer normal direction at s
$t(x)$	directional derivative of $u(\mathbf{x})$ along the outer normal direction at x
$U(s,x)$	kernel function and matrix of the first dual integral eguation
$U^e(s,x)$	degenerate kernel function of $U(s,x)$ for $R < \rho$
$U^i(s,x)$	degenerate kernel function of $U(s,x)$ for $R > \rho$
$u(s)$	potential on the source point s
$u(x)$	potential on the field point x
$< v >$	influence row vectors in $[T]$ by collocating CHIEF points
$< w >$	influence row vectors in $[U]$ by collocating CHIEF points
$Y_n(\cdot)$	the n-th order Bessel function of second kind
$Y_n'(\cdot)$	the derivative of $Y_n(\cdot)$
x	position vector of field point
x_1	position vector of field point
x_2	position vector of field point
∇^2	Laplacian operator
(R,θ)	polor coordinates of s
(ρ,ϕ)	polor coordinates of x
(ξ,η)	elliptical coordinate
Σ	diagonal matrix of SVD
Φ	left unitary matrix of SVD
Ψ	right unitary matrix of SVD

α	major axis of ellipse domain
β	minor axis of ellipse domain
τ_{12}	shearing stress
τ_{23}	shearing stress
κ	twist angle per unit length
ϕ_1	left singular vector from ϕ corresponging to the zero singular value
ϕ_i	spurious mode ϕ_s
ψ_1	right singular vector from ψ corresponging to the zero singular value
ψ^D	spurious mode ψ^D corresponding to the Dirichlet problem
ψ^N	spurious mode ψ^N corresponding to the Dirichlet problem

中文摘要

本文提出了邊界積分方程中退化問題的統一觀點, 此退化問題包含了退化尺度、退化邊界、內域眞假特徵值和外域虛擬頻率。而所有的退化問題均源自於影響係數矩陣的秩降現象。針對這些退化問題, 可應用 Fredholm 二擇一定理、奇異值分解法 (SVD), 以及奇異值分解法中的補充行、補充列的技巧來加以探討。邊界元素法中的影響係數矩陣經由奇異值分解所得到 的左酉及右酉單位向量矩陣, 利用 Fredholm 二擇一定理 與奇異值補充行、補充列的技巧 來檢視它們與眞假與虛擬特徵模態的關係。本文針對此退化問題提出一個統一的推導。在退化尺度的問題, 採用三種正規化的方法; 超強奇異積分式、加剛體運動項法、CHEEF 法來 處理秩數不足的問題。此外, 一個 更有效率的方法爲只需一個正規尺度即可找出退化尺度而不需試誤法。同時, 在二維 Laplace 問題證明出退化尺度的存在。我們亦證明出基本解加剛體運動項法在解退化尺度問題時, 新的退化尺度爲 原本退化尺度的e^{-c}倍。含束制條的振動薄膜退化邊界問題, 將採用 傳統邊界元素法伴隨奇異值分解法來解決, 這種方法取代了過去以對偶邊界元素法或多領域邊界元素法解退化邊界問題。內域眞假特徵值與外域虛擬頻率出現的機制 將一貫採用 Fredholm二擇一定理以及奇異值分解法來探討。在本文並提出 CHIEF 與 CHEEF 點數目和位置的判定準則。數值結果均驗證本方法的有效性。

Abstract

We provide a perspective on the degenerate problems, including degenerate scale, degenerate boundary, spurious eigensolution and fictitious frequency, in the boundary integral formulation. All the degenerate problems originate from the rank deficiency in the influence matrix. Both the Fredholm alternative theorem and singular value decomposition (SVD) technique are employed to study the degenerate problems. Updating terms and updating documents of the SVD technique are utilized. The roles of right and left unitary vectors of the influence matrices in BEM and their relations to true, spurious and fictitious modes are examined by using the Fredholm alternative theorem. A unified method for dealing with the degenerate problem in BEM is proposed. For the degenerate scale problem, three regularization techniques, hypersingular formulation, method of adding a rigid body mode and CHEEF concept, are employed to deal with the rank-deficiency problem. Instead of direct searching for the degenerate scale by trial and error, a more efficient technique is proposed to directly obtain the singular case since only one normal scale needs to be computed. The existence of degenerate scale is proved for the two-dimensional Laplace problem using the integral formulation. The addition of a rigid body term, c, in the fundamental solution can shift the original degenerate scale to a new degenerate scale by a factor e^{-c}. Instead of using either the multi-domain BEM or the dual BEM for degenerate-boundary problems, the eigensolutions for membranes with stringers are obtained in a single domain by using the conventional BEM in conjunction with the SVD technique. The occuring mechanism of both the spurious and fictitious eigensolutions are unified by using the Fredholm alternative theorem and SVD technique. The criterion to check the validity of CHIEF and CHEEF points is also addressed. Several examples are demonstrated to check the validity of the proposed method.

Chapter 1
Introduction

1.1 Degenerate problems in BEM

The boundary integral equation method (BIEM) and the boundary element method (BEM) have received much attention since Rizzo [121] proposed a numerical treatment of the boundary integral equation for elastostatics. Most of the efforts have been focused on the singular boundary integral equation for primary fields (*e.g.* potential u or displacement $\mathbf{u}$). For most problems, the formulation of a singular boundary integral equation for the primary field provides sufficient conditions to ensure a unique solution. In some cases, *e.g.*, those with Hermite polynomial elements [131], degenerate boundaries [66, 76, 77, 119], corners [33], the construction of a symmetric matrix [4, 5, 88], the improvement of condition numbers [31], the construction of an image system [31], the tangent flux or hoop stress calculation on the boundary [44], an error indicator in the adaptive BEM [103], fictitious (irregular) frequencies in exterior acoustics [97, 98], spurious eigenvalues in the real-part BEM [36, 100, 101], the imaginary-part BEM [39, 40] and the multiple reciprocity method (MRM) [48, 47, 136, 137], degenerate scale [41, 46, 57, 60, 72] and the Tikhonov formulation for inverse problems, it is found that the integral representation for a primary field can not provide sufficient constraints. In another words, the influence matrices are rank deficient. It is well known that the hypersingular equation plays an important role in the aforementioned problems. Many researchers have paid attention to the hypersingular equation. One can consult the review article on hypersingularity can be found in Chen and Hong [27]. The hypersingular formulation provides the theoretical bases for degenerate boundary problems. Totally speaking, four degenerate problems in BEM, degenerate scale, degenerate boundary, spurious eigenvalues and fictitious frequency, are encountered. In the following, the four rank-deficiency sources are reviewed as follows.

1.2 Degenerate scale for 2-D Laplace and Navier problems

It is well known that rigid body motion test or so called use of simple solution can be employed to examine the singular matrices in BEM for the strongly singular and hypersingular kernels in the problems without degenerate boundaries. Zero eigenvalues associated with rigid body modes are imbedded in the corresponding influence matrices. In such a case, singular matrix occurs physically and mathematically. The nonunique solution for a singular matrix is found to include a rigid body term for the interior Neumann (traction) problem. However, for a certain geometry, the influence matrix of the weakly singular kernel may be singular for the Dirichlet problem [53]. In another words, the numerical results may be unstable when the used scale is changed or the considered domain is expanded to a special size. The nonunique solution is not physically realizable but results from the zero eigenvalue of the influence matrix in the BEM. The special geometry dimension which results in a nonunique solution for a potential problem is called a degenerate scale by He [72] and Chen *et al.* [46]. The term “scale” stems from the fact that degenerate mechanism depends on the geometry size used in the BEM implementation. Some mathematicians [55, 60] coined it a critical value (C.V.) since it is mathematically realizable. For several specific boundary conditions, some studies for potential problems (Laplace equations) [46], plate problems (biharmonic equations) [60] and plane elasticity problems [41, 72] have been done. The difficulties due to nonuniqueness of solutions were overcome by the necessary and sufficient boundary integral formulation [72] and boundary contour method [143]. The degenerate scale problems in the BEM have been studied analytically by Kuhn [99] and Constanda [57] and numerical experiments have been performed [46]. Degenerate kernels and circulant matrices were employed to determine the eigenvalues for the influence matrices analytically in a discrete system for circular and annular problems [46]. The singularity pattern distributed along a ring boundary resulting in a null field can be obtained when the ring boundary is a degenerate scale. An annular region has also been considered for the harmonic equation [76] and the biharmonic equation [111] and the possible degenerate scales were investigated. Hypersingular formulation is an alternative to study the degenerate scale problems for simply-connected problems [41], since eigenvalues are never zero. Another simple approach is to superimpose a rigid body motion in the fundamental solution so that the zero eigenvalue can be shifted to be nonzero. However, this treatment results in another degenerate scale. By employing the CHIEF concept [14], a CHEEF approach was developed to obtain the independent constraint.

A unified method will be proposed to study the problem by using the Fredholm alternative theorem and SVD updating technique. Both the spurious mode (mathematically realizable) and rigid body mode (physically realizable) can be determined. The roles of left and right unitary matrices in SVD for BEM will be examined. In addition, a direct treatment in the matrix operation instead of adding a rigid body term in the fundamental solution can be derived.

1.3 Degenerate boundary in boundary value problems

For the problem with a degenerate boundary, the dual integral representation has been proposed for crack problems in elasticity by Hong and Chen [76, 77], and boundary element researchers [66, 67, 110, 119, 124, 133, 138] have increasingly paid attention to the second equation of the dual representation. The second equation, which is derived for the secondary field (*e.g.*, flux t or traction $\mathbf{t}$), is very popular now and is termed the hypersingular boundary integral equation. Hong and Chen [76] presented the theoretical bases of the dual integral equations in a general formulation which incorporates the displacement and traction boundary integral equations. Huang and So [80] extended the concept of the Hadamard principal value in the dual integral equations [76] to determine the dynamic stress intensity factors of multiple cracks. Gray [66, 67] also independently found the hypersingular integral representations for the Laplace equation and the Navier equation although he did not coin the formulation “dual”. Martin, Rizzo and Gonsalves [109] called the new kernel in the dual integral equations “hypersingular” while Kaya [92] earlier called the kernel “superstrong singularity”. Since the formulation was derived for the secondary field, by analogy with the term “natural boundary condition”, Feng and Yu [65, 139, 141] called the method “natural BEM” or “canonical integral equations”. Balas, Sladek and Sladek in their book [6] proposed a unified theory for crack problems by using the displacement boundary integral equation and another integro-differential equation for the traction field. Based on the dual integral representation for the degenerate boundary problems, Hong and Chen developed the dual BEM programs for crack [76] and potential flow problems with a cutoff wall [35]. Besides, Chen and his coworkers extended the dual BEM program for the Laplace equation and the Navier equation to three programs. One is for the Helmholtz equation by the dual MRM [37]. Another is for the Helmholtz equation by the complex-valued formulation [135, 136]. The other is for the modified Helmholtz equation [50]. A general purpose program, BEASY, was developed for crack problems by the Wessex Institute of Technology (WIT) and termed the "dual boundary element method (DBEM)" [119, 138]. This program has been extended to solve crack growth problems more efficiently by using the benefit of the single-domain approach [101, 138]. Chen and Hong [27], Mi and Aliabadi [110] extended two-dimensional cases to three-dimensional crack problems. A program implemented by Lutz *et al.* [107] was also reported. In the mathematical literature, the relationships between the boundary integral operators and various layer potentials are obtainable through the so-called Calderon projector [31]. Four identities to relate the four kernels have been constructed. The order of pseudo-differential operator for the integral equations on the circular case in the dual formulation was discussed by Amini [2], Chen and Chiu [25]. Detailed discussions can be found in [113, 115]. These mathematical problems were first studied by Hadamard [70] and Mangler [108]. The hypersingular integral equation was derived by Hadamard in solving the cylindrical wave equation by employing the spherical means of descent. The improper integral was then defined by Tuck [129] as the "Hadamard principal value". Almost at the same time of Hadamard's work, Mangler derived the same mathematical form in solving a thin airfoil problem. This is the reason why the improper integral of hypersingularity is called the "Mangler principal value" in theoretical aerodynamics [3]. This nonintegrable integral of hypersingularity [115] arises naturally in the dual boundary integral representations especially for problems with degenerate boundaries, *e.g.*, crack problems in elasticity [31, 76, 77], heat flow through a baffle [29], Darcy flow around a cutoff wall [127], a cracked bar under torsion [23], screen impinging in acoustics [21, 51, 48, 105, 127], antenna in electromagnetic wave [64], a thin breakwater [50] and aerodynamic problems of a thin airfoil [130]. Applications of the hypersingular integral equation in mechanics were discussed by Martin *et al.* [109] and by Chen and Hong [31]. Combining the singular integral equation, *e.g.*, Green's identity (scalar field) or Somigliana's identity (vector field), with the hypersingular integral equation, we can construct the dual integral equations according to the continuous and discontinuous properties of the potential as the field point moves across the boundary [35]. From the above point of view, the definition of the *dual (boundary) integral equations* is quite different from that of the *dual integral equations* given by Sneddon and Lowangrub [125] and Buecker [10], which, indeed, come from the same equation but different collocation points in crack problems of elastodynamics. The solution for the conventional dual integral equations was first studied by Beltrami [56]. The dual boundary integral equations for the primary and secondary fields defined and coined by Hong and Chen are generally independent of each other, and only for very special cases are they dependent [18].

To deal with the degenerate boundary problems, the hypersingular formulation is a powerful method in conjunction with the dual BEM. However, regularization for hypersingularity is required. To avoid hypersingularity, one alternative has been proposed by using the multi-domain approach of singular equation in sacrifice of introducing artificial boundary where the continuity and equilibrium conditions on the interface boundary are considered to condense the matrix. We may wonder whether it is possible to solve the degenerate problems by using only the singular equation in the single-domain approach. The SVD technique will be considered to achieve the goal.

1.4 Spurious eigensolutions for interior eigenproblems

For interior problems, eigendata are very important informations in vibrations and acoustics. According to the complex-valued boundary element method [21, 22, 43], the eigenvalues and eigenmodes can be determined. Nevertheless, complex arithmetic is required. To avoid complex arithmetic, many approaches including the multiple reciprocity method (MRM) [117], the real-part [100, 101, 37] and the imaginary-part BEMs [39, 63] have been proposed. For example, Tai and Shaw [126] employed only real-part kernel in the integral formulation. A simplified method using only the real-part or imaginary-part kernel was also presented by De Mey [63] and Hutchinson [81]. Although De Mey found that the zeros for a real-part of the complex determinant may be different from the determinant using the real-part kernel, the spurious eigensolutions were not discovered analytically. Chen and Wong [47] and Yeih *et al.* [135, 136] found the spurious eigensolutions analytically in the MRM using simple examples of rod and beam, respectively. Later, Kamiya *et al.* [86] and Yeih *et al.* [137] independently claimed that MRM is no more than the real-part BEM. Kang *et al.* [91] employed the Nondimensional Dynamic Influence Function method (NDIF) to solve the eigenproblem. Chen *et al.* [40] commented that the NDIF method is a special case of imaginary-part BEM. Kang and Lee also found the spurious eigensolutions and filtered out the spurious eigenvalues by using the net approach [89]. Later, they extended to solve plate vibration problems [90]. Chen *et al.* [19] proposed a double-layer potential approach to filter out the spurious eigenmodes. The reason why spurious eigenvalues occur in the real-part BEM is the loss of the constraints, which was investigated by Yeih *et al.* [137]. The spurious eigensolutions and fictitious frequencies arise from an improper approximation of the null space operator [123]. The fewer number of constraint equations makes the solution space larger. Spurious eigensolutions were also found in the Maxwell equation [9]. The spurious eigensolutions can be filtered out by using many alternatives, *e.g.*, the complex-valued BEM [22], the domain partition technique [12], the dual formulation in conjunction with the SVD updating techniques [26, 36, 37] and the CHEEF (Combined Helmholtz Exterior integral Equation Formulation) method [14]. Besides, the spurious eigensolution for the multiply-connected problem was found even though the complex-valued kernel was used [45].

A unified formulation to study the phenomenon will be proposed by using the Fredholm alternative theorem and SVD technique. SVD updating techniques in conjunction with the dual formulation will be employed to sort out the true and spurious eigenvalues. In addition, the relation between the left unitary vector in SVD and the spurious mode will be discussed.

1.5 Fictitious frequency in exterior acoustics

For exterior acoustics, the solution to the boundary is perfectly unique for all wave numbers. This is not the case for the numerical treatment of integral equation formulation, which breaks down at certain frequency known as irregular frequency or fictitious frequency. This problem is completely nonphysical because there are no discrete eigenvalues for the exterior problems. It was found that the singular (*UT*) equation results in fictitious frequencies which are associated with the interior eigenfrequency of the Dirichlet problems while the hypersingular (*LM*) equation produces fictitious frequencies which are associated with the interior eigenfrequency of the Neumann problems [18]. The general derivation was provided in a continuous system [18], and a discrete system was analytically studied using the properties of circulant for a circular case [20, 38]. Schenck [122] proposed a CHIEF (Combined Helmholtz Interior integral Equation Formulation) method, which is easy to implement and is efficient but still has some drawbacks. Burton and Miller [11] proposed an integral equation that was valid for all wave numbers by forming a linear combination of the singular integral equation and its normal derivative through an imaginary constant. In case of a fictitious frequency, the resulting coefficient matrix for the exterior acoustic problems becomes ill-conditioned. This means that the boundary integral equations are not linearly independent and the resulted matrix is rank deficient. In the fictitious-frequency case, the rank of the coefficient matrix is less than the number of the boundary unknowns. The SVD updating technique can be employed to detect the possible fictitious frequencies and modes by checking whether the first minimum singular value, σ_1, is zero or not [14].

By employing the Fredholm alternative theorem and SVD updating technique, the degenerate mechanism for the four numerical problems, degenerate boundary, degenerate scale, spurious eigenvalues and fictitious frequencies, will be studied. A unified formulation will be constructed to solve for rank-deficiency problems. Illustrative examples will be illustrated to check the validity of the proposed method.

1.6 Scope of the thesis

In this thesis, the degenerate problems, degenerate boundary, degenerate scale, spurious eigenvalues and fictitious frequencies, will be studied by using the BEM in conjunction with the Fredholm alternative theorem and SVD updating technique. The emphasis of each chapter are summarized below. In Chapter 2, a more efficient technique is proposed to directly obtain the singular case since only one normal scale needs to be computed without direct searching for the degenerate scale by trial and error. We will prove the existence of degenerate scale for the two-dimensional Laplace problem using the integral formulation. Besides, it is found that the addition of a rigid body term, c, in the fundamental solution can shift the original degenerate scale to a new degenerate scale by a factor e^{-c}. To deal with the numerical instability due to the degenerate scale, three approaches, method of adding a rigid body mode, hypersingular formulation and CHEEF method, will be applied to remove the zero singular value. In Chapter 3, instead of using either the multi-domain BEM or the dual BEM, the degenerate boundary eigenproblem will be solved by using the conventional BEM in conjunction with the SVD technique. Chapter 4 will focus on sorting out the true and spurious eigenvalues with the Fredholm alternative theorem and SVD techniques in conjunction with the dual BEM. In addition, we also review the four methods, the complex-valued formulation, the real-part, the imaginary-part BEMs and MRM. The possible occurence of spurious eigensolutions in the four approaches will be addressed. In Chapter 5, we obtain the ficitious modes in the singular vectors of SVD as well as the true eigenmodes for the interior problems at the same time once the updating matrix was decomposed by using the SVD technique. A criterion for checking the minimum number and validity of the CHIEF points will be studied analytically in the discrete system.

Chapter 2

Degenerate scale for torsion bar problems with arbitrary cross sections using the dual BEM

Summary

In this thesis, torsion bar problems are solved by using the dual BEM. It is found that a degenerate scale problem occurs if the conventional BEM is used. In this case, the influence matrix is rank deficient and numerical results become unstable. Both the circular and elliptical bars are studied analytically in the continuous system. In the discrete system, the Fredholm alternative theorem in conjunction with the SVD updating documents is employed to sort out the spurious mode which causes the numerical instability. Three regularization techniques, method of adding a rigid body mode, hypersingular formulation and CHEEF concept, are employed to deal with the rank-deficiency problem. The existence of degenerate scale is proved for the two-dimensional Laplace problem using the integral formulation. The addition of a rigid body term, c, in the fundamental solution can shift the original degenerate scale to a new degenerate scale by a factor e^{-c}. The torsion rigidities are obtained and compared with analytical solutions. Numerical examples including elliptical, square, triangular bars and circular bar with keyway under torsion, were demonstrated to show the failure of conventional BEM in case of the degenerate scale. After employing the three regularization techniques, the accuracy of the proposed approaches is achieved.

2-1 Introduction

During the last three decades, boundary element method (BEM) has been recognized as an acceptable tool for engineering analysis [8, 61]. However, there still exists some pitfalls imbedded in the BEM, e.g., rank-deficiency problems. The well-known one is the fictitious (irregular) frequency in the exterior acoustics. Burton and Miller [11] solved the problem by combining singular and hypersingular equations with an imaginary constant. Chen *et al.* [45] extended the Burton and Miller method to filter out the spurious eigenvalues in the multiply-connected eigenproblem. Schenck [122] proposed a Combined Helmholtz Interior integral Equation Formulation (CHIEF) method, which is easy to implement by applying the integral equation on a number of points located outside the domain of interest. It is efficient to overcome the problem of nonunique solutions in case of fictitious frequency, but it still has some drawbacks since the chosen point may fail. How to determine the number of points and how to choose their positions were discussed by Chen *et al.* [15]. In a similar way for the interior eigenproblem, the CHEEF technique [14] instead of the CHIEF concept was applied to filter out spurious eigenvalues successfully by adding constraints from the points outside the domain in the multiple reciprocity BEM [48], real-part BEM [100] and imaginary-part BEM [39]. Rank-deficiency problems also occur when BEM is applied to deal with crack or corner problems. Dual formulation in conjunction with the hypersingular equation has recieved attention in the last decade. A review article can be found in [34].

In the BEM implementation, the rigid body motion or so called constant potential test is always employed to examine the singular matrices of strongly singular kernels and hypersingular kernels for the problems without degenerate boundaries. Lutz *et al.* [107] termed it a simple solution. Based on this concept, diagonal terms of a singular influence matrix can be easily determined. Singular matrix occurs physically and mathematically in the sense that the nonunique solution for the singular matrix implies an arbitrary rigid body term for the interior Neumann (traction) problem. However, the influence matrix of the weakly singular kernel may be singular for the Dirichlet problem [53] when the geometry is special. The nonunique solution is not physically realizable but results from the zero singular value in the influence matrix by using the BEM. From the point of view of linear algebra, the problem also originates from the rank deficiency in the influence matrices. For example, the nonunique solution of a circle with a unit radius has been noted by Petrosky [118] and by Jaswon and Symm [84]. The special geometry which results in a nonunique solution for a potential problem is called “degenerate scale”. The term “scale” stems from the fact that the numerical instability of a unit circle of radius 1 m (1 cm) disappears if the radius of 100 cm (0.01 m) is used in the BEM implementation. Christiansen [54, 55] termed it a critical value (C.V.) since it is mathematically realizable. In real implementation, we need to avoid the number one for the circular radius using the normalized scale. The numerical difficulties due to nonuniqueness of solutions have been solved by using the necessary and sufficient boundary integral equation (NSBIE) [71, 73, 74, 72] and boundary contour method [143]. Also, the degenerate scale of multiply-connected problems was discussed for the Laplace equation by Tomlinson *et al.* [128]. The nonunique solution for the multiply-connected biharmonic problems was also studied by Mitra and Das [111]. Chen *et al.* [46] studied the degenerate scale for the simply-connected and multiply-connected problems by using the degenerate kernels and circulants in a discrete system for circular and annular cases. Mathematically speaking, the singularity pattern distributed along a ring boundary resulting in a null-field solution introduces a degenerate scale. This concept was also extended to study the spurious eigenvalues for annular cavities by Chen *et al.* [45]. The similar application to the two-dimensional elasticity was addressed in [41]. A rigorous study was proposed mathematically by Kuhn [99] and Constanda [57, 59] for the occurring mechanism of the degenerate scale. SVD technique has been used to detect the nonunique solution in case of degenerate scale [55]. Three regularization techniques will be employed to avoid the zero singular value. One alternative to treat the problem is to superimpose a rigid body term in the fundamental solution for the BEM formulation. Although the degenerate scale problem can be circumvented for the special geometry, the degenerate scale will be proved to move to another size. Another alternative of hypersingular formulation is employed to shift the zero eigenvalue in paying the price of determining the Hadamard principal value. By adopting the CHEEF concept for obtaining an independent constraint, we can also deal with the degenerate scale problems free of hypersingularity.

In this thesis, we will focus on the analytical investigation for the phenomenon of degenerate scales in the BEM for torsion problems in continuous and discrete systems. The degenerate scale for the elliptical bar under torsion will be derived analytically in a continuous system by using the elliptical coordinate. Circular domain is a special case for check. The degenerate kernel and circulant are employed to derive the degenerate scale in the continuous and discrete systems, respectively. Any simply-connected problem will be proven to have a degenerate scale. Also, the rigid body term c will be proved to move the original degenerate scale to the new degenerate scale by a factor of e^{-c}. In the discrete system, the Fredholm alternative theorem in conjunction with SVD updating document will be employed to find the degenerate scale and the corresponding spurious mode. The relation between the spurious mode and unitary vector in SVD will be constructed. Also, we will propose three alternatives, method of adding a rigid body mode, hypersingular formulation and CHEEF technique, to overcome the nonunique solution in the numerical implementation. Method of adding a rigid body mode in the fundamental solution can shift the zero singular value in the conventional BEM. Instead of using the conventional BEM, the second equation in the dual BEM, i.e., hypersingular formulation, can avoid the zero singular value. By using the CHEEF technique, the addition of a constraint by collocating the points outside the domain can promote the rank of the singular matrix. The optimum number and appropriate positions for the collocating points will be addressed. Numerical examples, torsion problems of elliptical, square, triangular bars and circular bar with keyway, will be demonstrated to see the numerical instability for the degenerate scale problems. The treament for the suppression of numerical instability will be done.

2-2 Dual boundary integral formulation and dual BEM for torsion problems

The torsion problem of a bar with an arbitrary cross section in Fig.2-1 can be formulated by the Poisson equation as follows [23, 120]:

$$\bigtriangledown^2 u^*(x_1, x_2) = -2, \qquad (x_1, x_2) \in D, \tag{2-1}$$

where u^* is the torsion (Prandtl) function, ∇^2 is the Laplacian operator and D is the domain. The boundary condition is

$$u^*(x_1, x_2) = 0, \qquad (x_1, x_2) \in B, \tag{2-2}$$

where B is the boundary. Since Eq.(2-1) contains the body source term which results in a domain integral by using the BEM, the problem can be reformulated to

$$\bigtriangledown^2 u(x_1, x_2) = 0, \qquad (x_1, x_2) \in D, \tag{2-3}$$

and the boundary condition is changed to

$$u(x_1, x_2) = \frac{(x_1^2 + x_2^2)}{2}, \qquad (x_1, x_2) \in B, \tag{2-4}$$

where the torsion function u^* can be obtained from u by superimposing $\tilde{u}$, $u = u^* + \tilde{u}$ and $\tilde{u} = \frac{(x_1^2+x_2^2)}{2}$.

This new model for the torsion problem using Eq.(2-3) is the Laplace equation subject to the Dirichlet data of Eq.(2-4), which is very easy to implement using the DBEM, e.g., the BEPO2D program can be used in this study, [31]. The torque, M_z, can then be determined by

$$M_z = \iint\limits_D (x_1\tau_{23} - x_2\tau_{13})\, dx_1\, dx_2, \tag{2-5}$$

where τ_{23} and τ_{13} are the shearing stresses determined by $\tau_{23} = -\kappa G \frac{\partial u^*}{\partial x_1}$ and $\tau_{13} = \kappa G \frac{\partial u^*}{\partial x_2}$, G is the shear modulus and κ denotes the twist angle per unit length.

By employing the Green's second identity and Eq.(2-1), the area integral in Eq. (2-5) can be transformed into a boundary integral and a domain integral as follows:

$$\begin{aligned} M_z &= \iint\limits_D (x_1\tau_{23} - x_2\tau_{13})\, dx_1\, dx_2 \\ &= -\kappa G \iint\limits_D (x_1\frac{\partial u^*}{\partial x_1} + x_2\frac{\partial u^*}{\partial x_2})\, dx_1\, dx_2 \\ &= -\kappa G \iint\limits_D (\nabla\tilde{u}\cdot\nabla u^*)\, dx_1\, dx_2 \\ &= -\kappa G \iint\limits_D \nabla\cdot(\tilde{u}\nabla u^*)\, dx_1\, dx_2 + \kappa G \iint\limits_D \tilde{u}\nabla^2 u^*\, dx_1\, dx_2 \\ &= -\kappa G \oint_B \tilde{u}\frac{\partial u^*}{\partial n}\, dB - \kappa G \iint\limits_D (x_1^2 + x_2^2)\, dx_1\, dx_2. \end{aligned} \tag{2-6}$$

The induced area integral of the second term on the right hand side of the equal sign in Eq.(2-6) can be reformulated into a boundary integral again by using the Gauss theorem as follows:

$$\begin{aligned} -\kappa G \iint\limits_D (x_1^2 + x_2^2)\, dx_1\, dx_2 &= \frac{-\kappa G}{16} \iint\limits_D \nabla^2\{(x_1^2 + x_2^2)^2\}\, dx_1\, dx_2 \\ &= \frac{-\kappa G}{16} \oint\limits_B \frac{\partial\{(x_1^2 + x_2^2)^2\}}{\partial n}\, dB. \end{aligned} \tag{2-7}$$

The torsion problem can be simulated by using the mathematical model of the Laplace equation as shown in Eq.(2-3). Now, we will consider the boundary integral formulation for numerical analysis. Using the Green's identity, the first equation of the dual boundary regular integral equations for the domain point x can be derived as follows:

$$2\pi u(x) = \int_B T(s,x)u(s)\,dB(s) - \int_B U(s,x)\frac{\partial u(s)}{\partial n_s}\,dB(s), \tag{2-8}$$

where

$$\begin{aligned} U(s,x) &\equiv ln(r), && (2\text{-}9)\\ T(s,x) &\equiv \frac{\partial U(s,x)}{\partial n_s}, && (2\text{-}10)\end{aligned}$$

in which r is the distance between the field point x and the source point s, and n_s is the normal vector for the boundary point s. After taking the normal derivative of Eq.(2-8), the second equation of the dual boundary regular integral equations for the domain point x can be derived:

$$2\pi\frac{\partial u(x)}{\partial n_x} = \int_B M(s,x)u(s)dB(s) - \int_B L(s,x)\frac{\partial u(s)}{\partial n_s}dB(s), \tag{2-11}$$

where

$$\begin{aligned} L(s,x) &\equiv \frac{\partial U(s,x)}{\partial n_x}, && (2\text{-}12)\\ M(s,x) &\equiv \frac{\partial^2 U(s,x)}{\partial n_x \partial n_s}, && (2\text{-}13)\end{aligned}$$

in which n_x is the normal vector for the field point x. Eqs.(2-8) and (2-11) are coined the dual boundary regular integral equations for the domain point x. The explicit forms of the kernel functions can be found in [31]. By tracing the field point x to the boundary, the dual boundary singular integral equations for the boundary point x can be derived:

$$\pi u(x) = C.P.V.\int_B T(s,x)u(s)dB(s) - R.P.V.\int_B U(s,x)\frac{\partial u(s)}{\partial n_s}dB(s), \tag{2-14}$$

$$\pi\frac{\partial u(x)}{\partial n_x} = H.P.V.\int_B M(s,x)u(s)dB(s) - C.P.V.\int_B L(s,x)\frac{\partial u(s)}{\partial n_s}dB(s), \tag{2-15}$$

where $R.P.V.$, $C.P.V.$ and $H.P.V.$ denote the Riemann principal value, Cauchy principal value and Hadamard or Mangler principal value, respectively. After discretizing the boundary into $2N$ boundary elements, Eqs.(2-14) and (2-15) reduce to

$$[U]_{2N\times 2N}\,\{t\}_{2N\times 1} = [T]_{2N\times 2N}\,\{u\}_{2N\times 1}\,, \tag{2-16}$$

$$[L]_{2N\times 2N}\,\{t\}_{2N\times 1} = [M]_{2N\times 2N}\,\{u\}_{2N\times 1}\,, \tag{2-17}$$

where $[U]$, $[T]$, $[L]$ and $[M]$ are the four influence matrices which can be found in [31], $\{u\}$ and $\{t\}$ are the boundary data for the primary and the secondary boundary variables, respectively.

To determine the torsion rigidity using Eq.(2-6), the following boundary integral can be integrated numerically as follows:

$$\oint_B \tilde{u}\frac{\partial u^*}{\partial n}\,dB = \oint_B \tilde{u}\frac{\partial u}{\partial n}\,dB - \oint_B \tilde{u}\frac{\partial \tilde{u}}{\partial n}\,dB = \sum_{j=1}^{2N} \tilde{u}_j[(\frac{\partial u}{\partial n})_j - (\frac{\partial \tilde{u}}{\partial n})_j]\,l_j, \tag{2-18}$$

where $(\frac{\partial u}{\partial n})_j$ is the normal derivative of u for the j^{th} boundary element, l_j is the length of the j^{th} boundary element and another boundary integral in Eq.(2-7) can be discretized as follows:

$$\oint_B \frac{\partial\{(x_1^2+x_2^2)^2\}}{\partial n}\,dB = 4\sum_{j=1}^{2N}(\frac{\partial \tilde{u}^2}{\partial n})_j\,l_j. \tag{2-19}$$

2-3 Proof of the existence for the degenerate scale of the two-dimensional Laplace problem using the integral formulation

Theorem 1: Existence theorem

[Proof] :

For any two-dimensional Laplace problem with a simply-connected domain, there exists a degenerate scale when we solve the problem by using the boundary integral formulation or BEM.

For two-dimensional potential problems, there exists a unique solution for $\psi_1(s)$ satisfying

$$u(x) = \int_B U(s,x)\psi_1(s)\;dB(s),\;\; x\in B, \tag{2-20}$$

where B is the normal boundary with the enclosing domain D. For simplicity, we can assume a constant potential field since it is a “simple solution” for the Laplace equation. Eq.(2-20) reduces to

$$1 = \int_B U(s,x)\,\psi_1(s)\;dB(s),\;\; x\in B. \tag{2-21}$$

When the degenerate scale B_d occurs, the nonunique solution of Eq.(2-21) implies that

$$0 = \int_{B_d} U(s,x)\,\psi_1(s)\;dB(s),\;\; x\in B_d, \tag{2-22}$$

has a nontrivial solution for $\psi_1(s)$, where B_d is the boundary of degenerate scale using the fundamental solution $U(s,x) = \ln(r)$. By expressing the boundary contour in terms of $f(x_1, x_2) = 0$, we have a new closed boundary curve

$$f(\frac{x_1}{d}, \frac{x_2}{d}) = 0, \tag{2-23}$$

where d is the expansion ratio. The two boundary curves, B and B_d, are shown in Fig.2-2(a). By mapping the nondegenerate (normal) boundary to the degenerate boundary, we have

$(x_1, x_2) \Rightarrow (x_1\, d, x_2\, d) = (x_1, x_2)\, d$,

$dB(s) \Rightarrow dB(s\, d) = dB(s)\, d$,

$U(s,x) \Rightarrow U(s\, d, x\, d) = U(s,x) + \ln(d)$,

$\psi_1(s) \Rightarrow \bar{\psi}_1(s\, d) = \psi_1(s)$.

According to mapping properties, the homogeneous Eq.(2-22) yields

$$0 = \int_{B_d} U(s\, d, x\, d)\ \bar{\psi}_1(s\, d)\ dB(s\, d). \tag{2-24}$$

In order to have a nontrivial solution for Eq.(2-24), we have

$$\begin{aligned} 0 \ &= \int_B d\ (U(s,x) + \ln(d))\ \psi_1(s)\, dB(s) \\ &= d + d\ \ln(d) \int_B \psi_1(s)\, dB(s) = d + d\ \ln(d)\ \Gamma, \end{aligned} \tag{2-25}$$

after using Eq.(2-21) and defining

$$\Gamma = \int_B \psi_1(s) dB(s). \tag{2-26}$$

According to Eq.(2-25), the degenerate scale occurs when the expansion ratio, d, satisfies

$$d = e^{-\frac{1}{\Gamma}}. \tag{2-27}$$

For determing the degenerate scale systematically from one trial on a normal scale, we provided a flowchart shown in Fig.2-2(b).

Here, a simple example of a circle with a radius, a, is demonstrated to verify Eq.(2-27). According to Eq.(2-21), we have

$$\psi_1(s) = \frac{1}{2\ \pi\ a\ \ln(a)}. \tag{2-28}$$

By using Eq.(2-26), we can determine

$$\Gamma = \int_B \frac{1}{2\ \pi\ a\ \ln(a)} dB(s) = \frac{1}{\ln(a)}. \tag{2-29}$$

Substituting Eq.(2-29) to Eq.(2-27), the expansion ratio is

$$d = e^{-\ln(a)} = \frac{1}{a}. \tag{2-30}$$

After expanding the radius a by multiplying the expansion ratio, $\frac{1}{a}$, the degenerate scale of radius with a unit length is proved. The numerical results for the circle are shown to match well with the analytical solutions in the second column of Table 2-1.

2-4 Proof of the expansion ratio of e^{-c} for the new degenerate scale after adding a rigid body term c in the fundamental solution

Theorem 2:

The boundary of $g(x_1, x_2) = 0$, which is a degenerate scale using the fundamental solution $(U(s, x) = \ln(r))$ is changed to a new degenerate scale of $g(\frac{x_1}{e^{-c}}, \frac{x_2}{e^{-c}}) = 0$ using the modified fundamental solution $(U^*(s, x) = \ln(r) + c)$.

[Proof] :

If the degenerate scale B_d ($g(x_1, x_2) = 0$) occurs, the fundamental solution $U(s, x)$ can be modified to $U(s, x) + c$ to avoid the singular case. In other words, there is a unique solution $\psi_1(s)$ for the following equation,

$$1 = \int_{B_d} [U(s, x) + c]\, \psi_1(s)\, dB(s). \tag{2-31}$$

In a similar way, we expand the normal boundary B_d $(U(s, x) = \ln(r) + c)$ in Eq.(2-31) to the "new degenerate scale", B_{d^*}, by using the modified fundamental solution as shown in Fig.2-2(a). The homogeneous Eq.(2-31) reduces to

$$0 = \int_{B_{d^*}} [U(s\, d^*, x\, d^*) + c]\, \psi_1(s\, d^*)\, dB(s\, d^*). \tag{2-32}$$

In the new degenerate scale, B_{d^*}, for the case of modified fundamental solution $(U(s, x) = \ln(r) + c)$, it means that Eq.(2-32) has a nontrivial solution. By using mapping properties, $dB(s\, d^*) =$

$d^* \, dB(s)$ and $U(s\,d^*, x\,d^*) = U(s,x) + \ln\, d^*$, Eq. (2-32) reduces to

$$\begin{aligned} 0 &= d^* \int_{B_d} [(U(s,x) + \ln(d^*) + c]\psi_1(s)dB(s) \\ &= d^* \ln(d^*) \int_{B_d} \psi_1(s)dB(s) + c\, d^* \int_{B_d} \psi_1(s)dB(s) + d^* \int_{B_d} U(s,x)\psi_1(s)dB(s). \end{aligned} \tag{2-33}$$

Since $\int_{B_d} U(s,x)\,\psi_1(s)\,dB(s) = 0$ in the original degenerate scale, Eq. (2-33) simplifies to

$$\begin{aligned} 0 &= d^* \ln(d^*) \int_{B_d} \psi_1(s)dB(s) + cd^* \int_{B_d} \psi_1(s)dB(s) \\ &= (\ln(d^*) + c) \int_{B_d} \psi_1(s)dB(s). \end{aligned} \tag{2-34}$$

The expansion ratio, d^*, satisfying

$$d^* = e^{-c}, \tag{2-35}$$

results in a new degenerate scale in Eq.(2-35). To demonstrate the accuracy of Eq.(2-35), a special case of circular bar will be disscussed in the following section in detail.

2-5 Mathematical analysis of the degenerate scale for an elliptical bar under torsion

For an elliptical bar under torsion as shown in Fig.2-3(a), the governing equation is also

$$\nabla^2 u(x_1,\, x_2) = 0, \quad (x_1,\, x_2)\; \in \;\; D. \tag{2-36}$$

To study the degenerate scale for an elliptical bar [104], we consider an infinite domain and use the elliptic coordinate ξ and η defined by

$$z = k\cosh\zeta, \qquad \zeta = \xi + i\eta. \tag{2-37}$$

where z is the complex plane $(x_1 + ix_2)$, k is a constant and

$$x_1 = k\cosh\xi\cos\eta, \qquad x_2 = k\sinh\xi\sin\eta. \tag{2-38}$$

The coordinate ξ is a constant and is equal to ξ_0 on the ellipse of the semiaxes $k\cosh\xi_0$ and $k\sinh\xi_0$ as shown in Fig.2-3(b). If the semiaxes are given as α and β, k and ξ_0 can be determined by

$$k = \sqrt{\alpha^2 - \beta^2}, \qquad \xi_0 = \tanh^{-1}\left(\frac{\alpha}{\beta}\right). \tag{2-39}$$

We assume u_i and u_e, for the interior and exterior potentials as shown in Fig.2-3(b), respectively,

$$\begin{aligned} u_i(\xi,\eta) &= c_1, && \text{(2-40)} \\ u_e(\xi,\eta) &= c_2 + c_3\xi, && \text{(2-41)} \end{aligned}$$

where the subscripts "i" and "e" denote the interior or exterior point separated by the elliptical boundary $\xi = \xi_0$, respectively. When ξ approaches infinity, we have the asymptotic form

$$r = |z| = |k\cosh\zeta| \simeq \frac{k}{2}e^{\xi}, \quad \text{(2-42)}$$

and

$$\xi \simeq \ln(r) - \ln(\frac{k}{2}). \quad \text{(2-43)}$$

When ξ approaches infinity, the exterior potential approaches $\ln r$ and the coefficient c_2 must be chosen as $c_3\ln(\frac{k}{2})$. The potential in the exterior domain is

$$u_e(\xi,\eta) = c_3(\xi + \ln(\frac{k}{2})). \quad \text{(2-44)}$$

On the other hand, when ξ approaches ξ_0 on the elliptical boundary, we have

$$\begin{aligned} u_e(\xi,\eta) &= c_3(\xi_0 + \ln(\frac{k}{2})) \\ &= c_3(\tanh^{-1}\left(\frac{\beta}{\alpha}\right) + \frac{1}{2}\ln(\alpha^2-\beta^2) - \ln(2)). && \text{(2-45)} \end{aligned}$$

after using Eq.(2-39). If we set

$$\tanh(x) = \left(\frac{e^{2x}-1}{e^{2x}+1}\right) = \chi, \quad \text{(2-46)}$$

we have

$$x = \frac{1}{2}\ln\left(\frac{1+\chi}{1-\chi}\right) = \tanh^{-1}\chi. \quad \text{(2-47)}$$

By setting χ to be $(\frac{\beta}{\alpha})$, we have

$$\tanh^{-1}\left(\frac{\beta}{\alpha}\right) = \frac{1}{2}\ln\left(\frac{\alpha+\beta}{\alpha-\beta}\right). \quad \text{(2-48)}$$

The exterior potential in Eq.(2-45) becomes

$$\begin{aligned} u_e(\xi,\eta) &= c_3(\frac{1}{2}\ln\left(\frac{\alpha+\beta}{\alpha-\beta}\right) + \frac{1}{2}\ln(\alpha^2-\beta^2) - \ln(2)) \\ &= c_3\ln\left(\frac{\alpha+\beta}{2}\right). && \text{(2-49)} \end{aligned}$$

For the continuity of displacement across the boundary, the displacement by approaching from the exterior domain must equal to the potential by approaching from the interior domain. We have

$$c_1 = \ln\left(\frac{\alpha+\beta}{2}\right)c_3, \tag{2-50}$$

and the potential can be written as

$$u_i(\xi,\eta) \;=\; c_3 \ln\left(\frac{\alpha+\beta}{2}\right), \tag{2-51}$$

$$u_e(\xi,\eta) \;=\; c_3[\xi + \frac{1}{2}\ln(\alpha^2-\beta^2) - \ln(2)]. \tag{2-52}$$

The degenerate scale occurs for the interior null field when the relationship between α and β is $\alpha+\beta = 2,$ i.e., $c_1 = 0$. In such a case, the strength of the singularity along the elliptical boundary can not be determined in BEM implementation. This is the reason why a degenerate scale occurs. The fields for u_i and u_e are shown in Fig.2-3(c) for contour and 3-D plots. It is found that the null field is obtained in the ellipse. From Eqs.(2-38) and (2-39), the tangent vector $\tilde{\mathbf{t}}$ and normal vector $\tilde{\mathbf{n}}$ in Fig.2-3(d) can be derived as follows:

$$\tilde{\mathbf{t}} = (-k\cosh\xi_0\sin\eta,\ k\sinh\xi_0\cos\eta), \tag{2-53}$$

$$\tilde{\mathbf{n}} = (k\sinh\xi_0\cos\eta,\ k\cosh\xi_0\sin\eta). \tag{2-54}$$

The exact solution for the normal flux on the boundary is

$$\begin{aligned}\frac{\partial u(\xi_0,\eta)}{\partial\tilde{\mathbf{n}}} = \psi_1(\eta) &= \frac{u(\xi_0+\triangle\xi_0,\eta)-u(\xi_0,\eta)}{r(\xi_0,\eta;\xi_0+\Delta\xi_0,\eta)}\\ &= \frac{1}{\sqrt{\beta^2\cos^2\eta+\alpha^2\sin^2\eta}}.\end{aligned} \tag{2-55}$$

where $r(\xi_0,\eta;\xi_0+\Delta\xi_0,\eta)$ is the distance between the two points (ξ_0,η) and $(\xi_0+\Delta\xi_0,\eta)$ in the elliptical coordinate, as shown in Fig.3-3(d). When α approaches β, the elliptical boundary becomes a circle and the degenerate scale is found to be $\alpha=\beta=1$. The result is the same in comparison with the degenerate scale in [84, 118]. Eq.(2-55) reduces to $\psi_1(\eta)=1$ for the circle.

2-6 Special case - circular bar with radius R

When α equals to β in the elliptical case, it becomes a circular bar. The null field of Fig.2-3(c) is simplified to Fig.2-4 where $u^e(\xi,\ \eta) = \ln\ r$ can be obtained from Eq.(2-49) by setting $\alpha=\beta=r$.

The degenerate scale occurs at the radius of one. In this case, $\phi(\eta) = \psi(\eta) = 1$. For the discrete system of $2N$ boundary elements, the influence matrix of $[U]$ is a symmetric circulant which can be decomposed by using SVD technique as

$$[U] = [\Phi][\Sigma][\Psi]^T, \tag{2-56}$$

where the singular values in the $[\Sigma]$ matrix are

$$\sigma_l = \begin{cases} 2\pi\, R \ln(R), & n = 0 \\ -\pi \frac{R}{|n|}, & n = \pm 1, \pm 2, \cdots, \pm(N-1), N, \end{cases} \tag{2-57}$$

After adding a rigid body term, c, in the fundamental solution, the influence matrix $[U]$ is modified to

$$[U^r] = [U] + c^* \{\phi_1\}\{\psi_1\}^T, \tag{2-58}$$

where uniform mesh results in

$$\{\phi_1\} = \{\psi_1\} = \frac{1}{\sqrt{2N}} \left\{ \begin{array}{c} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{array} \right\}_{2N \times 1} . \tag{2-59}$$

We can easily obtain

$$c^* = 2c\, N\, l = 2\pi\, r^* c. \tag{2-60}$$

In order to demonstrate that the rigid body term c can shift the degenerate scale $R = 1$ to another place $R = r^*$, the minimum singular value of the influence matrix $[U^r]$ becomes zero,

$$2\pi\, r^* \ln(r^*) + c^* = 0. \tag{2-61}$$

Using Eq. (2-60), we have

$$2\pi\, r^* \ln(r^*) + c\, 2\pi\, r^* = 0, \tag{2-62}$$

Eq. (2-62) yields

$$r^* = e^{-c}. \tag{2-63}$$

Eq.(2-35) is obtained again using the BEM. In the same way, we can prove Eq.(2-35) in the continuous system. First, we define a boundary integral operator $\mathcal{U}$ which maps one boundary density function $p(s)$ to another boundary density function $q(x)$ as

$$\mathcal{U}(p(s)) = \lambda\, q(x), \tag{2-64}$$

where the boundary integral operator, $\mathcal{U}$, is defined as

$$\mathcal{U}(\psi(s)) = \int_B U(s,x)\psi(s)\ dB(s),\ \ x \in B. \tag{2-65}$$

In this case, the associated eigenfunction for the zero eigenvalue is $\psi(s) = 1$, i.e..

$$\mathcal{U}(\psi(s)) = \lambda\psi(x) = \int_B U(s,x)\psi(s)\ dB(s) = 0,\ \ x \in B. \tag{2-66}$$

When the degenerate scale occurs, the eigenvalue, λ, is zero. By using the degenerate kernel function for the fundamental solution added by a rigid body term, c, [31], we have

$$\begin{aligned} U^*(s,x) &= U^*(R,\theta;\rho,\phi) && \text{(2-67)}\\ &= \ln R - \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{\rho}{R}\right)^m \cos(m(\theta-\phi)) + c, && \text{(2-68)}\end{aligned}$$

where $x = (\rho,\,\phi)$ and $s = (R,\,\theta)$. For the circular case of radius one, the zero singular value results in a degenerate scale. After adding a rigid body term, c, the minimum singular value shifts to

$$\sigma_1^* = 2\pi\, R\, \ln(R) + 2\pi\, R\, c. \tag{2-69}$$

We can obtain the radius with a unit length (free of rigid body term) is shifted to e^{-c} (after adding a rigid body term c) for keeping the zero singular value, see Fig.2-5. In order to demonstrate that the rigid body term c can shift the degenerate scale $(R = \rho = 1)$ to another place $(R = r^*)$. Eq. (2-69) can be rewritten as

$$2\pi\, r^*\, \ln(r^*) + 2\pi\, r^*\, c = 0, \tag{2-70}$$

Eq. (2-70) yields

$$r^* = e^{-c}. \tag{2-71}$$

2-7 Detection of degenerate scales and determination of spurious modes by using the SVD updating documents and the Fredholm alternative theorem

Fredholm alternative theorem:

The linear algebraic equation $[K]\{u\} = \{\bar{b}\}$ has a unique solution if and only if the only continuous solution to the homogeneous equation

$$[K]\{u\} = \{0\} \tag{2-72}$$

is $\{u\} \equiv \{0\}$. Alternatively, the homogeneous equation has at least one solution if the homogeneous adjoint equation

$$[K]^H \{\phi\} = \{0\} \tag{2-73}$$

has a nontrivial solution $\{\phi\}$, where $[K]^H$ is the transpose conjugate matrix of $[K]$ and $\{\bar{b}\}$ must satisfy the constraint $(\{\bar{b}\}^H \{\phi\} = 0)$. If the matrix $[K]$ is real, the transpose conjugate of a matrix is equal to its transpose only [62], i.e., $[K]^H = [K]^T$. By using the UT formulation, we have

$$[U]\ \{t\} = [T]\ \{u\} = \{\bar{b}\}. \tag{2-74}$$

According to the Fredholm alternative theorem, Eq. (2-74) has at least one solution for $\{t\}$ if the homogeneous adjoint equation

$$[U]^T \{\phi_1\} = \{0\}, \tag{2-75}$$

has a nontrivial solution $\{\phi_1\}$, in which the constraint $(\{\bar{b}\}^T \{\phi_1\} = 0)$ must be satisfied. By substituting $\bar{b} = [T]\{u\}$ in Eq. (2-74) into $\{\bar{b}\}^T \{\phi_1\} = 0$, we obtain

$$\{u\}^T [T]^T \{\phi_1\} = 0. \tag{2-76}$$

Since $\{u\}$ is an arbitrary vector for the Dirichlet problem, we have

$$[T]^T \{\phi_1\} = \{0\}, \tag{2-77}$$

where $\{\phi_1\}$ is the spurious mode. Combining Eq. (2-75) and Eq. (2-77) together, we have

$$\begin{bmatrix} [U]^T \\ [T]^T \end{bmatrix} \{\phi_1\} = \{0\} \ \text{ or } \ \{\phi_1\}^T \begin{bmatrix} [U] & [T] \end{bmatrix} = \{0\}. \tag{2-78}$$

Eq.(2-78) indicates that the two matrices have the same spurious mode $\{\phi_1\}$ corresponding to the same zero singular value when a degenerate scale occurs. The former one in Eq.(2-78) is a form of updating term and the latter one is a form of updating document. By using the SVD technique for the $[U]^T$ and $[T]^T$ matrices, we have

$$\begin{aligned} [U]^T &= [\Psi_U]\,[\Sigma_U]\,[\Phi_U]^T\,, \\ [T]^T &= [\Psi_T]\,[\Sigma_T]\,[\Phi_T]^T\,, \end{aligned} \tag{2-79}$$

where $\{\phi_1\}$ is imbedded in both the matrices, $[\Phi_U]$ and $[\Phi_T]$, with the corresponding zero singular value in the matrices, $[\Sigma_U]$ and $[\Sigma_T]$, respectively. Since $\{\phi_1\}$ is one of the left unitary vector of $[\Phi_U]$ matrix with respect to the zero singular value, we have

$$[U]^T\{\phi_1\} = 0\,\{\psi_1\}, \tag{2-80}$$

where $\{\phi_1\}$ and $\{\psi_1\}$ are the pair of nontrivial spurious modes which satisfy

$$[U]\{\psi_1\} = 0\,\{\phi_1\}. \tag{2-81}$$

The $\{\psi_1\}$ in Eq.(2-81) for the discrete system and $\psi(\eta)$ in Eq.(2-55) for the continuous system will be examined in the following numerical examples. To sum up, rigid body mode $\{1,\ 1,\ \cdots\ 1,\ 1\}^T$and spurious mode $\{\psi_1\}$ satisfy

$$[T]\left\{\begin{array}{c} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{array}\right\} = [U]\left\{\begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{array}\right\}, \tag{2-82}$$

$$[T]\left\{\begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{array}\right\} = [U]\,\{\psi_1\}\,, \tag{2-83}$$

respectively.

2-8 Three regularization techniques to deal with degenerate scale problems in BEM

2-8-1 Method of adding a rigid body mode

Since the $[U]$ matrix is singular in case of the degenerate scale, the modified fundamental solution can be added by a rigid body term c,

$$U^{*}(s,x)=U(s,x)+c. \tag{2-84}$$

The influence matrix $[U]$ is modified to $[U^{*}]$, where the component form for the element is

$$U^{*}_{ij}=U_{ij}+c\,l_{j}.\;\;(i,\;j=1,\cdots 2N) \tag{2-85}$$

The zero singular value in $[U]$ moves to a nonzero value for $[U^{*}]$. To demonstrate the effectiveness, the minimum singular value after the modified fundamental solution will be shown in the numerical examples.

2-8-2 Hypersingular formulation

Instead of using the Eq.(2-16) in the conventional BEM, the second equation of Eq.(2-17) in the dual BEM is used. To demonstrate the idea, the singular value for the $[L]$ matrix will be shown to be nonzero no matter what the expansion ratio is in the following numerical examples.

2-8-3 CHEEF method

Since the $[U]$ matrix is singular, the rank is deficient. In order to promote the rank, the independent constraint is required. To resort to the null field equation by collocating the point outside the domain, we have

$$< w > \;\; \{t\} = < v > \;\; \{u\}, \tag{2-86}$$

where $< w >$ and $< v >$ are the influence row vectors by collocating the exterior point in the null-field equation. By combining Eqs.(2-16) with (2-86), we have

$$\begin{bmatrix} [U]_{2N\times 2N} \\ < w >_{1\times 2N} \end{bmatrix} \left\{ \begin{array}{c} t \end{array} \right\}_{2N\times 1} = \begin{bmatrix} [T]_{2N\times 2N} \\ < v >_{1\times 2N} \end{bmatrix} \left\{ \begin{array}{c} u \end{array} \right\}_{2N\times 1}. \tag{2-87}$$

According to the Eq.(2-87), we can obtain the reasonable solution by using either the least squares method or the SVD technique.

2-9 Numerical examples

In this section, four cases including elliptical, square, triangular bars and circular bar with keyway are considered.

2-9-1 Elliptical bar

For the elliptical bar with axes $\alpha\ m$ and $\beta\ m$ $(\alpha = 3\beta)$ under torsion, the analytical solution for the conjugate warping function is [120]

$$u(x_1, x_2) = \frac{\alpha^2(2\beta^2 + x_1^2 - x_2^2) + \beta^2(-x_1^2 + x_2^2)}{2(\alpha^2 + \beta^2)}, \quad (x_1, x_2) \in D, \tag{2-88}$$

and the boundary flux is

$$\frac{\partial u}{\partial n} = -\frac{(\alpha^2 - \beta^2)(-\beta^2 x_1^2 + \alpha^2 x_2^2)}{(\alpha^2 + \beta^2)\sqrt{\beta^4 x_1^2 + \alpha^4 x_2^2}}. \tag{2-89}$$

The torsion rigidity, T_r, is

$$T_r = G\frac{\pi\alpha^3\beta^3}{\alpha^2 + \beta^2}. \tag{2-90}$$

The nontrivial boundary mode $\{\psi_1\}$ obtained in Eq.(2-81) in the BEM and the analytical solution $\psi_1(\eta)$ using Eq.(2-55) matched well in Fig.2-6. Good agreement for the numerical data of Eq.(2-78) and the exact solution for the spurious mode is obtained in Fig.2-6. Table 2-2 shows the torsional rigidity obtained by using different approaches. The conventional BEM can work well for the normal case. However, the numerical instability results in a deteriorated BEM solution when the degenerate scale $(\alpha + \beta = 2)$ occurs in the shadow area of Table 2-2. Good agreement was obtained in comparison with the analytical solutions after using the regularization techniques as shown in Table 2-2.

By using the conventional BEM, the zero singular value occurs in case of degenerate scale. After adding the rigid body term, c, in the fundamental solution, the zero singular value moves to another place by a factor e^{-c} instead of the original one as shown in Fig.2-7(a). To investigate how seriously the rank deficiency behaves, we plot the second minimum singular value versus the expansion ratio in Fig.2-7(b). It indicates that the rank is deficient by one only. This supports us that only one CHEEF point is sufficient. The zero singular value disappears in Fig.2-7(c) for the $[L]$ matrix in the hypersingular formulation. In order to avoid hypersingularity, the CHEEF method by collocating one point outside the domain can promote the rank as shown in Fig.2-7(d). Since no zero solution outside the domain also shown in Fig.2-3(c), the selected CHEEF points are always valid.

2-9-2 Square bar

For the square bar with area $4a^2\ m^2$ under torsion, the analytical solution for the conjugate warping function is [120]

$$u(x_1,x_2) = a^2 + \frac{1}{2}(x_1^2 - x_2^2) - \frac{32a^2}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^n\lambda_n\cosh(\lambda_n x_2)\cos(\lambda_n x_1)}{(2n+1)^3\cosh(\lambda_n a)},\ (x_1,x_2)\in D, \quad \text{(2-91)}$$

where

$$\lambda_n = (2n+1)\frac{\pi}{2a}, \qquad n = 0,1,2... \quad \text{(2-92)}$$

The boundary flux is

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial x_1}n_{x_1} + \frac{\partial u}{\partial x_2}n_{x_2}, \quad (x_1,x_2)\in B, \quad \text{(2-93)}$$

where

$$\frac{\partial u}{\partial x_1} = -x_1 - \frac{32a^2}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\lambda_n\cosh(\lambda_n x_2)\sin(\lambda_n x_1)}{(2n+1)^3\cosh(\lambda_n a)}, \quad \text{(2-94)}$$

$$\frac{\partial u}{\partial x_2} = x_2 - \frac{32a^2}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^{n}\lambda_n\sinh(\lambda_n x_2)\cos(\lambda_n x_1)}{(2n+1)^3\cosh(\lambda_n a)}, \quad \text{(2-95)}$$

and n_{x_1} and n_{x_2} are the components of the normal vector on the boundary.
The torsional rigidity, T_r, of a square bar is

$$T_r = 16k_1 G a^4 \tag{2-96}$$

where

$$k_1 = \frac{1}{3}(1 - \frac{192}{\pi^5}\Sigma_{n=0}^{\infty}\frac{\tanh(\lambda_n a)}{(2n+1)^5}). \tag{2-97}$$

Table 2-2 shows the torsional rigidity by using different approaches. In the same way, the conventional BEM (UT formulation) can not obtain the acceptable results for the case of the degenerate scale as shown in Table 2-2. Fig.2-8 shows the spurious modes of $\{\phi_1\}$ and $\{\psi_1\}$. In this case, no analytical solution can be compared with. By using the conventional BEM, the zero singular value occurs in case of the degenerate scale. After adding the rigid body term in the fundamental solution, the zero singular value moves to another degenerate scale instead of original one as shown in Fig.2-9(a). To investigate how seriously the rank deficiency behaves, we plot the second minimum singular value versus the expansion ratio in Fig.2-9(b). It indicates that rank is deficient by one only. This supports us that only one CHEEF point is required. By employing the hypersingular equation in the dual BEM, it is found that the singular value of $[L]$ matrix for any scale is nonzero as shown in Fig.2-9(c). In order to avoid hypersingularity, the CHEEF concept by collocating one point outside the domain can promote the rank as shown in Fig.2-9(d).

2-9-3 Triangular bar

For the equilateral triangular bar with the height h m under torsion, the analytical solution for the conjugate warping function is [120]

$$u(x_1, x_2) = -\frac{1}{2h}(3x_2x_1^2 - x_2^3 + hx_2^2 - hx_1^2 + h^2x_2), \;\; (x_1, x_2) \in D, \tag{2-98}$$

and the boundary flux is

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial x_1}n_{x_1} + \frac{\partial u}{\partial x_2}n_{x_2}, \quad (x_1, x_2) \in B, \tag{2-99}$$

where

$$\frac{\partial u}{\partial x_1} = -\frac{1}{2h}(6x_2x_1 - 2hx_1), \tag{2-100}$$

$$\frac{\partial u}{\partial x_2} = -\frac{1}{2h}(3x_1^2 - 3x_2^2 + 2hx_2 + h^2), \tag{2-101}$$

The torsion rigidity, T_r, is

$$T_r = G\frac{\sqrt{3}}{45}h^4. \tag{2-102}$$

Table 2-3 shows the torsional rigidity by using different approaches. In the same way, the conventional BEM (UT formulation) can not obtain the acceptable results for the case of degenerate scale as shown in Table 2-3. Fig.2-10 shows the spurious modes of $\{\phi_1\}$ and $\{\psi_1\}$. In this case, no analytical solution can be compared with. By using the conventional BEM, the zero singular value occurs in case of the degenerate scale. After adding the rigid body term in the fundamental solution, the zero singular value moves to another degenerate scale instead of original one as shown in Fig.2-11(a). To investigate how seriously the rank deficiency behaves, we plot the second minimum singular value versus the expansion ratio in Fig.2-11(b). It indicates that rank is deficient by one only. It is found that the singular value of $[L]$ matrix in the hypersingular equation for any scale is nonzero as shown in Fig.2-11(c). In order to avoid hypersingularity, the CHEEF method by collocating one point outside the domain can promote the rank as shown in Fig.2-11(d).

2-9-4 Circular bar with keyway

For the circular bar with keyway under torsion, the analytical solution for the conjugate warping function is [120]

$$u(x_1, x_2) = ax_1(1 - \frac{b^2}{x_1^2 + x_2^2} + \frac{1}{2}b^2), \quad (x_1, x_2) \in D, \tag{2-103}$$

and the boundary flux is

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial x_1}n_{x_1} + \frac{\partial u}{\partial x_2}n_{x_2}, \quad (x_1, x_2) \in B, \tag{2-104}$$

where

$$\frac{\partial u}{\partial x_1} = -\frac{2ab^2x_1^2}{(x_1^2+x_2^2)^2}, \tag{2-105}$$

$$\frac{\partial u}{\partial x_2} = \frac{2abx_1x_2}{(x_1^2+x_2^2)^2}. \tag{2-106}$$

The torsion rigidity, T_r, is

$$T_r = 2Ga^4k_2, \tag{2-107}$$

where

$$k_2 = \frac{1}{24}(\sin 4\gamma + 8\sin 2\gamma + 12\gamma) - \frac{1}{2}(\frac{b}{a})^2(\sin 2\gamma + 2\gamma) + \frac{4}{3}(\frac{b}{a})^3(\sin\gamma) + \frac{1}{4}(\frac{b}{a})^4\gamma \tag{2-108}$$

in which

$$2\cos\gamma = \frac{b}{a}. \tag{2-109}$$

Table 2-3 shows the torsional rigidity by using different approaches. In the same way, the conventional BEM (UT formulation) can not obtain the acceptable results for the case of degenerate scale as shown in Table 2-3. Fig.2-12 shows the spurious modes of $\{\phi_1\}$ and $\{\psi_1\}$. In this case, no analytical solution can be compared with. By using the conventional BEM, the zero singular value occurs in case of the degenerate scale. After adding the rigid body term in the fundamental solution, the zero singular value moves to another degenerate scale instead of original one as shown in Fig.2-13(a). To investigate how seriously the rank deficiency behaves, we plot the second minimum singular value versus the expansion ratio in Fig.2-13(b). It indicates that rank is deficient by one only. By employing the hypersingular equation in the dual BEM, it is found that the singular value of $[L]$ matrix for any scale is nonzero as shown in Fig.2-13(c). In order to avoid hypersingularity, the CHEEF method by collocating one point outside the domain can promote the rank as shown in Fig.2-13(d).

2-10 Conclusions

In this chapter, the numerical instability for torsion problems by using the conventional BEM was addressed. Instead of direct searching for the degenerate scale by trial and error, a more efficient technique is proposed to directly obtain the singular case since only one normal scale needs to be computed. The degenerate scale for the torsion bar with an elliptical section was derived analytically in the continuous system using the elliptical coordinate. For the discrete system, the source of numerical instability is found to be the spurious modes (left and right unitary vectors in SVD with respect to the zero singular value) which were obtained by using the Fredholm alternative theorem and SVD updating document. To deal with the numerical instability due to the degenerate scale, three approaches, method of adding a rigid body mode, hypersingular formulation and CHEEF method, were successfully applied to remove the zero singular value. Good agreement between the BEM results and the analytical solutions were obtained if the regularization techniques are used. Numerical examples, including a circular bar, an elliptical bar, a square bar, triangular bar and a circular bar with keyway were demonstrated to check the validity.

Chapter 3

Eigenanalysis for membranes with stringers using BEM in conjunction with SVD technique

Summary

It is well known that either the multi-domain BEM or the dual BEM can solve boundary value problems with degenerate boundaries. In this chapter, the eigensolutions for membranes with stringers are obtained in a single domain by using the conventional BEM in conjunction with the SVD technique. By adopting the SVD technique for rank revealing, the nontrivial boundary mode can be detected by the successive zero singular values which are not due to the degeneracy of degenerate boundary. The boundary modes are obtained according to the right unitary vectors with respect to the zero singular values in the SVD. Three examples, a single-edge stringer, a double-edge stringer and a central stringer in a circular membrane, are considered. The results of the present method, are compared with those of the multi-domain BEM, the dual BEM, the DtN method, the FEM (ABAQUS) and analytical solutions if available. Good agreement is obtained. The goal to deal with the eigenproblem in a single domain without hypersingularity is achieved.

3-1 Introduction

A large amount of boundary value problems (BVPs) were solved efficiently by using the boundary element method (BEM) since Rizzo [121] discretized the integral equations for elastostatics in 1967. Over twenty years, the main applications were limited in BVPs without degenerate boundaries. Since the degenerate boundary results in rank deficiency for the conventional BEM, the multi-domain BEM was utilized to solve the nonunique solution by introducing an artificial boundary in the last two decades, *e.g.*, cutoff wall [102], thin barrier [106] and crack problems [7]. However, the eigenproblem with a degenerate boundary was not solved by using the multi-domain BEM to the authors' best knowledge. The drawback of the multi-domain approach is obvious in that the artificial boundary is arbitrary, and thus not qualified as an automatic scheme. In addition, a larger system of equations is required since the degrees of freedoms on the interface are put into the system. For half plane or infinite problem, the artificial boundary is not finite. The three shortcomings encourage researchers to deal with the degenerate boundary problem by using the dual BEM with hypersingularity in the last decades, *e.g.*, Hong and Chen [35, 76], Gray [66, 67] and Kirkup [94, 95, 96] independently derived the hypersingular formulation for the degenerate boundary problems. Aliabadi and his coworkers [1, 110, 119] have published many papers on its applications to fracture mechanics. One can consult the review article by Chen and Hong [27]. We may wonder is it possible to find the eigensolution in a single domain with a degenerate boundary approach without using the hypersingular equation.

In this thesis, we will solve the membrane eigenproblems with stringers using the multi-domain BEM and a new method. By employing only the conventional BEM instead of the dual BEM, the eigenvalue will be detected in a single domain by finding the successive zero singular values using the rank revealing technique of SVD. Three cases, a single-edge stringer, a double-edge stringer and a central stringer, will be considered. Also, the FEM using ABAQUS, the DtN method, the dual BEM and analytical solutions if available will be utilized in comparison with the present solutions of both the multi-domain BEM and the new method.

3-2 Integral formulation and boundary element implementation for the membrane eigenproblem with stringers

Consider a membrane eigenproblem as shown in Fig.3-1(a), (b) and (c), which has the following governing equation:

$$\nabla^2 u(x) + k^2 u(x) = 0,\ x\ \ in\ \ D, \tag{3-1}$$

where D is the domain of interest, x is the domain point, $u(x)$ is the displacement and k is the wave number. The boundary conditions are given as follows:

$$u(x) = 0,\ x\ \ on\ \ B_1, \tag{3-2}$$

$$\frac{\partial u(x)}{\partial n_x} = 0,\ x\ \ on\ \ B_2, \tag{3-3}$$

where B_1 is the essential boundary with the specified homogeneous displacement, B_2 is the natural boundary with homogeneous normal flux in the n_x direction, and B_1 and B_2 comprise the whole boundary of the domain D. For the stringer B_1 can be composed of stringer (degenerate boundary) C^+ and C^- as shown in Fig.3-1(a), (b) and (c). For the homogeneous boundary conditions, we can determine the critical wave number k by using the BEM.

The first equation of the dual boundary integral equations for the domain point can be derived from Green's third identity [43] :

$$2\pi u(x) = \int_B T(s,x)u(s)dB(s) - \int_B U(s,x)\frac{\partial u(s)}{\partial n_s}dB(s),\ x \ \in \ D, \tag{3-4}$$

where $U(x,s)$ is the fundamental solution which satisfies

$$\nabla^2 U(x,s) + k^2 U(x,s) = \delta(x-s),\ x \ \in \ D, \tag{3-5}$$

in which $\delta(x-s)$ is the Dirac-delta function, and $T(s,x)$ is defined by

$$T(s,x) \equiv \frac{\partial U(s,x)}{\partial n_s}, \tag{3-6}$$

in which n_s is the outward directed normal at the boundary point s. By moving the field point x in Eq.(3-4) to the boundary, the first dual boundary integral equation for the boundary point can be obtained as follows:

$$\pi u(x) = C.P.V. \int_B T(s,x)u(s)dB(s) - R.P.V. \int_B U(s,x)\frac{\partial u(s)}{\partial n_s}dB(s),\ x \ \in \ B, \tag{3-7}$$

where $C.P.V.$ is the Cauchy principal value and $R.P.V.$ is the Riemann principal value. The boundary integral equation can be discretized by using N constant boundary elements for B, and the resulting algebraic system (UT formulation: conventional BEM) can be obtained as

$$[T]\{u\} = [U]\{t\}, \tag{3-8}$$

where $t = \frac{\partial u}{\partial n}$, [ ] denotes a square matrix with dimension N by N, { } is a column vector for the boundary data and the elements of the square matrices are, respectively,

$$U_{ij} \ = \ R.P.V. \int_{B_j} U(s_j, x_i)dB(s_j), \tag{3-9}$$

$$T_{ij} \ = \ -\pi\delta_{ij} + C.P.V. \int_{B_j} T(s_j, x_i)dB(s_j), \tag{3-10}$$

where B_j denotes the j^{th} boundary element and δ_{ij} is the Kronecker delta.

3-3 Review of the multi-domain BEM and the dual BEM for the eigenproblem with a degenerate boundary

3-3-1 Multi-domain BEM

Since the degenerate boundary on C^+ and C^- as shown in Fig.3-2(a) produces double unknowns, Eq.(3-8) can provide an additional equation by collocating the point x on C^+ or C^-. Instead of obtaining the independent equations by using the hypersingular formulation [43], the multi-domain BEM is one alternative. By dividing the domain into two subdomains (index 1 and 2) and using the conventional BEM for each subdomain, we have the two equations from Eq.(3-8) as follows,

$$\begin{bmatrix} T_{cc}^1 & T_{cf}^1 \\ T_{fc}^1 & T_{ff}^1 \end{bmatrix} \begin{Bmatrix} u_c^1 \\ u_f^1 \end{Bmatrix} = \begin{bmatrix} U_{cc}^1 & U_{cf}^1 \\ U_{fc}^1 & U_{ff}^1 \end{bmatrix} \begin{Bmatrix} t_c^1 \\ t_f^1 \end{Bmatrix}, \tag{3-11}$$

and

$$\begin{bmatrix} T_{cc}^2 & T_{cf}^2 \\ T_{fc}^2 & T_{ff}^2 \end{bmatrix} \begin{Bmatrix} u_c^2 \\ u_f^2 \end{Bmatrix} = \begin{bmatrix} U_{cc}^2 & U_{cf}^2 \\ U_{fc}^2 & U_{ff}^2 \end{bmatrix} \begin{Bmatrix} t_c^2 \\ t_f^2 \end{Bmatrix}, \tag{3-12}$$

where the superscripts 1 and 2 are the labels of the subdomains and the subscripts c and f denote the complementary and interface sets for u and t, respectively. Since the unknown pairs of $\{u_f^1\}$, $\{u_f^2\}$, $\{t_f^1\}$ and $\{t_f^2\}$ are introduced in the artificial boundary as shown in Fig.3-2(a), two constraints of the continuity and equilibrium conditions are necessary,

$$\{u_f^1\} = \{u_f^2\}, \tag{3-13}$$

and

$$\{t_f^1\} = -\{t_f^2\}. \tag{3-14}$$

By assembling the Eqs.(3-11) and (3-12) and using Eqs.(3-13) and (3-14), we have

$$\begin{bmatrix} U_{cc}^1 & U_{cf}^1 & 0 \\ U_{fc}^1 & U_{ff}^1 & 0 \\ 0 & -U_{cf}^2 & U_{cc}^2 \\ 0 & -U_{ff}^2 & U_{fc}^2 \end{bmatrix} \begin{Bmatrix} t_c^1 \\ t_f^1 \\ t_c^2 \end{Bmatrix} = \begin{bmatrix} T_{cc}^1 & T_{cf}^1 & 0 \\ T_{fc}^1 & T_{ff}^1 & 0 \\ 0 & T_{cf}^2 & T_{cc}^2 \\ 0 & T_{ff}^2 & T_{fc}^2 \end{bmatrix} \begin{Bmatrix} u_c^1 \\ u_f^1 \\ u_c^2 \end{Bmatrix}. \tag{3-15}$$

By collecting the unknown variables, $\{t_c^1\}$, $\{t_f^1\}$, $\{t_c^2\}$, $\{u_f^1\}$ for the Dirichlet eigenproblem and the known homogeneous boundary conditions, $\{u_c^1\}$ and $\{u_c^2\}$, to the left and right hand sides of the equal sign, respectively, Eq.(3-15) is reformulated to

$$\left[\begin{array}{c} U_{MD} \end{array}\right] \left\{\begin{array}{c} t_c^1 \\ t_f^1 \\ t_c^2 \\ u_f^1 \end{array}\right\} = \{0\}, \tag{3-16}$$

where $\{u_c^1\} = \{u_c^2\} = 0$ for the Dirichlet boundary condition are substituted and

$$\left[\begin{array}{c} U_{MD} \end{array}\right] = \left[\begin{array}{cccc} U_{cc}^1 & U_{cf}^1 & 0 & T_{cf}^1 \\ U_{fc}^1 & U_{ff}^1 & 0 & T_{ff}^1 \\ 0 & -U_{cf}^2 & U_{cc}^2 & T_{cf}^2 \\ 0 & -U_{ff}^2 & U_{fc}^2 & T_{ff}^2 \end{array}\right]. \tag{3-17}$$

By plotting the determinant of the matrix, $[U_{MD}]$, versus k, we can find the eigenvalue where the determinant drops to a local minimum in the direct-searching scheme.

3-3-2 Dual BEM [43]

Instead of using the multi-domain BEM, the dual BEM is also one alternative for the degenerate-boundary problem. By adding independent constraints, differential operator can be introduced. This is the key idea of the dual BEM. After taking the normal derivative with respect to Eq.(3-4), the second equation of the dual boundary integral equations for the domain point can be derived:

$$2\pi\frac{\partial u(x)}{\partial n_x} = \int_B M(s,x)u(s)dB(s) - \int_B L(s,x)\frac{\partial u(s)}{\partial n_s}dB(s), \ x \ \in \ D, \tag{3-18}$$

where the two kernels are

$$L(s,x) \equiv \frac{\partial U(s,x)}{\partial n_x}, \tag{3-19}$$

$$M(s,x) \equiv \frac{\partial^2 U(s,x)}{\partial n_x \partial n_s}. \tag{3-20}$$

By moving the field point x in Eq.(3-18) to the boundary, the second one of dual boundary integral equations for the boundary point can be obtained as follows:

$$\pi \frac{\partial u(x)}{\partial n_x} = H.P.V. \int_B M(s,x)u(s)dB(s) - C.P.V. \int_B L(s,x)\frac{\partial u(s)}{\partial n_s}dB(s), \ x \ \in \ B, \quad \text{(3-21)}$$

where $H.P.V.$ is the Hadamard (Mangler) principal value. After boundary element descretization, we have

$$[M]\{u\} = [L]\{t\}, \quad \text{(3-22)}$$

where

$$L_{ij} = \pi\delta_{ij} + C.P.V. \int_{B_j} L(s_j, x_i)dB(s_j), \quad \text{(3-23)}$$

and

$$M_{ij} = H.P.V. \int_{B_j} M(s_j, x_i)dB(s_j). \quad \text{(3-24)}$$

For the membrane eigenproblem with stringers, the homogeneous Dirichlet boundary condition is considered. After determining the influence coefficients and substituting the boundary conditions, we can obtain the transcendental eigenequations as follows:

$$[U(k)]\{t\} = \{0\}, \quad \text{(3-25)}$$

$$[L(k)]\{t\} = \{0\}, \quad \text{(3-26)}$$

where $\{t\}$ is the boundary mode for $t = \frac{\partial u}{\partial n}$, and the wave number, k, is embedded in each element of the matrices, $[U]$ and $[L]$. By employing the direct-searching scheme for the determinant of $[U]$ or $[L]$, trivial data are obtained for the plot of determinant versus k since the two matrices are singular for any value of k. In other words, either UT or LM method alone fails to solve the eigenproblem.

By combining the dual equations on the degenerate boundary when x collocates on C^+ or C^-, the nontrivial eigensolution exists when the determinant of the combined influence matrix is zero by using the direct-searching method. Since either one of the two equations, UT or LM, for the normal boundary S as shown in Fig.3-1(a) can be selected, two alternative approaches, $UT + LM$ and $LM + UT$, are proposed for the combined influence matrices as follows:

The $UT + LM$ method has the eigenequation

$$[K_{UL}] \left\{ \begin{array}{c} t_{j_S} \\ t_{j_{C+}} \\ t_{j_{C-}} \end{array} \right\} = \{0\}, \tag{3-27}$$

where

$$[K_{UL}] = \begin{bmatrix} U_{i_S j_S} & U_{i_S j_{C+}} & U_{i_S j_{C-}} \\ U_{i_{C+} j_S} & U_{i_{C+} j_{C+}} & U_{i_{C+} j_{C-}} \\ L_{i_{C+} j_S} & L_{i_{C+} j_{C+}} & L_{i_{C+} j_{C-}} \end{bmatrix}, \tag{3-28}$$

the subscripts, i_S and i_{C^+}, denote the collocation points on the S and C^+ boundaries, respectively, and the subscripts, j_S and j_{C^+}, denote the element ID on the S and C^+ boundaries, respectively. The $LM + UT$ method has the eigenequation

$$[K_{LU}] \left\{ \begin{array}{c} t_{j_S} \\ t_{j_{C+}} \\ t_{j_{C-}} \end{array} \right\} = \{0\}, \tag{3-29}$$

where

$$[K_{LU}] = \begin{bmatrix} L_{i_S j_S} & L_{i_S j_{C+}} & L_{i_S j_{C-}} \\ L_{i_{C+} j_S} & L_{i_{C+} j_{C+}} & L_{i_{C+} j_{C-}} \\ U_{i_{C+} j_S} & U_{i_{C+} j_{C+}} & U_{i_{C+} j_{C-}} \end{bmatrix}. \tag{3-30}$$

By plotting the determinants of $[K_{UL}]$ or $[K_{LU}]$ versus k, eigenvalues can be found by using the direct-searching scheme.

3-4 Direct-searching scheme by using determinant and singular value in BEM

3-4-1 Multi-domain BEM

The eigenvalue k can be obtained by direct searching the determinant versus k, such that

$$det[U_{MD}(k)] = 0, \tag{3-31}$$

where $[U_{MD}(k)]$ is defined in Eq.(3-17). The numerical results will be elaborated on later. After determining the eigenvalues, the boundary mode can be obtained by setting a normalized value to be one in an element for the nontrivial vector. By substituing the eigenvalue and boundary mode into Eq.(3-4), the interior mode can be obtained.

3-4-2. Dual BEM

In the same way, the eigenvalue k can be obtained from

$$det[K_{UL}(k)] = 0 \quad \text{or} \quad det[K_{LU}(k)] = 0, \tag{3-32}$$

where $[K_{UL}(k)]$ and $[K_{LU}(k)]$ are defined in Eqs.(3-28) and (3-30), respectively. The interior mode can be obtained in the same way as the multi-domain BEM does.

3-4-3 UT BEM+SVD

The aforementioned two methods, either the multi-domain BEM or the dual BEM is well known for degenerate boundary problems in the literature. Here, we propose a new approach to deal with the eigenproblem using the UT BEM and SVD. For the Dirichlet eigenproblem, the boundary element mesh on the degenerate boundary was shown in Fig.3-2(b). The influence matrix $[U(k)]$ is rank deficient due to two sources, the degeneracy of stringers and the nontrivial mode for the eigensolution. Since N_d constant elements locate on the stringer, the matrix $[U(k)]$ results in N_d zero singular values $(\sigma_1 = \sigma_2 \cdots = \sigma_{N_d} = 0)$. The next $(N_d + 1)^{th}$ zero singular value $\sigma_{N_d+1} = 0$ originates from the nontrivial eigensolution. To detect the eigenvalues, the $(N_d + 1)^{th}$ zero singular value versus k can be plotted to find the drop where the eigenvalue occurs.

Since the SVD technique is adopted for rank revealing, the decomposition is reviewed as follow:
Given a matrix $[K]$, SVD can decompose into

$$[K(k)]_{M\times P} = [\Phi]_{M\times M}[\Sigma]_{M\times P}[\Psi]^H_{P\times P}, \tag{3-33}$$

where $[\Phi]$ is a left unitary matrix constructed by the left singular vectors $(\{\phi_i\},\ \ i=1,2,\ldots M)$, and $[\Sigma]$ is a diagonal matrix which has singular values $\sigma_1,\sigma_2,\cdots,\sigma_{P-1}$ and σ_P allocated in a diagonal line as

$$[\Sigma]=\begin{bmatrix} \sigma_P & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}_{M\times P} , \tag{3-34}$$

in which $\sigma_P \geq \sigma_{P-1}\cdots \geq \sigma_1$ and $[\Psi]^H$ is the complex conjugate transpose of a right unitary matrix constructed by the right singular vectors $(\{\psi_i\},\ \ i=1,2,\ldots P)$. As we can see in Eq.(3-34), there exist at most P nonzero singular values.

By employing the SVD technique to determine the eigenvalue, we can obtain the boundary mode at the same time by extracting the right singular vector $\{\psi\}$ in the right unitary matrix $[\Psi]$ of SVD with respect to the near zero or zero singular value by using

$$[K]\{\psi_i\}=\sigma_i\{\phi_i\}\ \ i=1,2,3\cdots P\ . \tag{3-35}$$

If the q^{th} singular value, σ_q, is zero, in Eq.(3-35) we have

$$[K]\{\psi_q\}=0\{\phi_q\}=\{0\},\ q\leq P\ . \tag{3-36}$$

According to Eq.(3-36), the nontrivial boundary mode is found to be the right singular vector, $\{\psi_q\}$, in the right unitary matrix. Therefore, the step to determine nontrivial boundary mode in the multi-domain BEM and dual BEM is avoided by setting a reference value. Here, UT BEM+SVD employed the influence $[U]$ for $[K]$ in Eq.(3-8) for the Dirichlet eigenproblem.

3-5 Numerical examples

We next consider the three problems illustrated in Fig.3-1(a)-(c), which have been solved by Givoli and Vigdergauz [68] and Chen *et al.* [43]. A circular membrane is given with a radius

R. For simplicity, we set $R = 1m$. In this study, the conventional BEM (UT formulation) in conjunction with SVD is employed. In order to check the validity, the results of UT BEM+SVD and the multi-domain BEM are compared with those of the exact solution, the DtN method, the dual BEM and the ABAQUS (FEM) results. The conventional boundary element meshes for these cases are shown in Fig.3-3(a), (b) and (c) and the multi-domain boundary element meshes are shown in Fig.3-4(a), (b) and (c) for the single-edge, the double-edge and the central stringers, respectively.

Case 1. Single-edge stringer with length $a = 1$:

Using the conventional BEM (UT formulation) in conjunction with SVD, the $(\sigma_{N_d+1})^{th}$ zero singular value obtained by using Eq.(3-33) for $[U]$ matrix, $([K] = [U])$ is plotted versus the wave number in Fig.3-5(a). The curve drops at the eigenvalues. By using the dual BEM and the multi-domain BEM, the determinants in Eqs.(3-32) and (3-31) versus the wave number are also shown in Fig.3-5(b) and (c), respectively, without using the SVD technique [43]. Good agreement for the former eigenvalues in Fig.3-5(a), (b) and (c) are made. The DtN method missed some eigenvalues as disscussed in [43], since symmetry and anti-symmetry are not fully considered. In addition, the exact eigenvalues satisfying $J_{\frac{n}{2}}(k), n = 1, 2, 3 \cdots$, and the FEM results using ABAQUS are compared with those of the UT BEM+SVD, the dual BEM (DBEM) and the multi-domain BEM in Table 3-1(a). For this case, the number of boundary elements, N_d, on the degenerate boundary is 5. Since the $(N_d + 1)^{th}$ zero singular value, σ_{N_d+1}, originates from the nontrivial boundary mode, Fig.3-6(a) shows the $\{\psi_{N_d+1}\}$ along the boundary for the former eight eigenvalues. It is found that $\{\psi_{N_d+1}\}$ matched well with the exact boundary eigensolutions which are $(-1)^n \sin(\frac{n\theta}{2}),\ n = 1, 2, \cdots$, as predicted analytically in [43]. For the former eight eigenvalues, the first right singular vector $\{\psi_1\}$ corresponding to the first zero singular value $(\sigma_1 = 0)$ along the boundary in Fig.3-6(b), also indicate that the element of boundary mode $\{\psi_1\}$ are trivial except on the degenerate boundary. Since the former N_d zero singular values $(\sigma_1 = \sigma_1 = \cdots = \sigma_{N_d} = 0)$ originate from the degenerate boundary, the corresponding right singular vectors $(\{\psi_1\} \sim \{\psi_{N_d}\})$ are found to be trivial except on the degenerate boundary as shown in Fig.3-7, for the case of $k = 3.09$. In other words, Fig.3-7 reveals that the former five zero singular values stems from the degeneracy due to stringers. The former eight modes by using the UT BEM+SVD are compared well with those of FEM as shown in Fig.3-8.

Case 2. Double edge stringer with length $a = 0.5$:

Using the conventional BEM (UT formulation) in conjunction with SVD, the $(N_d + 1)^{th}$ zero singular value obtained by using Eq.(3-33) for $[U]$ matrix, $([K] = [U])$ is plotted versus the wave number in Fig.3-9(a). The curve drops at the eigenvalues. By using the dual BEM and the multi-domain BEM, the determinants in Eqs.(3-32) and (3-31) versus the wave number are also shown in Fig.3-9(b) and (c), respectively. Good agreement for the eigenvalues in Fig.3-9(a), (b) and (c) is obtained. In addition, the FEM results by using ABAQUS are compared with those using UT BEM+SVD, the dual BEM and the multi-domain BEM in Table 3-1(b). The former eight modes by using the UT BEM+SVD are compared with those of the FEM as shown in Fig.3-10.

Case 3. Central stringer with length $a = 0.8$:

Using the conventional BEM (UT formulation) in conjunction with SVD, the $(N_d + 1)^{th}$ zero singular value obtained by using Eq.(3-33) for $[U]$ matrix, $([K] = [U])$ is plotted versus the wave number in Fig.3-11(a). The curve drops at the eigenvalues. By using the dual BEM and the multi-domain BEM, the determinants in Eqs.(3-32) and (3-31) versus the wave number are also shown in Fig.3-11(b) and (c), respectively. Good agreement for the eigenvalues in Fig.3-11(a), (b) and (c) is obtained. The FEM results by using ABAQUS are compared with those using the UT BEM+SVD, the dual BEM and the multi-domain BEM in Table 3-1(c). The former eight modes by using the UT BEM+SVD are compared with those of the FEM as shown in Fig.3-12.

3-6 Conclusions

Instead of using either the multi-domain BEM or the dual BEM, the conventional BEM was successfully utilized to solve the degenerate boundary eigenproblem in conjunction with the SVD technique. Not only hypersingularity can be avoided but also a single domain is required. By detecting the successive zero singular values, the eigenvalues were found and the boundary eigenmodes were obtained according to the corresponding right unitary vectors. Good agreement among the results of present method, the FEM (ABAQUS), DtN method, the multi-domain BEM, the dual BEM and analytical solutions if available was obtained. The goal to solve the eigenproblem using the singular formulation in a single domain was achieved. In addition, the boundary mode and eigenvalue can be obtained at the same time once the influence matrix was decomposed by using the SVD.

Chapter 4

On the true and spurious eigensolutions for eigenproblems using the Fredholm alternative theorem and SVD

Summary

The appearance of spurious eigensolutions for interior eigenproblems is examined by employing the complex-valued formulation, the real-part, the imaginary-part BEMs and the multiple reciprocity method in a unified manner. In this chapter, the Fredholm alternative theorem and SVD updating techniques in conjunction with the dual formulation are employed to deal with the eigenproblem. Numerical examples given circular domains are illustrated to see the validity of the present formulation.

4-1 Introduction

Solving eigenproblems by using BEM has been studied by many researchers. Many methods including the complex-valued boundary element method [43], the multiple reciprocity method (MRM) [37], the real-part [37, 100] and the imaginary-part BEMs [63] have been proposed. Althuogh the real-part BEM can obtain the true eigenvalue, this leads to spurious roots in addition to the correct ones. Hutchinson [81] has investigated the mode shapes in order to identify and reject the spurious ones. Chen *et al.* used the residue method to identify the true solution by substituting the possible eigensolution into dual equations. One may wonder is it possible to recognize the true or spurious eigenvalues without determining the mode shapes in advance. In order to achieve this purpose, Chen and his coworkers [21] have studied the interior eigenproblems and published many papers [22, 39, 101]. Among them, *e.g.,* domain partition method, SVD updating method, CHEEF method, and GSVD technique were employed to sort out the true eigensolutions. Besides, Chen and Wong [47], and Yeih *et al.* [135, 136] found analytically the spurious eigensolutions for a rod and a beam in the MRM. In addition, Kamiya *et al.* [86] and Yeih *et al.* [137] independently claimed that MRM is no more than the real-part BEM. Kang *et al.* [91] employed the

Nondimensional Dynamic Influence Function method (NDIF) to solve the eigenproblem. Chen *et al.* [40] commented that the NDIF method is a special case of imaginary-part BEM. Kang and Lee also found the spurious eigensolutions and filtered out the spurious eigenvalues by using the net approach [89]. Later, they extended to solve plate vibration problems [90]. Chen *et al.* [19] proposed a double-layer potential approach to filter out the spurious eigenmodes. In this chapter, a unified formulation will be presented, including using the Fredholm alternative theorem and SVD techniques in conjunction with the dual formulation for sorting out the true and spurious eigenvalues. A circular case is used to examine the validity of the present formulation.

4-2 Problem statement and the methods of solution

The governing equation for the eigenproblem is the Helmholtz equation as follows:

$$\nabla^2 u(x) + k^2 u(x) = 0, \;\; x \;\; in \;\; D. \tag{4-1}$$

where D is the domain of interest, x is the domain point, k is the wave number and $u(x)$ is the displacement or acoustic pressure for the vibration problem or acoustic problem, respectively.

On the basis of the dual formulation, the unified null-field integral formulation for the Helmholtz equation using the direct method can be written as

$$0 = \int_B T(s,x)u(s)dB(s) - \int_B U(s,x)t(s)dB(s), \quad x \in D^e \tag{4-2}$$

$$0 = \int_B M(s,x)u(s)dB(s) - \int_B L(s,x)t(s)dB(s),, \quad x \in D^e \tag{4-3}$$

where D^e is the complementary domain of D, $x = (\rho, \phi)$ is a field point and $x = (R, \theta)$ is a source point, $t(s) = \frac{\partial u(s)}{\partial n_s}$, $U(s,x)$ is the fundamental solution and the explicit forms for the four methods as shown bellow:

Direct BEM	Complex-valued BEM	Real-part BEM	Imaginary-part BEM	MRM
$U(s,x)$	$\frac{-i\pi H_0^{(1)}(kr)}{2}$	$\frac{\pi Y_0(kr)}{2}$	$\frac{\pi J_0(kr)}{2}$	$\frac{\pi}{2}\bar{Y}_0(kr)$

where $r = |\mathbf{s} - \mathbf{x}|$, $H_0^{(1)}(kr)$ is the first kind Hankel function with zeroth order and $J_0(kr)$ and $Y_0(kr)$ are the zeroth order Bessel functions of first kind and second kind, respectively. The fundamental solution of the MRM is

$$\frac{\pi}{2}\bar{Y}_0(kr) = (\ln r)\sum_{n=0}^{\infty} p_n(kr)^{2n} + \sum_{n=0}^{\infty} q_n(kr)^{2n} \tag{4-4}$$

$$= \frac{\pi}{2}Y_0(kr) - [\ln\frac{k}{2} + \gamma]J_0(kr), \tag{4-5}$$

in which γ is the Euler constant, $p_n = \frac{(-1)^n}{4^n(n!)^2}$ and $q_n = \frac{(-1)^(n+1)}{4^n(n!)^2}(1 + \frac{1}{2} + \frac{1}{3}\cdots + \frac{1}{n})$. Another kernel functions are derived by

$$T(\mathbf{s},\mathbf{x}) = \frac{\partial U(\mathbf{s},\mathbf{x})}{\partial n_\mathbf{s}},$$

$$L(\mathbf{s},\mathbf{x}) = \frac{\partial U(\mathbf{s},\mathbf{x})}{\partial n_\mathbf{x}},$$

$$M(\mathbf{s},\mathbf{x}) = \frac{\partial^2 U(\mathbf{s},\mathbf{x})}{\partial n_\mathbf{s} n_\mathbf{x}}.$$

The true and spurious eigensolution were solved by using the degenerate kernel, Fourier series and circulants in continuous and discrete systems. Four approaches, the complex-valued formulation, the real-part, the imaginary-part BEMs and MRM, are summarized and the occurrence of true and spurious eigensolutions is also reviewed in the following subsection.

4-2-1 True eigensolutions by using the complex-valued BEM

By using the UT and LM formulations for the Dirichlet eigenproblem, the eigenequations are derived for the circular problem, respectively

$$UT: \quad [J_\ell(k\rho) + iY_\ell(k\rho)]J_\ell(k\rho) = 0, \tag{4-6}$$

and

$$LM: \quad [J'_\ell(k\rho) + iY'_\ell(k\rho)]J_\ell(k\rho) = 0. \tag{4-7}$$

The true eigenvalues are the roots of $J_\ell(k\rho) = 0$ for the common part in the eigenequations of Eqs.(4-6) and (4-7).

For the Neumann problem, the eigenequations are derived, respectively

$$UT: \quad [J_\ell(k\rho) + iY_\ell(k\rho)]J'_\ell(k\rho) = 0, \tag{4-8}$$

and

$$LM: \quad [J'_\ell(k\rho) + iY'_\ell(k\rho)]J'_\ell(k\rho) = 0. \tag{4-9}$$

The true eigenvalues are the roots of $J'_\ell(k\rho) = 0$ for the common part in the eigenequations of Eqs.(4-8) and (4-9).

4-2-2 True and spurious eigensolutions by using the real-part BEM

By employing the real-part kernels in the UT and LM equations for the Dirichlet eigenproblem, we obtain the eigenequations,

$$UT: \quad Y_\ell(k\rho)J_\ell(k\rho) = 0, \quad \ell = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N, \tag{4-10}$$

$$LM: \quad Y'_\ell(k\rho)J_\ell(k\rho) = 0, \quad \ell = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N, \tag{4-11}$$

respectively. The k value satisfying Eqs.(4-10) or (4-11) may be spurious eigenvalues of union set ($Y_\ell(k\rho) = 0$ or $Y'_\ell(k\rho) = 0$) or true eigenvalues of intersection set ($J_\ell(k\rho) = 0$) to satisfy both Eqs.(4-10) and (4-11).

For the Neumann problem, we obtain the eigenequations,

$$UT: \quad Y_\ell(k\rho)J'_\ell(k\rho) = 0, \quad \ell = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N, \tag{4-12}$$

$$LM: \quad Y'_\ell(k\rho)J'_\ell(k\rho) = 0, \quad \ell = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N, \tag{4-13}$$

respectively. The k values satisfying Eqs.(4-12) or (4-13) may be spurious eigenvalue of union set ($Y_\ell(k\rho) = 0$ or $Y'_\ell(k\rho) = 0$) or true eigenvalue of intersection set ($J'_\ell(k\rho) = 0$) to satisfy both Eqs.(4-12) and (4-13).

4-2-3 True and spurious eigensolutions by using the imaginary-part BEM

By employing the imaginary-part kernels in the UT and LM equations for the Dirichlet eigenproblem, we obtain the eigenequations,

$$UT: \quad J_\ell(k\rho)J_\ell(k\rho) = 0, \quad \ell = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N, \tag{4-14}$$

$$LM: \quad J'_\ell(k\rho)J_\ell(k\rho) = 0, \quad \ell = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N, \tag{4-15}$$

respectively. The k values satisfying Eqs.(4-14) or (4-15) may be spurious eigenvalues of union set ($J_\ell(k\rho) = 0$ or $J'_\ell(k\rho) = 0$) or true eigenvalues of intersection set ($J_\ell(k\rho) = 0$) to satisfy both Eqs.(4-14) and (4-15).

For the Neumann problem, we obtain the eigenequations,

$$UT: \quad J_\ell(k\rho)J'_\ell(k\rho) = 0, \quad \ell = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N, \tag{4-16}$$

$$LM: \quad J'_\ell(k\rho)J'_\ell(k\rho) = 0, \quad \ell = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N, \tag{4-17}$$

respectively. The k values satisfying Eqs.(4-16) or (4-17) may be spurious eigenvalues of union set ($J_\ell(k\rho) = 0$ or $J'_\ell(k\rho) = 0$) or true eigenvalues of intersection set ($J'_\ell(k\rho) = 0$) to satisfy both Eqs.(4-16) and (4-17).

4-2-4 True and spurious eigensolutions by using the MRM

By employing the MRM kernels in the UT and LM equations for the Dirichlet eigenproblem, we can summarize the eigenequations as follows [36],

$$\text{True eigenequation:} \;\; J_n(k\rho) = 0, \tag{4-18}$$

By using the direct MRM (UT) formulation, we have

$$\text{Spurious eigenequation:} \;\; \frac{\pi}{2}Y_0(k\rho) - (\ln\frac{k}{2} + \gamma)J_0(k\rho) = 0. \tag{4-19}$$

By employing the MRM kernels in UT and LM equations for the Neumann problem, we can summarize the true eigenequation as follows,

$$\text{True eigenequation:} \;\; J_n^{'}(k\rho) = 0, \tag{4-20}$$

By using the direct MRM (LM), we have

$$\text{Spurious eigenequation:} \;\; \frac{\pi}{2}Y_0^{'}(k\rho) - (\ln\frac{k}{2} + \gamma)J_0^{'}(k\rho) = 0. \tag{4-21}$$

4-3 Extraction of the spurious eigensolutions by using the Fredholm alternative theorem and SVD updating techniques

Fredholm alternative theorem:

The linear algebraic equation $[K]\{u\} = \{\bar{b}\}$ has a unique solution if and only if the only continuous solution to the homogeneous equation

$$[K]\{u\} = \{0\} \tag{4-22}$$

is $\{u\} \equiv \{0\}$. Alternatively, the homogeneous equation has at least one solution if the homogeneous adjoint equation

$$[K]^T\{\phi\} = \{0\} \tag{4-23}$$

has a nontrivial solution $\{\phi\}$, where $[K]^T$ is the transpose conjugate matrix of $[K]$ and $\{\bar{b}\}$ must satisfy the constraint $(\{\bar{b}\}^T\{\phi\} = 0)$. By using the UT formulation, we have

$$[U(k)]\ \{t\} = [T(k)]\ \{u\} = \{\bar{b}\}. \tag{4-24}$$

According to the Fredholm alternative theorem, Eq.(4-24) has at least one solution for $\{t\}$ if the homogeneous adjoint equation

$$[U(k_s)]^T\{\phi_1\} = \{0\}, \tag{4-25}$$

has a nontrivial solution $\{\phi_1\}$, in which the constraint $(\{\bar{b}\}^T\{\phi_1\} = 0)$ must be satisfied. By substituting $\{\bar{b}\} = [T(k)]\{u\}$ in Eq. (4-24) into $\{\bar{b}\}^T\{\phi_1\} = 0$, we obtain

$$\{u\}^T [T(k_s)]^T \{\phi_1\} = 0. \tag{4-26}$$

Since $\{u\}$ is an arbitrary vector for the Dirichlet problem, we have

$$[T(k_s)]^T\{\phi_1\} = \{0\}, \tag{4-27}$$

where $\{\phi_1\}$ is the spurious mode. Combining Eq. (4-25) and Eq. (4-27) together, we have

$$\begin{bmatrix} [U(k_s)]^T \\ [T(k_s)]^T \end{bmatrix} \{\phi_1\} = \{0\} \ \text{ or } \ \{\phi_1\}^T \begin{bmatrix} [U(k_s)] & [T(k_s)] \end{bmatrix} = \{0\}. \tag{4-28}$$

Eq.(4-28) indicates that the two matrices have the same spurious mode $\{\phi_1\}$ corresponding to the same zero singular value for the spurious eigenvalue k_s. The former one in Eq.(4-28) is a form of updating term and the latter one is a form of updating document. By using the real-part BEM (UT formulation) in conjunction with the Fredholm alternative theorem and SVD updating techniques, the spurious eigenvalue k_s satisfies

$$\begin{bmatrix} [U_R(k_s)]^T \\ [T_R(k_s)]^T \end{bmatrix} \{\phi_R^{(UT)}\} = \{0\}, \tag{4-29}$$

where the subscript R denotes the real part.

In the hypersingular formulation (LM method), the spurious eigenvalue satisfies

$$\begin{bmatrix} [L_R(k_s)]^T \\ [M_R(k_s)]^T \end{bmatrix} \{\phi_R^{(LM)}\} = \{0\}. \tag{4-30}$$

By using the imaginary-part BEM, the spurious eigenvalue satisfies

$$\begin{bmatrix} [U_I(k_s)]^T \\ [T_I(k_s)]^T \end{bmatrix} \{\phi_I^{(UT)}\} = \{0\}, \tag{4-31}$$

where the subscript I denotes the imaginary part. In the hypersingular formulation of imaginary-part BEM, the spurious eigenvalue satisfies

$$\begin{bmatrix} [L_I(k_s)]^T \\ [M_I(k_s)]^T \end{bmatrix} \{\phi_I^{(LM)}\} = \{0\}. \tag{4-32}$$

4-4 Extraction of the true eigensolutions by using the Fredholm alternative theorem and SVD updating techniques

For the Dirichlet eigenproblem, the true eigenvalue k_t satisfies

$$\begin{bmatrix} [U_R(k_t)] \\ [L_R(k_t)] \end{bmatrix} \{\psi_R^{(UL)}\} = \{0\}, \tag{4-33}$$

and

$$\begin{bmatrix} [U_I(k_t)] \\ [L_I(k_t)] \end{bmatrix} \{\psi_I^{(UL)}\} = \{0\}, \tag{4-34}$$

by using the real-part and imaginary-part BEMs, respectively.

For the Neumann problem, the true eigenvalue can be sorted out by using

$$\begin{bmatrix} [T_R(k_t)] \\ [M_R(k_t)] \end{bmatrix} \{\psi_R^{(TM)}\} = \{0\}, \tag{4-35}$$

and

$$\begin{bmatrix} [T_I(k_t)] \\ [M_I(k_t)] \end{bmatrix} \{\psi_I^{(TM)}\} = \{0\}, \tag{4-36}$$

by using the real-part and imaginary-part BEMs, respectively.

General speaking, the SVD structure for the four influence matrices in the dual BEM are unified in Tables 4-1(a) and 4-1(b) when $k = k_s$ and $k = k_t$, respectively. .

4-5 Numerical examples

Both the Dirichlet and Neumann eigenproblems for a circular domain with radius a m are considered here. The true and spurious eigenvalues are shown in Tables 4-2 and 4-3 by employing various approaches, the real-part and the imaginary-part BEMs as well as singular and hypersingular formulations.

In Table 4-2, the real-part BEM is used for the interior eigenproblem and twenty constant elements are adopted on the boundary. For the Dirichlet eigenproblem, the true eigenvalues, $J_n(ka) = 0$, can be found by checking the same dropping positions in the the figures of the local minimum singular value obtained from $[U]$ and $[L]$ matrices. For the Neumann eigenproblem, the true eigenvalues $J_n'(ka) = 0$, are also found in the similar way by checking local minimum singular value obtained from the $[T]$ and $[M]$ matrices. The local minimum singular value obtained from the updating matrices, $[U \;\; L]^T$, and $[T \;\; M]^T$ occurs in the true eigenvalues. It is found that $[U]$ and $[T]$ matrices have the same spurious eigenvalues of $Y_n(ka) = 0$ by using the singular formulation. In the hypersingular formulation, $[L]$ and $[M]$ matrices have the same spurious eigenvalues of $Y_n'(ka) = 0$. The updating matrices, $[U \;\; T]$, and $[L \;\; M]$ can sort out the spurious eigenvalues, $Y_n(ka) = 0$ and $Y_n'(ka) = 0$ by using the SVD, respectively.

By using the imaginary-part BEM, only eight constant elements are used on the boundary in order to avoid the ill-conditioned matrix. The results are shown in Table 4-3. For the Dirichlet eigenproblem, the true eigenvalues, $J_n(ka) = 0$, can be found by checking the same dropping positions in the the figures of the local minimum singular value obtained from $[U]$ and $[L]$ matrices. For the Neumann eigenproblem, the true eigenvalues $J_n'(ka) = 0$, are also found in the similar way by checking local minimum singular value obtained from the $[T]$ and $[M]$ matrices. The local minimum singular value obtained from the updating matrices, $[U \;\; L]^T$, and $[T \;\; M]^T$ occurs in the true eigenvalues. It is found that $[U]$ and $[T]$ matrices have the same spurious eigenvalues of $J_n(ka) = 0$ by using the singular formulation. In the hypersingular formulation, $[L]$ and $[M]$ matrices have the same spurious eigenvalues of $J_n'(ka) = 0$. The updating matrices, $[U \;\; T]$, and $[L \;\; M]$ can sort out the spurious eigenvalues, $J_n(ka) = 0$ and $J_n'(ka) = 0$ by using the SVD, respectively. In this case, spurious multiplicity appears since spurious since spurious eigenvalues are equal to true ones.

The true and spurious eigenvalues by using MRM are shown in Table 4-4. It is found that all the figures drop at the positions as predicted analytically in Eqs.(4-29)$\sim$(4-36).

4-6 Conclusions

By using the Fredholm alternative theorem and SVD techniques in conjunction with the dual formulations, the true and spurious eigenvalues in the complex-valued formulation, the real-part, the imaginary-part BEMs and MRM are sorted out successfully. The numerical results agree well with the analytical prediction. Although Table 4-2, 4-3 and 4-4 match well with the analytical prediction, it is worth mentioning that the imaginary-part BEM becomes ill-conditioned once the number of element increaesd. Ill-conditioned behavior is inherent in the regular formulation and deserves further study.

Chapter 5
Fictitious frequency revisited

Summary

The nonexistence and nonuniqueness problems associated with integral equation methods for exterior acoustics are revisited. Based on the Fredholm alternative theorem in conjunction with the SVD updating technique, the fictitious frequency and mode can be extracted. After selecting the CHIEF points, we can obtain the influence row vectors. A criterion in selecting the minimum number of CHIEF points and their positions is developed to check the validity by testing the orthogonality condition between the influence row vector and right unitary vector. It is proved in the discrete system that the source of numerical instability originates from the zero division by zero by using the generalized coordinates of unitary vectors in SVD. A flowchart to detect the fictitious frequency and to overcome the numerical instability by the CHIEF method is plotted and implemented in our program. Radiation problems of a cylinder and a square rod are demonstrated to see the validity of the present formulation.

5-1 Introduction

Boundary element method has been used for solving radiation and scattering problems [15, 20] for many years. The fictitious-frequency problems in the exterior acoustics have the same rank-deficiency mechanism as the spurious eigenvalue appears in the interior eigenproblem when the multiple reciprocity BEM, the real-part or the imaginary-part BEM is employed. In a fictitious-frequency problem of the exterior acoustics, the ill-conditioned matrices occurring in the BEM [49] are linearly dependent, *i.e.*, they are rank deficient. For this problem, Schenck [122] proposed the CHIEF (Combined Helmholtz Interior integral Equation Formulation) method by collocating the point outside the domain as an auxiliary constraint to promote the rank of influence matrices. Chen *et al.* extended the CHIEF method to CHEEF method for overcoming the spurious eigenvalues. However, this method still has some drawbacks. If the CHIEF point locates on or near the nodal line of interior modes, it can not provid a valid constraint [85]. To overcome this problem, Chen *et al.* [15] presented the analytical study to select the valid CHIEF points for the circular case using circulants. For the same purpose to general cases, a criterion for checking the validity of the selected CHIEF points will be addressed in detail by employing the Fredholm alternative theorem and the SVD updating techniques. Numerical examples will be demonstrated to see the validity of the present formulation.

5-2 Problem statement and review of the CHIEF method

In this section, the CHIEF method for the two-dimensional Helmholtz equation is briefly summarized here. The governing equation for the exterior acoustics is

$$\nabla^2 u(x) + k^2 u(x) = 0, \;\; x \;\; in \;\; D, \tag{5-1}$$

where $u(x)$ and k are the acoustic pressure and the wave number, respectively. To solve the problem by using the boundary integral formulation, we have

$$\pi u(x) = C.P.V. \int_B T(s,x)u(s)dB(s) - R.P.V. \int_B U(s,x)\frac{\partial u(s)}{\partial n_s}dB(s), \tag{5-2}$$

where x is the field point, s is the source point, n_s is the normal vector for the boundary point s, $C.P.V.$ and $R.P.V.$ denote the Cauchy principal value and Riemann principal value, respectively. By discretizing the boundary integral formulation (BIE) in Eq.(5-2) into N constant elements, the linear algebraic equation can be obtained

$$[U]\{t\} = [T]\{u\}, \tag{5-3}$$

where $[U]$ and $[T]$ are the influence matrices [44]. For the ficitious frequency case, the influence matrix is singular, *i.e.*, the rank is deficient. In order to promote the rank, the CHIEF method by collocating the points outside the domain as auxiliary constraints was successfully applied to deal with this problem. By collocating the point outside the domain for the null-field BIE, the additional constraint is

$$< w > \;\; \{t\} = < v > \;\; \{u\}, \tag{5-4}$$

where $< w >$ and $< v >$ are the influence row vectors by collocating the point in the null-field equation. By combining Eq.(5-3) with Eq.(5-4), we have the over-determined system

$$\left[\begin{array}{c} [U] \\ < w > \end{array} \right] \left\{ \ t \ \right\} = \left[\begin{array}{c} [T] \\ < v > \end{array} \right] \left\{ \ u \ \right\}, \tag{5-5}$$

if the sufficient CHIEF points are provided.

5-3 Detection of the fictitious frequency and ficitious mode in BEM for exterior acoustics using the Fredholm alternative theorem and SVD technique

Fredholm alternative theorem:

The linear algebraic equation $[K]\{u\} = \{b\}$ has a unique solution if and only if the continuous solution to the homogeneous equation

$$[K]\{u\} = \{0\}, \tag{5-6}$$

is $\{u\} \equiv \{0\}$. Alternatively, the homogeneous equation has at least one solution if the homogeneous adjoint equation

$$[K]^H\{\phi\} = \{0\} \tag{5-7}$$

has a nontrivial solution $\{\phi\}$, where $[K]^H$ is the transpose conjugate matrix of $[K]$ and $\{b\}$ must satisfy the constraint $(\{b\}^H\{\phi\} = 0)$. By using the UT formulation, we have

$$[U(k)] \ \{t\} = [T(k)] \ \{u\} = \{b\}. \tag{5-8}$$

According to the Fredholm alternative theorem, Eq.(5-8) has at least one solution for $\{t\}$ if the homogeneous adjoint equation

$$[U(k_f)]^H \{\phi_1\} = \{0\} \tag{5-9}$$

has a nontrivial solution $\{\phi_1\}$, where k_f is the fictitious wave number. For the Dirichlet problem, the constraint $(\{b\}^H\{\phi_1\} = 0)$ must be satisfied. By substituting $\{b\} = [T(k_f)]\{u\}$ in Eq. (5-8) into $\{b\}^H\{\phi_1\} = 0$, we obtain

$$\{u\}^H [T(k_f)]^H \{\phi_1\} = 0. \tag{5-10}$$

Since $\{u\}$ is an arbitrary vector, we have

$$[T(k_f)]^H \{\phi_1\} = \{0\}, \tag{5-11}$$

where $\{\phi_1\}$ is the ficitious mode. Combining Eq.(5-9) and Eq.(5-11) together, we have

$$\begin{bmatrix} [U(k_f)]^H \\ [T(k_f)]^H \end{bmatrix} \{\phi_1\} = \{0\} \text{ or } \{\phi_1\}^H \begin{bmatrix} [U(k_f)] & [T(k_f)] \end{bmatrix} = \{0\}. \tag{5-12}$$

Eq.(5-12) indicates that the two matrices have the same spurious mode $\{\phi_1\}$ corresponding to the same zero singular value when rank deficiency occurs in case of ficitious frequency. The former one in Eq.(5-12) is a form of updating term and the latter one is a form of updating document.

By using the singular and hypersingular formulations, the fictitious wave number, k_f of a multiplicity P, satisfies

$$\begin{bmatrix} [U_i(k_f)]^H \\ [T_i(k_f)]^H \end{bmatrix} \{\phi_j\} = \{0\}, \quad j = 1, 2, \cdots, P \tag{5-13}$$

$$\begin{bmatrix} [L_i(k_f)]^H \\ [M_i(k_f)]^H \end{bmatrix} \{\phi_j\} = \{0\}, \quad j = 1, 2, \cdots, P \tag{5-14}$$

where the subscript i denotes the use of interior degenerate kernel for the exterior problem.

5-4 Mathemetical structure for the updating matrix

According to the SVD technique, Eq.(5-13) results in

$$[U_i]\{\psi_j^{(U)}\} = 0\{\phi_j\} = \{0\}, \quad j = 1, 2, \cdots, P \tag{5-15}$$

$$[T_i]\{\psi_j^{(T)}\} = 0\{\phi_j\} = \{0\}, \quad j = 1, 2, \cdots, P \tag{5-16}$$

where $\{\psi_j^{(U)}\}$ and $\{\psi_j^{(T)}\}$ are the right unitary vectors for $[U]$ and $[T]$, $\{\phi_j\}$ are the common left unitary vectors. By using the updating term for deriving the true boundary modes $\{\psi_j^D\}$ in the interior Dirichlet eigenproblem, we have

$$\begin{bmatrix} [U_e] \\ [L_e] \end{bmatrix} \{\psi_j^D\} = \{0\}, \quad j = 1, 2, \cdots, P \tag{5-17}$$

where the subscript e denotes the use of exterior degenerate kernel for the interior problem. Since the kernel functions have the symmetry and transponse symmetry properities, we have

$$U_e(s,x) = U_i(x,s) \quad \text{or} \quad [U_e] = [U_i] \quad \text{symmetry,} \tag{5-18}$$

and

$$L_e(s,x) = T_i(x,s) \quad \text{or} \quad [L_e] = [T_i] \quad \text{transponse symmetry.} \tag{5-19}$$

By using Eqs.(5-18) and (5-19), Eq.(5-17) reduces to

$$\begin{bmatrix} [U_i] \\ [T_i] \end{bmatrix} \{\psi_j^D\} = \{0\} \qquad j = 1, 2, \cdots, P. \tag{5-20}$$

Comparing Eq.(5-20) with Eqs.(5-15) and (5-16), we find

$$\{\psi_j^{(U)}\} = \{\psi_j^{(T)}\} = \{\psi_j^D\} \qquad j = 1, 2, \cdots, P. \tag{5-21}$$

It means that the $[U_i]$ and $[T_i]$ matrices for the exterior acoustics, have the same right singular vectors ($\{\psi_j^D\}$) as the $[U_e]$ and $[L_e]$ matrices have for the interior Dirichlet eigenproblem.

In order to examine the left and right singular vectors in the singular matrix, Eq.(5-13) can be rewritten as follows:

$$\begin{bmatrix} [U_i(k_f)]^H \\ [T_i(k_f)]^H \end{bmatrix}_{2N\times N} \{\phi_j\}_{N\times 1} = 0 \begin{Bmatrix} \psi_j^D \\ \psi_j^D \end{Bmatrix}_{2N\times 1} \qquad j = 1, 2, \cdots, P. \tag{5-22}$$

Generally speaking, the matrix of Eq.(5-22) can be decomposed into

$$\begin{bmatrix} [U_i]^H \\ [T_i]^H \end{bmatrix}_{2N\times N} = [\Psi^D]_{2N\times 2N}[\Sigma]_{2N\times N}[\Phi]^H_{N\times N}, \tag{5-23}$$

where

$$[\Psi^D]_{2N\times 2N} = \begin{bmatrix} \{\psi_1^D\} & \cdots & \{\psi_P^D\} & | & \{\psi_{P+1}\} & \cdots & \{\psi_{2N}\} \\ \{\psi_1^D\} & \cdots & \{\psi_P^D\} & | & \{\psi_{P+1}\} & \cdots & \{\psi_{2N}\} \end{bmatrix}_{2N\times 2N}, \tag{5-24}$$

$$
[\Sigma]_{2N\times N} = \begin{bmatrix} [0]_{P\times P} & \cdots & \cdots & 0 \\ \vdots & \sigma_{P+1} & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & \sigma_N \\ 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix}_{2N\times N} \tag{5-25}
$$

and

$$
[\Phi]^H_{N\times N} = \begin{bmatrix} \{\phi_1\} & \cdots & \{\phi_P\} & | & \{\phi_{P+1}\} & \cdots & \{\phi_N\} \end{bmatrix}^H_{N\times N}, \tag{5-26}
$$

Eq.(5-23) indicates that all the ficitious modes $\{\phi_i\},\ \ 1 \le i \le P$, and the true modes $\{\psi_i^D\},\ \ 1 \le i \le P$, are obtained at the same time once the updating matrix is decomposed by SVD technique.

In other words, the SVD structure for the four influence matrices in the dual BEM can be unified in Table 5-1.

5-5 Source of numerical instability - zero division by zero

The analytical study and numerical experiments for the optimum numbers and proper positions of the selected CHIEF points have been proposed by Chen *et al.* [15] for a circular case. However, we will extend to the general case in a discrete system. In the case of the fictitious frequency of multiplicity P, P CHIEF points are needed. One can obtain P fictitious modes by using Eq.(5-13). The source of number instability is proved as follows:

According to the right unitary vectors $\{\psi_i\}$ for $[T]$ and $[U]$ matrices, we can express the boundary data into

$$
\{u\} = \sum_{i=1}^{N} \beta_i \{\psi_i^{(T)}\} = [\Psi^{(T)}]\{\beta\}, \tag{5-27}
$$

$$
\{t\} = \sum_{i=1}^{N} \alpha_i \{\psi_i^{(U)}\} = [\Psi^{(U)}]\{\alpha\}, \tag{5-28}
$$

where N is the number of unknowns, α_i and β_i are the generalized coordinates. By using the SVD technique, Eq.(5-8) can be rewritten to

$$[\Phi^{(U)}][\Sigma^{(U)}]\{\alpha\} = \{b\}, \tag{5-29}$$

By pre-multiplying the regular mode $\{\phi_i^{(U)}\}^H, \ \ P+1 \leq i \leq N$, to both sides of Eq.(5-29), we have

$$\sigma_i^{(U)}\alpha_i = \{\phi_i^{(U)}\}^H\{b\}, \quad P+1 \leq i \leq N. \tag{5-30}$$

Since the singular values $\sigma_i^{(U)}, \ \ P+1 \leq i \leq N$, are nonzero, the generalized coordinates $\alpha_i, \ \ P+1 \leq i \leq N$, can be determined by

$$\alpha_i = \frac{1}{\sigma_i^{(U)}}\{\phi_i^{(U)}\}^H\{b\}, \quad P+1 \leq i \leq N. \tag{5-31}$$

By pre-multiplying the fictitious mode $\{\phi_i^{(U)}\}^H, \ \ 1 \leq i \leq P$, to both sides of Eq.(5-29) and using orthogonal propurty, we have

$$\sigma_i^{(U)}\alpha_i = \{\phi_i^{(U)}\}^H\{b\}, \quad 1 \leq i \leq P. \tag{5-32}$$

Since the singular values $\sigma_i^{(U)}, \ \ 1 \leq i \leq P$, are zero, the coefficients $\alpha_i, \ \ 1 \leq i \leq P$, can not be determined due to zero division by zero from Eq.(5-31) in the fictitious case of multiplicity P.

It is interesting to find that the generalized coordinates, $\alpha_1, \alpha_2, \cdots$ and α_P are the terms of zero division by zero in Eq.(5-32) since

$$\{\phi_i^{(U)}\}^H[T]\{\bar{u}\} = 0, \quad P+1 \leq i \leq N, \tag{5-33}$$

after using $\{b\} = [T]\{\bar{u}\}$ and $[T]^H\{\phi_i^{(U)}\} = 0$.

5-6 A criterion to check the validity of CHIEF points

For the fictitious frequency of a multiplicity P, the generalized coordinates $\alpha_1, \alpha_2, \cdots, \alpha_{P-1}$ and α_P can not be determined from Eq.(5-31). By choosing P CHIEF points, we have additional constraints

$$\left[\begin{array}{c|c} U_{PP} & U_{PK} \end{array}\right]\left\{\begin{array}{c} \alpha \end{array}\right\} = \left[\begin{array}{c|c} T_{PP} & T_{PK} \end{array}\right]\left\{\begin{array}{c} \beta \end{array}\right\}, \tag{5-34}$$

where the subscripts P and K denote the degree of freedom separated by the fictitious set $(1, 2, \cdots, P)$ and the regular set $(P+1, P+2, \cdots, N)$. The elements in $[U_{PP}]$, $[U_{PK}]$, $[T_{PP}]$ and $[T_{PK}]$ are definded as

$$(U_{PP})_{ij} = < w_i > \{\psi_j^{(U)}\}, \quad 1 \leq i, j \leq P, \tag{5-35}$$

$$(U_{PK})_{ij} = < w_i > \{\psi_j^{(U)}\}, \; 1 \leq i \leq P, \quad P+1 \leq j \leq N, \tag{5-36}$$

$$(T_{PP})_{ij} = < v_i > \{\psi_j^{(T)}\}, \quad 1 \leq i, j \leq P, \tag{5-37}$$

$$(T_{PK})_{ij} = < v_i > \{\psi_j^{(T)}\}, \; 1 \leq i \leq P, \quad P+1 \leq j \leq N. \tag{5-38}$$

Since $\alpha_{P+1}, \alpha_{P+2}, \cdots$, and α_N can be determined by Eq.(5-31), and the influence row vectors $< w_i >, \; i = 1, 2, \cdots, P$, can be obtained by collocating the CHIEF points, Eq.(5-34) reduces to

$$\left[\, U_{PP} \,\right] \left\{ \begin{array}{c} \alpha_1 \\ \vdots \\ \alpha_P \end{array} \right\} = \left[\, T_{PP} \;\; | \;\; T_{PK} \,\right] \left\{ \begin{array}{c} \beta_1 \\ \vdots \\ \beta_N \end{array} \right\} - \left[\, U_{PK} \,\right] \left\{ \begin{array}{c} \alpha_{P+1} \\ \vdots \\ \alpha_N \end{array} \right\} = \{f\}, \tag{5-39}$$

The terms of the right hand side of the equal sign can be calculated as a load vector $\{f\}$ since their values can be determined. The unknown vector, $\{\alpha\}_{P\times 1}$ is solvable once the determinant of the matrix $[U_{PP}]$ is nonzero as follows:

$$det \left\| \begin{array}{ccc} < w_1 > \{\psi_1^{(U)}\} & \cdots & < w_1 > \{\psi_P^{(U)}\} \\ \vdots & \ddots & \vdots \\ < w_P > \{\psi_1^{(U)}\} & \cdots & < w_P > \{\psi_P^{(U)}\} \end{array} \right\| \neq 0. \tag{5-40}$$

Whether the number of CHIEF point is sufficient or not depends on the multiplicity P, *i.e.*, we need at least P CHIEF points for the fictitious frequency of a multiplicity P to determine the P coefficients $(\alpha_1 \cdots \alpha_p)$. Once the P CHIEF points are selected, their validity depends on the nonzero determinant of Eq.(5-40).

For the special case of multiplicity one $(P = 1)$, Eq.(5-40) reduces to

$$< w > \{\psi_1^{(U)}\} \neq 0. \tag{5-41}$$

By collocating the interior point, the magnitude of the determinant represent the inner product of the influence row vector and the interior mode since $\{\psi^{(U)}\}$ is the true boundary mode of the

Dirichlet eigenproblem. The value is equal to the distribution of interior mode.
For the special case of multiplicity two $(P = 2)$, Eq.(5-40) reduces to,

$$det \begin{Vmatrix} < w_1 > \{\psi_1^{(U)}\} & < w_1 > \{\psi_2^{(U)}\} \\ < w_2 > \{\psi_1^{(U)}\} & < w_2 > \{\psi_2^{(U)}\} \end{Vmatrix} \neq 0. \quad (5\text{-}42)$$

In the following examples, both the multiplicity one $(P = 1)$ and two $(P = 2)$ will be disscussed for cylinder and square rod radiators.

For the Neumann problem, we can also provide the same criterion in a similar way by replacing $< w >$ and $\{\psi_j^{(U)}\}$ with $< v >$ and $\{\psi_j^{(T)}\}$, respectively.

5-7 Numerical examples

Case 1: infinite cylinder radiation

An exterior acoustic problem of a circular boundary with radius $a = 1\ m$ for the Dirichlet cylinder conditions is considered here. According to the flowchart illustrated in Fig.5-1, the Fredholm alternative theorem and SVD updating techniques are employed to detect the fictitious frequencies as shown in Fig.5-2. It is found that $[U_i]$ and $[T_i]$ matrices have the same ficitious poles of $J_n(ka) = 0$. The spurious poles agree with the true poles of the interior Dirichlet eigenproblem. For the hypersingular formulation, $[L]$ and $[M]$ matrices also have the same fictitious poles of $J_n^{'}(ka) = 0$ which are the true eigenvalues for the Neumann problem. After checking the multiplicity of the fictitious pole, two cases of multiplicity one $(k = J_0^{(2)} = 0)$, and multiplicity two $(k = J_1^{(1)} = 0)$, are adopted for demonstrating the validity of the present formulation.

1. Multiplicity of one $(k = J_0^{(2)})$:

For selecting all the possible CHIEF points, their positions locate inside the circle as shown in Fig.5-3(a). In this case, the determinants of Eq.(5-41) were calculated for each interior point and were plotted as shown in Fig.5-3(b). Contour plot shows the distribution of the magnitude of the real and imaginary parts of determinant. The selected CHIEF point of the darker color is vailder than the point with the whiter color. The failure points are found on the nodal line with white color and the results matched well with the analytical prediction [15].

2. Multiplicity of two ($k = J_1^{(1)}$):

In this case, one CHIEF point was fixed first and then consider the other CHIEF points as a variable. For selecting all the possible CHIEF points for the variable point, their positions located inside the circle as shown in Fig.5-3(b). The value of determinants were calculated by changing the second CHIEF points position in the interior rigion and were plotted in Fig.5-4. Contour plot shows the distribution of the magnitude of the real and imaginary parts of determinants. The selected CHIEF point of the darker color is vailder than the point with the whiter color. The failure points are found on the nodal line with white color and the results matched well with the analytical data [15].

Case 2: infinite square radiation

An exterior acoustic problem of a square boundary with lateral length $a = 2\ m$ for the Dirichlet boundary conditions is considered here. The Fredholm alternative theorem and SVD updating techniques are employed to detect the fictitious frequencies as shown in Fig.5-5.

1. Multiplicity of one ($k = 2.22$) and ($k = 4.44$) :

For selecting all the possible CHIEF points, their positions located inside the square as shown in Fig.5-6(a). In this case, the determinants of Eq.(5-41) were calculated one by one and were plotted as shown in Fig.5-6(b). Contour plot shows the distribution of the magnitude of the real and imaginary parts of determinant. The selected CHIEF point of the darker color is vailder than the point with the whiter color. The failure points are found on the nodal line with white color.

2. Multiplicity of two ($k = 3.51$):

In this case, one CHIEF point was fixed first and then consider the other CHIEF points as a variable. For selecting all the possible CHIEF points, their positions spread inside the square as shown in Fig.5-6(a). The determinants were calculated by changing the second CHIEF points location in the square and were plotted in Fig.5-7. Contour plot shows the distribution of the magnitude of the real and imaginary parts of determinants. The selected CHIEF point of the darker color is vailder than the point with the whiter color. The failure points are found on the nodal line with white color.

5-8 Conclusions

In order to overcome the rank-deficiency problem due to fictitious frequency, the CHIEF method was revisited and reformulated in a unified manner by using the Fredholm alternative theorem and SVD technique. The ficitious modes were obtained in the singular vectors of SVD as well as the true eigenmodes for the interior problems at the same time once the updating matrix was decomposed by using the SVD technique. Besides, the minimum number of CHIEF points was also addressed. A criterion for checking the validity of the CHIEF points was presented analytically in the discrete system. In addition, the source of numerical instability due to fictitious frequencies was found to originate from the zero divison by zero. Numerical examples of the cylinder and square rod radiators were demonstrated to see the validity of the unified formulation.

Chapter 6
Conclusions and further research

6-1 Conclusions

Four degenerate problems in the BEM were reviewed in this thesis. Mathematically speaking, the numerical problems originate from the rank deficiency of the influence matrix. Their rank-deficiency mechanisms were found and the numerical instability was solved in a unified manner by using the Fredholm alternative theorem and SVD techniques. From this study, several conclusions can be summarized as follows:

1. For the interior eigenproblem and exterior problem, spurious (fictitious) mode and true mode were separated to be imbedded in the left and right unitary vectors, respectively. after decomposing the influence matrix using the SVD updating techniques, Fredholm alternative theorem was adopted to obtain the updating documents in SVD.

2. In Chapter 2, it has been proved that the degenerate scale occured in the Dirichlet problem of 2-D Laplace problems by using the BEM. The conventional BEM (UT formulation) can not obtain acceptable results for the torsion bar problems with the degenerate scale. For an arbitrary cross section, instead of direct searching for the degenerate scale by trial and error, a more efficient technique was proposed to directly determine the degenerate scale since only one normal scale needs to be computed. Three regularization techniques, method of adding a rigid body mode, the hypersingular formulation and the CHEEF method, were successfully applied to overcome the rank-deficiency problem caused by the degenerate scale. Also, the added term "c" of a rigid body mode in the fundamental solution of BEM has been proven to shift to another degenerate scale by a factor of "e^{-c}".

3. The degenerate boundary has been solved by using the multi-domain BEM and the dual BEM. However, for the multi-domain BEM, one important drawback is incapability of dealing with infinite domain or semi-infinite domain problems. For the dual BEM, hyper-singular integrals must be handled. In Chapter 3, a new method, conventional BEM (UT

equation) in conjunction with the SVD techniques, for solving the degenerate boundary problem, was presented. The mechanism of rank-deficiency for the degenerate boundary problem stems form two sources, one is the degenerate boundary and the other is nontrivial eigensolution.

4. For interior eigenproblems, some constraints are lost if either the real-part or the imaginary-part dual BEM was used. In other words, the appearance of the spurious eigenvalue originates from the selected numerical methods, *e.g.*, UT equation, LM equation, single-layer method and double-layer method. For the real-part or imaginary-part dual BEM, the Fredholm alternative theorem in conjunction with SVD updating techniques was employed to extract the spurious eigenvalues for singular and hypersingular formulations. Besides, true eigenvalues can be detected for the Dirichlet or Neumann problem by using the SVD technique for the dual BEM.

5. In Chapter 5, a criterion was developed to check the validity of the selected CHIEF points by testing the orthogonality condition between the influence vector of collocation point and right singular vector. For exterior problems, the number of the required CHIEF points depend on the multiplicity of the corresponding fictitious eigenvalue. The fictitious mode can be extracted by using the SVD updating technique. The value of the inner product provides the valid (nonzero) or invalid (zero) information. Numerical results agree well.

6-2 Further research

There are several researches need further investigation as follows:

1. Although the degenerate scale occurs in the Dirichlet problem of simply two-dimensional Laplace problems by using the BEM, there is no proof of the occurrence of degenerate scale for the problem with the mixed-type boundary condition.

2. In a continuous system, the added term "c" of a rigid body mode in the fundamental solution of BEM has been proven to shift to another degenerate scale by a factor of "e^{-c}". However, the proof may be extended to the discrete system.

3. On the basis of the success of Chapter 3, the degenerate boundary problem for the Laplace equation may be solved by using the conventional BEM (UT formulation) and the SVD techniques. Also, the mathematical relation between the present method and the multi-domain BEM should be constructed.

4. The main drawback of the imaginary-part BEM seems to produce ill-conditional matrices. While this is sometimes the case, it is hoped that further research can alleviate the drawback. In addition, the mathematical equivalence between the imaginary-part BEM, the Trefftz method, the edge function method and the boundary collocation method (BCM) needs further investigation.

5. Whether the spurious (ficitious) modes in the UT and LM formulations are the same or not deserves further study.

6. In chapter 5, a criterion for a circular domain has been developed to check the validity of the selected CHIEF point by testing the orthogonality condition between the influence vector and fictitious mode. This criterion could be tested for problems with general boundaries.

7. Although we have proved the existence of degenerate scale in BEM for 2-D Laplace problem, the uniqueness theorem needs further examination.

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